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GLOBAL STABILITY CONDITIONS ON THE PLANE

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Abstract

The paper considers price adjustment on the plane and derives global stability conditions for such dynamics. First, we examine the well-known Scarf Example, to obtain and analyze a global stability condition for this case. Next, for a general class of excess demand functions, a set of conditions is identified which guarantee not only convergence to some equilibrium but also robustness of these properties.

Key words: global stability conditions, dynamics on the plane, excess demand functions, Dulac's Criterion.

JEL Classification Numbers: C62, C61, D50.

1 Introduction

In economic theory, stability conditions have not been given much importance while in matters of economic policy, such conditions are sometimes assumed at the outset without much ado. In matters of theory, it is well established that excess demand functions are not restricted substantially by routine assumptions such as Walras Law or Homogeneity of degree zero in prices. Consequently since almost "anything goes", the entire topic of stability of equilibrium is relegated to texts and forgotten about.

One of the first things in international trade policy, for example, is to assume the well known Marshall-Lerner Condition, which is nothing other than a *local* stability condition. This is at least a recognition of the fact that unless this condition is met, attainment of equilibrium cannot be ensured, at least locally. For policy considerations, often, it has been standard to assume that markets will clear and attain equilibrium. Since these may entail convergence from arbitrary initial configurations, what one must investigate are *global* stability conditions. There is however, *a priori*, hardly any theoretical reason to assume that without any such condition, convergence is assured. That there may be seemingly robust difficulties, for the stability of the tatonnement, has been usually taken for granted, first, due to the examples in [15] and [5] and then, due to the contributions of [3] and [16].

This dichotomy about the treatment of stability questions, between theory and policy, indicates that while it is implausible to have a theory of adjustment on disequilibrium prices which works for every type of excess demand function, it is meaningful to enquire what conditions would identify a set of excess demand functions which will lead to stable equilibria.

Apart from the intrinsic interest in such an exercise, there is another reason why we should be interested in such stability conditions. Recent work in experimental economics ([1]) has shown that predictions made by tatonnement processes are in fact quite accurate. So even if tatonnement processes may not converge for every conceivable set of excess demand functions, it would be of some importance to identify the excess demand functions for which they do indeed work. This then is the rationale for carrying out the analysis reported in this paper.

We also confine ourselves to identifying such conditions (stability conditions) in the context of adjustment processes on the plane mainly because tools available are best suited towards that objective. Thus basically we have had to restrict attention to systems of excess demand functions involving three goods; one of these goods is identified as numeraire and since prices of other goods are considered relative to this numeraire, the considered adjustment process will define a motion on the plane. There are some conditions which are available for such considerations such as the one in [14] and [11] but, as we hope to show, it is possible to considerably weaken these conditions.

One of the most well known examples of instability is provided by [15]; we consider this as a point of departure; by choosing a numeraire, we restrict the dynamics to be on the plane; it is demonstrated that there are closed orbits: this is the counterpart of the results in [15]. Next, we analyze a stability condition which ensures that the unique equilibrium is globally stable. If this condition is violated then the solution is shown to become unbounded. This exercise allows us to identify the two problems that we will have to encounter in the general case: the first is to guarantee the boundedness of the trajectory or the solution; and the second is to ensure the absence of closed orbits. We next provide, in the light of this experience, a set of restrictions which would guarantee that price adjustment will lead to an equilibrium. And we consider price adjustment which could in principle be triggered off from any initial price configuration. Thus we provide conditions for global stability of equilibrium.

2 The Scarf Contribution

Consider[1] an exchange model where there are three individuals $h = 1, 2, 3$ and three goods $j = 1, 2, 3$. The utility functions and endowments are as under:

$$U^1(q_1, q_2, q_3) = min(q_1, q_2);\ w^1 = (1, 0, 0)$$

$$U^2(q_1, q_2, q_3) = min(q_2, q_3);\ w^2 = (0, 1, 0)$$

$$U^3(q_1, q_2, q_3) = min(q_1, q_3);\ w^3 = (0, 0, 1)$$

Routine calculations lead to the following excess demand functions, where good 3 is treated as numeraire (i.e., $p_3 = 1$):

$$Z_1(p_1, p_2) = \frac{p_1(1 - p_2)}{(1 + p_1)(p_1 + p_2)}$$

$$Z_2(p_1, p_2) = \frac{p_2(p_1 - 1)}{(1 + p_2)(p_1 + p_2)}$$

and the tatonnement process, for this example is given by

$$\dot{p}_i = Z_i(p_1, p_2)\ i = 1, 2 \tag{1}$$

[1] We provide an analysis of the Scarf example which is somewhat different from the one in [15]. This would set up the groundwork for the later analysis. In particular, it should be pointed out that Scarf did not use a numeraire.

Notice that equilibrium for this exchange model (and for the process defined above) is given by $p_1 = 1, p_2 = 1$. It would be helpful to transform variables by setting $x_i = p_i - 1$ for $i = 1, 2$. With this change in variables, our process becomes

$$\dot{x}_1 = -\frac{x_2(1 + x_1)}{(x_1 + 2)(x_1 + x_2 + 2)}\ , \dot{x}_2 = \frac{x_1(1 + x_2)}{(x_2 + 2)(x_1 + x_2 + 2)} \tag{2}$$

Given an arbitrary initial $x^o = (x_1^o, x_2^o)$, how does the solution $x(t, x^o)$ to (2) behave as $t \to \infty$? We consider this question, next.

2.1 A Closed Orbit

Defining $v : R \to R$ by

$$v(x) = \frac{x^2}{2} + x - \ln(1 + x)$$

which is continuously differentiable for all x such that $1 + x > 0$, one may note that[2]

1 *For x small, $v(x) \approx x^2$.*

Next define $V(x) = V(x_1, x_2) = v(x_1) + v(x_2)$. It is straightforward to show that:

2 *$V(x)$ is strictly convex function and assumes a global minimum at $(0, 0)$; thus $V(x) > V(0, 0) = 0$ if $x \neq (0, 0)$.*

Further, it is easy to check that

3 *Along the solution $x(t, x^o)$ to (2), $\dot{V} = 0$ provided $x_i(t, x^o) > -1$ for $i = 1, 2$.*

These preliminary steps allows to furnish a complete answer to the question framed in the last section.

First of all, note that since $V(t) = V(x_1(t), x_2(t)) = V(x_1^o, x_2^o)$ for all t, it follows that the solution or trajectory $x(t, x^o) = (x_1(t), x_2(t))$ is bounded and each $x_i(t)$ is bounded away from -1: since if either of these conditions is violated, $V(t)$ would tend to $+\infty$. Hence the ω-limit set corresponding to x^o, $L_\omega(x^o)$, is non-empty and compact; also, $(0, 0) \notin L_\omega(x^o)$ if $x^0 \neq (0, 0)$ (remember, $(0, 0)$ is the equilibrium for the system) hence by the Poincaré-Bendixson theorem[3] $L_\omega(x^o)$ must be a closed orbit. This means that either we have a *limit cycle* or the trajectory $x(t, x^o)$ itself is a closed orbit.

If there is a limit cycle $\mathcal{L}$, then by virtue of the Claim 3, it follows that for any $y \in \mathcal{L}, V(y) = V(x^o)$; further, in such circumstances, there would be a neighborhood $\mathcal{N}$ of x^o

[2]For details, [12] and [13]; we mention the steps, for the sake of ease of reference.

[3]See, for instance, [10] p. 248.

such that for any solution $x(t,y)$ originating from any $y \in \mathcal{N}$, $x(t,y) \rightarrow \mathcal{L}^4$. Consequently, we must have $V(y) = V(x^o) \forall y \in \mathcal{N}$: this of course, is not possible, since the function V cannot be constant on an open set. Hence no such limit cycle exists. And the solution $x(t,x^o)$ must be a closed orbit. Thus we have shown the following to be true:

4 *For any initial configuration x^o, the solution to (2), $x(t,x^o)$ is a closed orbit around the equilibrium $(0,0)$. The equation to a typical orbit is $V(x) = V(x^o)$.*

We examine, next, a perturbation of this example.

2.2 Convergence

Consider a parameter say b, which stands for the amount of second good which individual 2 owns completely. Thus $b = 1$ would revert back to the example considered above. We continue to treat good 3 as the numeraire and then compute excess demand functions for the non-numeraire commodities for the case at hand; it turns out that these are given, using the same notation as above, by the following expressions:

$$Z_1(p_1,p_2) = \frac{p_1(1-p_2)}{(1+p_1)(p_1+p_2)}$$

$$Z_2(p_1,p_2) = \frac{p_2(p_1-b)+(1-b)p_1}{(1+p_2)(p_1+p_2)}$$

Consequently the system (1) now takes the form:

$$\dot{p_1} = \frac{p_1(1-p_2)}{(1+p_1)(p_1+p_2)} \text{ and } \dot{p_2} = \frac{p_2(p_1-b)+(1-b)p_1}{(1+p_2)(p_1+p_2)} \tag{3}$$

Once more standard computations ensure that the **unique equilibrium** is given by

$$p_1^* = \frac{b}{2-b} = \theta \text{ say}, p_2^* = 1$$

Thus it may be noted that our choice of the parameter places a restriction on its magnitude

$$0 < b < 2;$$

and we shall take it that this is met. Notice also that when $b = 1$, $\theta = 1$ too, and we have the earlier situation. That there have been some changes to the stability property of equilibrium is evident from computing characteristic roots :[5] Some tedious calculations reveal that the characteristic roots of the relevant matrix at equilibrium are given by:

$$\frac{1}{8}(-b+b^2 \pm \sqrt{b}\sqrt{\{-32+49b-26b^2+5b^3\}}).$$

Consequently, one may claim:

[4]See, for instance, [10] p.251.

[5]In fact it was shown in [12] that $b = 1$ provides a point of Hopf Bifurcation for the process (3).

5 *For the process (3), $(\theta, 1)$ is a locally asymptotically stable equilibrium if and only if $b < 1$; for $b > 1$, the equilibrium is locally unstable.*

A much stronger assertion is possible[6]:

6 *For the system (3), the unique equilibrium $(\theta, 1)$ is globally asymptotically stable whenever $b < 1$; and any trajectory with $(p_1^o, p_2^o) > (0, 0)$ as initial point remains within the positive orthant. When $b > 1$, any solution with an arbitrary non-equilibrium initial point is unbounded.*

Proof. We first note that for the system (3) **there can be no closed orbit in** $\Re^2_{++}$ so long as b is different from unity. For this purpose we shall use, Dulac's Criterion[7]. Now consider the function:

$$f(p_1, p_2) = \frac{(p_1 + p_2)(1 + p_1)(1 + p_2)}{p_1 . p_2}$$

on $\Re^2_{++}$. Notice that:

$$\frac{\partial f(p_1, p_2) Z_1(p_1, p_2)}{\partial p_1} + \frac{\partial f(p_1, p_2) Z_2(p_1, p_2)}{\partial p_2} = -(1 - b)/p_2^2$$

Thus $b \neq 1$ implies that Dulac's Criterion is satisfied by this choice of $f(p_1, p_2)$ and consequently there can be no closed orbits when $b \neq 1$. Applying next, the Poincaré-Bendixson Theorem, it follows that for any initial $p^o \in \Re^2_{++}$, the unique equilibrium $p^\star = (\theta, 1) \in L_\omega(p^o)$ provided the ω-limit set is non-empty.

Recall that for $b > 1$, the unique equilibrium is unstable; consequently no solution can enter a small enough neighborhood of $p^\star$; consequently, in this situation, $L_\omega(p^o)$ must be empty, if $p^o \neq p^\star$; thus the trajectories must be unbounded.

When $b < 1$, the unique equilibrium $p^\star$ is locally asymptotically stable; so if $L_\omega(p^o) \neq \emptyset$, $p^\star \in L_\omega(p^o) \Rightarrow p^\star = L_\omega(p^o)$; since once having entered a small enough neighborhood of the equilibrium, the trajectory cannot leave. Thus all that we need to guarantee convergence is that trajectories are bounded when $b < 1$.

[6]See [13], p. 89-90. We provide an alternative approach which will indicate what we have to accomplish in the more general case.

[7]See, [2], p. 305. This criterion looks for a function $f(p_1, p_2)$ which is continuously differentiable on some region R and for which

$$\frac{\partial f(p_1, p_2) h_1(p_1, p_2)}{\partial p_1} + \frac{\partial f(p_1, p_2) h_2(p_1, p_2)}{\partial p_2}$$

is of constant sign on R (not identically zero), then there is no closed orbit for the system $\dot{p_i} = h_i(p_1, p_2), i = 1, 2$ on the region R.

This last step may be accomplished by considering the function[8]:

$$W(p_1, p_2) = 2(1-b)p_1 + (2-b)p_1^2/2 - b\log p_1 + p_2^2/2 - \log p_2$$

and noting that its time derivative, along any solution to the system (3):

$$\dot{W} = \{((2-b)p_1 - b)(1+p_1)\}\frac{\dot{p_1}}{p_1} + (p_2^2 - 1)\frac{\dot{p_2}}{p_2}$$

$$= -(1-p_2)^2 \frac{p_1(1-b)}{p_2(p_1+p_2)} \leq 0$$

whenever $b < 1$. Thus for $b < 1$, $W(p_1(t), p_2(t)) \leq W(p_1^o, p_2^o)\forall t$, where we write $(p_1(t), p_2(t))$ as the solution to (3). Note that if $p_i(t) \to +\infty$, for some i, $W(p_1(t), p_2(t)) \to +\infty$ and the boundedness **and** positivity of the solution are established. This establishes the claim. •

There are thus two things to be noted from the above result: first that choosing a value of b **different** from unity negates the existence of a closed orbit; and a value of b **less than** unity is required to ensure that trajectories remain bounded. In a sense to be made precise below, these are the two aspects we need to account for if we are interested in identifying global stability conditions.

3 General Global Stability Conditions

If there are three goods and one of them is the numeraire, then the price adjustment equations of the type used for the Scarf example introduces dynamics on the plane. For motion on the plane, along with Poincaré-Bendixson Theorem, there is a result reported in [14] and its refinement [11]. We show next that it is possible to substantially weaken the conditions under which a global stability result may be deduced. This would allow us to conclude global stability for a competitive equilibrium as well as providing a general stability result which would be of some general interest, as well.

Consider the following systems of equations:

$$\dot{x} = f(x,y) \text{ and } \dot{y} = g(x,y) \tag{4}$$

where the functions f, g are assumed to be of class $\mathcal{C}^1$ on the plane $\Re^2$. For any pair of functions $f(x,y), g(x,y)$ let $J(f,g)$ or simply J, if the context makes it clear, stand for the Jacobian[9]:

[8] One may show that this function is, in addition, a Liapunov function for the system (3); see, for instance [13], p. 89-90.

[9] f_x for any function f will refer to the partial derivative of f with respect to the variable x.

$$\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

Consider, next, the following:

- O1: There is an unique equilibrium $(\bar{x}, \bar{y})$ to (4).
- O2: Trace of $J(f,g) = f_x + g_y < 0$ for all $(x,y) \in \Re^2$.
- O3: Determinant of $J(f,g) = f_x.g_y - f_y.g_x > 0$ for all $(x,y) \in \Re^2$
- O4: Either $f_x.g_y \neq 0$ for all $(x,y) \in \Re^2$ or $f_y.g_x \neq 0$ for all $(x,y) \in \Re^2$.

Under the conditions O1 - O4, Olech's Theorem, [14], shows that the unique equilibrium $(\bar{x}, \bar{y})$ is *globally asymptotically stable.* The contribution in [11] provides conditions which, in addition, guarantee that the solution remains positive, a requirement which has great importance to economic theory.

We shall use the setting of the tatonnement to investigate motion on the plane and for this purpose we introduce the notion of the excess demand functions $Z_i(p_1, p_2, p_3) : \Re^3_{++} \to \Re, i = 1, 2, 3$ which are required to satisfy the following:
A. Each $Z_i(.)$ is continuously differentiable with continuous partial derivatives and is bounded from below on $\Re^3_{++}$; further for any $(p_1, p_2, p_3) \in \Re^3_{++}, p_1.Z_1(.) + p_2.Z_2(.) + p_3.Z_3(.) = 0$ (Walras Law); and further for any $(p_1, p_2, p_3) \in \Re^3_{++}, \forall i, Z_i(\lambda p_1, \lambda p_2, \lambda p_3) = Z_i(p_1, p_2, p_3)$ for any $\lambda > 0$ (Homogeneity of degree zero in the prices); finally, for any sequence, $P^s = (p^s_1, p^s_2, p^s_3) \in \Re^3_{++},\ p^s_i = 1, \forall s$ for some index i, say $i = i_o$ and $||P^s|| \to +\infty$ as $s \to +\infty \Rightarrow Z_{i_o}(P^s) \to +\infty$[10](Boundary Condition).

The conditions listed under **A** are all routine; however they do imply some consequences of interest. First of all under these conditions, the set of equilibria for the economy $E = \{p \in \Re^3_{++} : Z_i(p) = 0 \forall i\} \neq \emptyset$; an independent demonstration of this assertion would follow as a by product of the analysis of the dynamics.

To study the dynamics on the plane, we shall investigate the solutions to a system of equations of the following type:

$$\dot{p_i} = h_i(p), i = 1, 2 \text{ with } p_3 \equiv 1 \tag{5}$$

where the functions $h_i(p)$ are assumed to satisfy the following: (we write $p = (p_1, p_2) \in \Re^2_{++}$)
B $h_i(p) = Z_i(p_1, p_2, 1), i = 1, 2.$

[10] $||x||$ stands for $\sqrt{(x_1^2 + x_2^2 + x_3^2)}$, when $x = (x_1, x_2, x_3)$.

Thus the equation (5) defines motion on the positive quadrant of the plane. A typical trajectory or solution to (5) from an initial $p^o \in \Re^2_{++}$ will be denoted by $\phi_t(p^o)$; the price configuration will be $(\phi_t(p^o), 1)$ for each instant t; this is just to signify that the numeraire (the third good) price is always kept fixed at unity. Also we note that any equilibrium for the dynamical system (5), say $\bar{p}$ where $h_i(\bar{p}) = 0, i = 1, 2$, implies that $(\bar{p}, 1)$ is an equilibrium for the economy, in the sense that $(\bar{p}, 1) \in E$ and conversely. We shall denote the equilibrium for (5) by E_R.

We are interested in the structure of the ω-limit set $L_\omega(p^o)$ i.e., the limit points of the trajectory $\phi_t(p^o)$ as $t \to +\infty$. On the plane, the structure of ***non-empty*** ω-limit sets is known to be one of the following[11]:

i. Consists of a single equilibrium or

ii. Consists of one closed orbit or

iii. an union of equilibria and paths tending to them.

It is because this classification offers some hope of obtaining general results that we shall investigate this situation more closely. We need to guarantee that the solution remains within the positive quadrant, which was the main item of concern in [11], as we mentioned above; then we need to guarantee that the ω-limit sets are non-empty; this will be accomplished by ensuring that the solution or trajectories are **bounded**; if a meaningful set of conditions allow us to rule out possibilities listed at (ii) and (iii), we have then a stability result. The conditions [14] mentioned above contain one such set of conditions; these need to be refined a bit if we want to ensure positivity as has been indicated in [11]. As should be apparent, even for motion on the plane, the requirements are fairly stringent.

First, we note:

7 *For each $i = 1, 2$, there exists $\varepsilon_i > 0$ such that $Z_i(p_i, p_j, 1) > 0$ if $p_i \leq \varepsilon_i$ for any p_j, $j \neq i$, $j = 1, 2$.*

Proof: Suppose to the contrary that there is no such ε_1 i.e., for any sequence $p_1^s > 0 \forall s, p_1^s \to 0$ as $s \to +\infty$, it is possible to find some $p_2^s > 0$ such that $Z_1(p_1^s, p_2^s, 1) \leq 0$ for all s large enough, say $s > S_1$. Consider the sequence $q^s = (1, p_2^s/p_1^s, 1/p_1^s)$ notice that $||q^s|| \to +\infty$ as $s \to +\infty$; hence by the boundary condition, $Z_1(q^s) \to +\infty$ or by homogeneity, $Z_1(p_1^s, p_2^s, 1) \to +\infty$; thus for all s large enough, say for $s > S_2$, $Z_1(p_1^s, p_2^s, 1) > 0$; we thus arrive at a contradiction for $s > Max(S_1, S_2)$. This establishes the claim. •

Given the above claim, note that any trajectory of (5), $\phi_t(p^o) = (p_1(t), p_2(t))$, say, where

[11] See, for instance, [2], p. 362.

$p_i^o > \varepsilon_i, i = 1, 2$ satisfies $p_i(t) > \varepsilon_i$ for all $t > 0$. Thus the trajectory remains within the positive orthant. This is important enough to be noted separately.

8 *Given* $\boldsymbol{A}$ *and* $\boldsymbol{B}$*, the solution* $\phi_t(p^o)$ *from any* $p^o > (0, 0)$ *remains within the positive orthant for all* t *and remains bounded away from the axes.*

The next step is to note that the solution has to remain within some bounded region. We claim:

9 *Under* $\boldsymbol{A}$ *and* $\boldsymbol{B}$*, any solution* $\phi_t(p^o)$ *to (5) for* $p^o > (0, 0)$ *remains bounded.*

Proof: Suppose $||\phi_t(p^o)|| \to +\infty$ as $t \to +\infty$; then by the boundary condition, $Z_3(\phi_t(p^o), 1) \to +\infty$ as $t \to +\infty$. Writing the solution as $p_1(t), p_2(t))$, we have by virtue of Walras Law:

$$p_1(t).Z_1(\phi_t(p^o), 1) + p_2(t).Z_2(\phi_t(p^o), 1) = -Z_3(\phi_t(p^o), 1) \to -\infty$$

as $t \to +\infty$. Recall that the excess demands $Z_i(.)$ are bounded below; hence, it follows that for some $i = 1, 2$, $p_i(t).Z_i(\phi_t(p^o), 1) \to -\infty$; this is possible only when $p_i(t) \to +\infty$ **and** $Z_i(.) < 0$. Thus for all $t \geq T$, say, $p_i(t).Z_i(.) < 0$ which means that for all $t \geq T$, $\dot{p_i} < 0$ or hence $p_i(t) \leq p_i(T)$ for all $t \geq T$: a contradiction. This establishes the claim. •

On the basis of the above claims, we know then that there is a rectangular region $R = \{(p_1, p_2) : \varepsilon_i \leq p_i \leq M_i\}$ in the positive quadrant within which the solution gets trapped. Incidentally, this fact together with Poincaré's theory of indices for singular points[12], implies that R contains equilibria; i.e., $E_R \neq \emptyset$; recall that, by virtue of our assumptions on excess demands, $(p_1, p_2, 1) \in E \Leftrightarrow (p_1, p_2) \in E_R$.

We shall assume now the following:

C i. Trace of the Jacobian $J(h_1, h_2)$ is not identically zero on R nor does it change sign on R.

C ii. On the set E_R, the Jacobian $J(h_1, h_2)$ has a non-zero trace and a non-zero determinant.

Notice that while the contributions in [14] and [11] demand an unique equilibrium, we do not. They demand a lot of other restrictions as well[13]. We have of course the properties of the excess demand function in A which have helped us to isolate a region such as R; **C i** and **C ii** appear weaker than the requirements demanded [11] and [14]. **C ii** ensures that the equilibria in E_R have characteristic roots with real parts non-zero: this ensures that all equilibria for the dynamic system (5) are hyperbolic or nondegenerate or simple[14]. Thus

[12]See, for instance, [2] p. 305.

[13]See, for example conditions listed as O1-O4, above.

[14]See, for instance, [6] p. 13; this helps in determining the nature of the fixed points locally by considering the linearized version.

the only fixed points are either focii or nodes (Poincaré index +1 for both) or saddle points (Poincaré index - 1)[15]. It follows that E_R contains a finite odd number of equilibria since the sum of the indices of all must add up to +1[16].

Proposition 1 *Under A, B and C, for any $p^o \in R$, $L_\omega(p^o) = p^\star \in E_R$. Thus all solutions converge to an equilibrium.*

Proof: Consider any $p^o \in R$ and the trajectory $\phi_t(p^o)$: the solution to (5); by virtue of the Claim 8, the ω-limit set $L_\omega(p^o)$ is not empty. Again by the criterion of Bendixson [17], **C i** implies that there can be no closed orbits in the region R. Thus there can be no limit cycle and hence the Poincaré-Bendixson Theorem implies that $L_\omega(p^o) \cap E_R \neq \emptyset$. It follows therefore that $p^\star \in L_\omega(p^o)$ for some $p^\star \in E_R$; consequently there is a subsequence $\{t_s\}, t_s \to +\infty$ as $s \to +\infty$ such that $\phi_{t_s}(p^o) \to p^\star$ as $s \to +\infty$.

Since we know that the only types of equilibria are focii, nodes and saddle-points, the characteristic roots of the Jacobian $J(h_1, h_2)$ at $p^\star$ have real parts either both positive or negative, or they are real and of opposite signs, given **C ii**.

In the first case, there would be an open neighborhood $N(p^\star)$ which no trajectory or solution could enter; consequently since our trajectory $\phi_t(p^o)$ does enter *every* neighborhood of $p^\star$, it follows that at $p^\star$, the characteristic roots of the Jacobian, if complex, have real parts negative; and if real, then at least one must be negative. Thus the fixed point $p^\star$ is either a sink or at worst, saddle-point. If it is a sink, then any trajectory once having entered a small neighborhood of the equilibrium, can never leave. Consequently, the trajectory $\phi_t(p^o)$ has no other limit point. Thus $L_\omega(p^o) = p^\star$. In the case of a saddle-point, there is only a single trajectory which converges to the equilibrium; if p^o happens to be on this trajectory, $L_\omega(p^o) = p^\star$ but otherwise it is not possible for a trajectory to have a saddle-point as a limit point. In any case therefore, the trajectory must **converge** to an equilibrium, as claimed. •

[15]See, for instance [2] p.301 or [6] p. 51. Consider the characteristic roots of the Jacobian evaluated at equilibrium. A focus is an equilibrium or fixed point with the characteristic roots are complex conjugates; the equilibrium is a stable focus when the real parts of these roots are negative; it is an unstable focus when the real parts are both positive; the equilibrium is called a node when these characteristic roots are both real and of the same sign; again it is a stable node if the real roots are both negative and an unstable node if the real roots are positive. Sometimes stable focii and nodes are called sinks; unstable nodes and focii are called sources. A saddle-point is an equilibrium when the characteristic roots are both real but of opposite sign.

[16]See, for instance, [2] p. 305.

[17]See, for example, [6] p. 44.

We provide, next a set of remarks which highlight the implications of the above result.

Remark 1 *The above result provides a set of conditions under which an adjustment on prices on disequilibrium, in the direction of excess demand, will always lead to an equilibrium. Notice also that these conditions guarantee that there will always be at least one sink i.e., an equilibrium at which the Jacobian has characteristic roots with real parts negative. To see this note that if no such equilibrium existed, then the only equilibria are saddle-points and sources. Also in aggregate they are finite in number and moreover, as argued above, no trajectory can come close to sources; so the only possibility for a limit is a saddle-point; but each saddle-point has only one trajectory leading to it and there are an infinite number of possible trajectories. Thus there must be a sink.*

More importantly:

Proposition 2 *Under* ***A****,* ***B*** *and* ***C****, if there is a unique equilibrium, it must be globally asymptotically stable.*

Remark 2 *As mentioned above, there must be at least one equilibrium where the characteristic roots have real parts negative. Hence the trace of the Jacobian at that equilibrium must be negative; further, since the trace at that equilibrium will be negative and the trace cannot change sign nor can it be zero at equilibria, it follows that the trace of the Jacobian at every equilibrium must be negative.*

Consequently, we have:

Proposition 3 *Under* ***A****,* ***B*** *and* ***C****, at every equilibrium, the sum of the characteristic roots of the Jacobian will be negative*[18].

Remark 3 *If we consider* $h_i(p_1, p_2)$ *to have the same sign as* $Z_i(p_1, p_2, 1), i = 1, 2$ *then the assumptions in* $\boldsymbol{C}$ *are restrictions placed on the functions* h_i*. Of course these become difficult to interpret. One may show that the Jacobian of* (h_1, h_2) *at equilibria is related to the Jacobian of* (Z_1, Z_2)*, where the partial derivatives are with respect to* (p_1, p_2)*, again at equilibria by means of the following:*

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \cdot \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

where all partial derivatives are evaluated at an equilibrium and $d_i > 0, i = 1, 2$ *are some positive numbers. This would provide some link between the equilibria for the dynamic process and equilibria for the economy.*

[18]Thus, if roots are complex, the real parts must be negative.

Let us reconsider the system (4); assume that the set of equilibria for this system $E = \{(x, y) : f(x, y) = 0, g(x, y) = 0\}$ is non-empty. The following general result follows from our analysis:

Proposition 4 *If*
i. There is a rectangular region $R = \{(x, y) : 0 \leq x \leq M, 0 \leq y \leq N\}$ such that any trajectory of (4) on the boundary of R is either inward pointing or coincides with the boundary;
ii. Trace of $J(f, g)$ is not identically zero and does not change sign in the positive quadrant;
iii. On the set E, the trace and determinant of the Jacobian $J(f, g)$ do not vanish;
then any trajectory $\phi_t(x^o, y^o)$ where $(x^o, y^o) > (0, 0)$ converges to a point of E.

A final remark considers the weakening of the assumption **C i**.

Remark 4 *If we can find a function $\theta(p_1, p_2)$ which is continuously differentiable on the region R and for which*

$$\frac{\partial\theta(p_1, p_2)h_1(p_1, p_2)}{\partial p_1} + \frac{\partial\theta(p_1, p_2)h_2(p_1, p_2)}{\partial p_2}$$

is of constant sign on R, then there is no closed orbit for the system (5) on the region R[19].

In some situations, the above may provide a weakening of the condition **C i**. It may be recalled that the sole purpose of **C i** was to rule out closed orbits in R. If, for example, $h_i(p_1, p_2) = Z_i(p_1, p_2, 1) = p_i.g_i(p_1, p_2), i = 1, 2$, then we may replace C i. by requiring that $p_1 g_{11}(p_1, p_2) + p_2 g_{22}(p_1, p_2)$ be of constant sign on R; note that we do not require the trace of $J(p_1 g_1, p_2 g_2)$ being constant on R. This follows by virtue of the fact that we may consider $\theta(p_1, p_2) = p_1^{-1} p_2^{-1}$ and then the condition in Remark 4 is satisfied for this choice of $\theta(p_1, p_2)$[20].

4 Conclusion

The above analysis shows, first of all, that cyclical behavior around equilibrium, noted by Scarf, is not robust particularly with reference to perturbation of the endowments. In an identical set-up, results exist which show that a redistribution of the goods among

[19] This is Dulac's criterion; the Bendixson's Criterion is a special case when $\theta(p_1, p_2) = 1$. Recall that the perturbation for the Scarf example, we used the Dulac's criterion to rule out the existence of closed orbits.

[20] Of course, the application of Dulac's criterion raises problems similar to the search for Liapunov functions.

individuals may also help to restore stability to the Scarf example: [8] and [9] both contain illuminating results in this connection[21].

More importantly, in the realm of micro-economic theory, it is well known that the substitution effects are all in the proper direction and it is income effects which may ruin stability. The Scarf example is an effort at ruling out all substitution effects; the instability noted by Scarf might then have been assumed to be due to this fact; that this is not the case may be seen by our result, since without introducing any substitution effects, the economy has a globally stable equilibrium when $b < 1$.

A more recent paper [1], points out the existence of an endowment distribution which leads to global stability. The identified endowment distribution is the one where each individual has an unit of the good that he is not interested in: that is individual one has a unit of good 3; individual 2 has an unit of good 1 and individual 3 has an unit of good 2[22]. The important and significant part of the contribution made in [1] lies in their discovery that experiments conducted with agents with similar preferences and endowments, but engaging in double auctions would lead to price movements which are predicted by the tatonnement model. Thus the results provided by the tatonnement process, they argue, should be looked at with greater care because they seem to predict what price adjustments might actually occur.

As we showed in our analysis of the Scarf Example, the perturbation allowed us to get rid of closed orbits; for convergence, we needed to show that the solution was bounded. One of the reasons for our being able to obtain such a different result was due to the fact that at the original equilibrium, the relevant matrix had purely complex characteristic roots, with zero real parts. It is not surprising that in such a situation, a perturbation changed the real parts of the characteristic root from zero to positive or negative.

Notice that **C ii** rules out the Jacobian of the excess demand functions from having characteristic roots with zero real part or from being singular; both serve to ensure that the properties we observe are robust; non-singularity of the Jacobian, some times called regularity preserves static properties of the equilibria of the economic system for small changes in parameters; the trace being non-zero at equilibrium, preserves the dynamic properties from small changes in parameters. While **C i** rules out the trace of the Jacobian from changing

[21] The interesting contribution in [4] looks at a slightly different question. It is shown that a different price mechanism is able to attain equilibrium for the Scarf example; the price mechanism is discrete and considers a weighted average of past prices, together with the current level of excess demand in determining revised prices.

[22] See in this connection, the example in [5] with two goods and two individuals with similar tastes.

signs on the positive quadrant which eliminates cycles. Thus **C i** rules out periodic behavior and **C ii** ensures **robustness**. These two together imply that the process will always lead to an equilibrium, provided trajectories are bounded; the particular equilibrium approached will depend on the initial configuration of prices, of course. It is also important to note that if there is a unique equilibrium, then that has to be globally asymptotically stable. Thus the feature of the original Scarf example, of a unique equilibrium which cannot be attained, is removed. However, these conclusions are for motion on the plane. Their interest lie in the fact that in many applications in economics, only such motions are considered.

In [7], there is an enquiry relating to the following questions: if a market is stable by itself, can it be rendered unstable from the price adjustment in other markets ? Alternatively, if a market is unstable when taken by itself, can it be rendered stable by the price adjustment in the other markets ? To both an answer was provided in the negative. Notice that **C i** essentially ensured (together with **C ii**) that the trace of the relevant Jacobian remained negative; notice that this would be implied by assuming that $Z_{ii} < 0$ for each i, that is when each market when taken in isolation, was stable. This in turn has been seen to imply that the markets together must also be globally stable. Under certain conditions, **C i** may be weakened further; this involves the existence of a function $\theta()$ satisfying Dulac's criterion, as in the case of the perturbation of the Scarf example. But this is a matter of serendipity rather than design.

General results in this area are difficult to obtain due to **two** reasons: first of all, the excess demand functions are not expected, *a priori* to satisfy any other property apart from Homogeneity of degree zero in the prices and Walras Law; secondly, dynamics in dimensions greater than 2 may be quite difficult to pin down. Even on the plane, a variety of dynamic motions are possible. It is not surprising that in higher dimensions matters become a lot more complicated and Walras Law and Homogeneity do not help too much. And consequently, we must impose additional restrictions which may be called **global stability conditions** ; we have shown that an easy such condition for the Scarf example is $b < 1$. For the general case, on the plane, the conditions in **C** serve the same purpose.

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