Exploiting Gossen's second law: a simple proof of the Euler equation and the maximum principle

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Abstract: We offer simple and intuitive proofs of the Euler equation and the maximum principle based on Gossen's Second Law, one of the best known results in economics.

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1. Introduction

Gossen's Second Law, literally stating that the marginal utility of one extra dollar (MUD) spent on each consumption good is the same for all the consumption goods as required by budget-constrained utility maximization, is unquestionably one of the best known results to students in economics. On the other hand, in the inter-temporal decision context, Euler's equation proves to be a far more powerful tool, from which one can readily obtain Gossen's second law in its inter-temporal version wherein the same goods (or services) at different times are formally viewed as different goods defined by the date and hence MUD remains the same across time. This short article aims to show that one can indeed *reverse* the reasoning, making use of Gossen's second law to prove the Euler equation without resorting to the calculus of variations. (The proof of the Euler equation using the calculus of variations is found in almost any textbook in mathematical economics, see e.g., Lancaster 1987, pp.377-9 and Léonard and Van Long 1995, pp.170-1). Furthermore, by similar argument, the maximum principle can also be established. Our approach has an obvious advantage: it is essentially based on one basic result found in any textbook on intermediate microeconomics, suggesting that, in addition to the familiar exercises of establishing theorems/propositions in economics by using mathematical reasoning, economic intuition may sometimes help establish theorems in mathematics as well. Serving as a nice example, Gossen's second law enables one to gain far more insights into optimization problems than conventionally presumed.

2. Intuition

Consider

$$V = \underset{x(t)\in C^1(t_0,T)}{Max} \int_{t_0}^{T} F(x(t), \dot{x}(t), t)dt \qquad \text{where } F \in C^2 \qquad (1)$$

where *T* could be infinity. Let $y(t) \equiv \dot{x}(t)$. One may interpret problem (1) as utility optimization in that the time-dependent "utility" *F* derives from the amount of goods X_t, $x(t)$, and the amount of Y_t, $y(t)$, for any $t \in [t_0, T]$. Suppose, for convenience, that $x(t)$ is measured in dollars. Note the utility interpretation of problem (1) holds regardless of whether or not *F* increases with $x(t)$ or $\dot{x}(t)$. For the sake of illustration, we may view the decision horizon of problem (1) as a period from year t_0 to year *T*. To simplify notation, the solution to problem (1) is denoted as $x(t)$ still in the rest of this section.

Hypothetically adding one extra dollar at any point of time $t*$ in the *X-Type* commodity space $\{x(t), t \in [t_0, T]\}$ leads to the same change in *V*, by Gossen's second law. But such a change in year $t*$ automatically causes a change in $x(t)$ for any $t > t*$, but has no effect on $x(t)$ for any $t < t*$. That is:

$$x(t) \to \begin{cases} x(t)+\$1 \; when \; t \geq t^* \\ x(t) \quad when \; t < t^* \end{cases},$$

This is illustrated in Figure 1.

Figure 1 inserted here.

What about $\dot{x}(t)$? Intuitively speaking, the change rate in $x(t)$, viz. $\dot{x}(t)$, can be understood as the difference in $x(t)$ between two successive years. For any year before t^*, there is no change in $x(t)$ and hence $\dot{x}(t)$ remains unchanged. For any year after t^*, all $x(t)$ increase by 1, thus $\dot{x}(t)$ also remains unchanged. For the particular year t^*, the amount of X_{t^*} suddenly jumps by one unit, resulting in an increase in the growth rate, $\dot{x}(t)$, by one. Thus,

$$\dot{x}(t) \to \begin{cases} \dot{x}(t)+1 \quad for \quad t = t^* \\ \dot{x}(t) \qquad otherwise \end{cases},$$

As a consequence, the change in V caused by a hypothetical increase in $x(t)$ by one is $\int_t^T \frac{\partial F}{\partial x(s)} ds + \frac{\partial F}{\partial \dot{x}(t)}$, which, by Gossen's second law, must be the same for any $t \in [t_0, T]$ and differentiation of which wrt time t consequently equals zero. Hence, the Euler equation, $\frac{d}{dt}(\frac{\partial F}{\partial \dot{x}(t)}) - \frac{\partial F}{\partial x(t)} = 0$.

3. A new proof of the Euler equation

We now present a rigorous treatment of the above economic intuition. As above, the optimal solution to (1) is denoted as $\{x(t), t \in [t_0, T]\}$. Arbitrarily choose $t^* \in (t_0, T)$ and increase $x(t^*)$ by an arbitrarily small number δ but no change is deliberately made about $\dot{x}(t)$ for all $t \in [t_0, T]$. Thus, after such a marginal change is made about $x(t)$ at t^*, $x(t)$ also increases by δ for any $t \geq t^*$, (refer to Figure 1) and remains unchanged for any $t < t^*$. Note the hypothetically adjusted $\{\tilde{x}(t;t^*,\delta), t \in [t_0, T]\} \equiv \{x(t),$ for $t < t^*$ and $x(t) + \delta,$ for $t \geq t^*\}$ is not continuous at point t^* and` hence $\tilde{x}(t;t^*,\delta) \notin C^1(t_0, T)$ for any $\delta > 0$. Or, more precisely, $\tilde{x}(t;t^*,\delta)$ is right-continuous but not left-continuous at $t = t^*$ for any $\delta > 0$. Letting $\delta \to 0$, the component of the effect of change in $x(t^*)$ through changing $\{x(t), t \in [t_0, T]\}$ on V is thus

$$\int_{t^*}^{T} \frac{\partial F}{\partial x(t)} dt \tag{2}$$

Clearly, $\forall t \neq t^*$. $\dot{\tilde{x}}(t;t^*,\delta) = \dot{x}(t)$, hence $d\dot{\tilde{x}}(t;t^*,\delta)/d\delta = 0$. Consider an arbitrarily small interval $[t^* - \varepsilon, t^*]$ where ε is a small positive number. Let $x(t;\varepsilon,t^*,\delta) \equiv x(t) + \frac{t-(t^*-\varepsilon)}{\varepsilon} \cdot \delta$ for any $t \in [t^* - \varepsilon, t^*]$. Then, $\dot{x}(t;\varepsilon,t^*,\delta) = \dot{x}(t) + \delta/\varepsilon$, hence $d\dot{x}(t;\varepsilon,t^*,\delta)/d\delta = 1/\varepsilon$, $\forall t \in (t^* - \varepsilon, t^*)$. Thus, $d\dot{x}(t;\varepsilon,t^*,\delta)/d\delta$ becomes a delta function as $\varepsilon \to 0$. But $d\dot{x}(t;\varepsilon,t^*,\delta)/d\delta$ is the same as $d\dot{\tilde{x}}(t;t^*,\delta)/d\delta$ at point $t = t^*$. Hence,

$d\dot{\tilde{x}}(t;t^*,\delta)/d\delta$ is also a delta function, for any value of δ, including, in particular, zero; that is, $d\dot{\tilde{x}}(t;t^*,\delta)/d\delta|_{\delta=0}$ is a delta function. Note $\frac{\partial F}{\partial \dot{x}(t)}$ is continuous for any $t \in (t_0, T)$, due to $F \in C^2$ and $x(t) \in C^1$. Also notice that $\tilde{x}(t;t^*,\delta)|_{\delta=0} = x(t)$ and $\dot{\tilde{x}}(t;t^*,\delta)|_{\delta=0} = \dot{x}(t)$ for any $t \in (t_0, T)$. Thus $\frac{\partial F}{\partial \dot{\tilde{x}}(t;t^*,\delta)}|_{\delta=0}$ is a continuous function. By the substitution property of the delta function (see, *e.g.*, Tuckwell 1988, p.51), the effect of change in $x(t^*)$ through changing $\dot{x}(t^*)$ on V equals,

$$\int \{ \frac{\partial F}{\partial \dot{\tilde{x}}(t;t^*,\delta)}|_{\delta=0} \cdot \frac{d\dot{\tilde{x}}(t;t^*,\delta)}{d\delta}|_{\delta=0} \} dt = \frac{\partial F}{\partial \dot{x}(t^*)} \tag{3}$$

Thus, it follows from (2) and (3) that the effect on V of an infinitesimal change in $x(t^*)$, denoted as $\partial V / \partial(x_{t^*} \circ \delta)$ for notational convenience, is

$$\partial V / \partial(x_{t^*} \circ \delta) = \int_{t^*}^{T} \frac{\partial F}{\partial x(t)} dt + \frac{\partial F}{\partial \dot{x}(t^*)} \tag{4}$$

Gossen's second law requires the above must be the same for any value of t^*. Differentiation of the RHS of (4) wrt t^* thus equals zero, i.e., $-\frac{\partial F}{\partial x(t^*)} + \frac{d}{dt}(\frac{\partial F}{\partial \dot{x}(t)})|_{t=t^*} = 0$, $\forall t^* \in (t_0, T)$. The Euler equation is established. Note our proof applies to multi-dimensional Euler equations as well, for which the analysis is essentially the same, yet entailing more cumbersome notations.

4. A new proof of the maximum principle

We consider the control problem with a fixed value of the state variable at the terminal-point (e.g., Léonard and Van Long 1995, Chapter 4),

$$V = \underset{u(t)}{Max} \int_{t_0}^{T} F(x(t), u(t), t)dt \quad \text{where } F \in C^1$$

$$\text{s.t., } \dot{x}(t) = f(x(t), u(t), t), \; x(t_0) = x_0 \text{ and } x(T) = x_T \tag{5}$$

The solution to problem (5) is denoted as $\hat{u}(t)$ and the corresponding state values over $t \in [t_0, T]$ as $\hat{x}(t)$. Clearly, for any arbitrary differentiable function $\pi(t)$,

$$V = \underset{u(t)}{Max} \int_{t_0}^{T} \{F(\hat{x}(t), u(t), t) + \pi(t)[f(\hat{x}(t), u(t), t) - \dot{\hat{x}}(t)]\}dt$$

$$\text{s.t., } \dot{\hat{x}}(t) = f(\hat{x}(t), u(t), t), \; \hat{x}(t_0) = x_0 \text{ and } \hat{x}(T) = x_T \tag{6}$$

As analysed above, Gossen's second law requires that, to adopt the notation introduced in equation (4), $\partial V / \partial(u_t \circ \delta)$ be constant. But, $\partial V / \partial(u_t \circ \delta)$ $= \int_{t}^{T} \{\frac{\partial F}{\partial u(s)} + \pi(s) \cdot \frac{\partial f}{\partial u(s)}\}ds$, of which differentiation wrt t thus equals zero; that is,

$$\frac{\partial F}{\partial u(t)} + \pi(t) \cdot \frac{\partial f}{\partial u(t)} = 0 \tag{7}$$

We signify the solution of (6) as $u(t) = u(\hat{x}(t), \dot{\hat{x}}(t), t)$ for the sake of notational neatness.[1] Problem (5) can thus be equivalently formulated as

[1] A careful reader might be concerned with the possibility that $u(t)$ may also depend on the values of $\hat{x}(t')$ and $\dot{\hat{x}}(t')$ for some $t' \neq t$, even on the path $\{(\hat{x}(s), \dot{\hat{x}}(s)), s \in [t_0, T]\}$. But that is not a problem, for apparently our argument below applies to such a general case.

$$V = \underset{x(t)}{Max} \int_{t_0}^{T} \{F(x(t), u(x(t), \dot{x}(t), t) + \pi(t)[f(x(t), u(x(t), \dot{x}(t), t), t) - \dot{x}(t)]\} dt \quad (8)$$

The solution to (8) is still signified as $x(t)$ hereafter to simplify notation. A similar argument to that in the preceding subsection on $\frac{d}{d\delta} \dot{\tilde{x}}(t; t^*, \delta)\big|_{\delta=0}$ being a delta function at $u(t) = u(\hat{x}(t), \dot{\hat{x}}(t), t)$ t^*, where $\{\tilde{x}(t; t^*, \delta),\ t \in [t_0, T]\} \equiv \{x(t),$ for $t < t^*$ and $x(t) + \delta,$ for $t \geq t^*\}$ for any $\delta \geq 0$, yields, $\partial V / \partial(x_t \circ \delta)$ $= \int_t^T \{\frac{\partial F}{\partial x(s)} + \pi(s) \cdot \frac{\partial f}{\partial x(s)}\} ds - \pi(t)$ $+ \int_t^T \{\frac{\partial F}{\partial u(s)} + \pi(s) \cdot \frac{\partial f}{\partial u(s)}\} \cdot \frac{\partial u}{\partial x(s)} ds$ $+ [\frac{\partial F}{\partial u(t)} + \pi(t) \cdot \frac{\partial f}{\partial u(t)}] \frac{\partial u}{\partial \dot{x}(t)}$. In the light of (7), $\partial V / \partial(x_t \circ \delta)$ $= \int_t^T \{\frac{\partial F}{\partial x(s)} + \pi(s) \cdot \frac{\partial f}{\partial x(s)}\} ds - \pi(t)$, which by Gossen's second law must be the same for any t, and of which differentiation wrt t therefore equals zero. Hence, $\dot{\pi}(t) = -\frac{\partial F}{\partial x(t)} - \pi(t) \cdot \frac{\partial f}{\partial x(t)}$. Of course, $\dot{x}(t) = f(x(t), u(t), t)$ holds. QED.

References

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Figure 1: Marginal change in $x(t^*)$**.**