HYPERBOLIC DISCOUNTING IS RATIONAL: VALUING THE FAR FUTURE WITH UNCERTAIN DISCOUNT RATES

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Hyperbolic discounting is rational: Valuing the far future with uncertain discount rates

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Conventional economics supposes that agents value the present vs. the future using an exponential discounting function. In contrast, experiments with animals and humans suggest that agents are better described as hyperbolic discounters, whose discount function decays much more slowly at large times, as a power law. This is generally regarded as being time inconsistent or irrational. We show that when agents cannot be sure of their own future one-period discount rates, then hyperbolic discounting can become rational and exponential discounting irrational. This has important implications for environmental economics, as it implies a much larger weight for the far future.

Key words: Hyperbolic discounting, environment, time consistent, exponential discounting, geometric random walk, term structure of interest rates

JEL: D91, G12

When we address a problem such as global warming we are forced to compare the benefit or harm of an action today, such as an investment in an alternative energy technology, against its consequences in the far future, such as environmental improvement. Consider a consumption stream $x = (x_0, x_1, x_2, \ldots)$ for an agent who gets instantaneous utility $u(x_t)$ from consuming at time t. Contemplating her future consumption from the point of view of time $s \geq 0$, her utility $U_s(x)$ is usually assumed to be a sum of the form

$$(1) \qquad U_s(x) = u(x_s) + \sum_{\tau=1}^{\infty} D_s(\tau) u(x_{s+\tau}).$$

$D_s(\tau)$ is the *discount function* at time s associated with consumption at a future time $s + \tau$, where $\tau \geq 1$. It weights the relative importance of the future vs. the present, and is usually assumed to be a decreasing function.

Why should we discount the future? Bohm-Bawerk (1889,1923) and Fisher (1930) argued that men were naturally impatient, perhaps owing to a failure of the imagination in conjuring the future as vividly as the present. Another justification for declining $D_s(\tau)$ in τ, given by Rae (1834,1905), is that people are mortal, so survival probabilities must enter the calculation of the benefits of future potential consumption. There are many possible reasons for discounting, as reviewed by Dasgupta (2004, 2008). Most economic analysis assumes exponential discounting $D_s(\tau) = D(\tau) = \exp(-r\tau)$, as originally posited by Samuelson (1937) and put on an axiomatic foundation by Koopmans (1960).

A natural justification for exponential discounting comes from financial economics and the opportunity cost of foregoing an investment. A dollar at time s can be placed in

the bank to collect interest at rate r, and if the interest rate is constant, it will generate $\exp(r(t-s))$ dollars at time t. A dollar at time t is therefore equivalent to $\exp(-r(t-s))$ dollars at time s. Letting $\tau = t - s$, this motivates the exponential discount function $D_s(\tau) = D(\tau) = \exp(-r\tau)$, independent of s.

Real people and animals, in contrast, do not use exponential discounting, but rather give more weight to events that are very immediate or very distant in time, and less weight at intermediate times[1]. This kind of attitude toward time is referred to as *hyperbolic discounting*, and is often written in the functional form

$$(2) \qquad\qquad D_s(\tau) = D(\tau) = (1 + \alpha\tau)^{-\beta},$$

where α and β are constants[2]. This functional form is generally believed to fit empirical data better than an exponential. In the limit as $\tau \to \infty$, $D(\tau)$ is proportional to $\tau^{-\beta}$, i.e. it follows a power law. We call any function that behaves this way in the limit *asymptotically hyperbolic*. Such a discount function puts much greater weight on the far future than any exponential discount function. This is particularly true when $\beta < 1$; in this case the integral from $t = t'$ to $t = \infty$ is infinite for any time $t' \geq 0$, hence there is an infinite weight on the far future.

A dramatic example of hyperbolic discounting was provided by Thaler (2005). He asked a group of subjects how much money they would be willing to accept in the future in lieu of receiving \$15 immediately. The average responses were: One month later \$20, one year later \$50, and ten years later \$100. The exponential function fits Thaler's data poorly, as can be seen by writing the discount factors in the form $D(\tau) = K^\tau$. Assuming $u(x) = x$ and measuring time in months, $D(1) = 15/20 = .75^1$, $D(12) = 15/50 = .9^{12}$, and $D(120) = 15/100 = .98^{120}$. The value of K needed varies from 0.75 to 0.98, in contrast to the constant value predicted under exponential discounting. The hyperbolic functional form, in contrast, fits the Thaler data quite well, with $\beta \approx 0.5$.

To better understand the implications of choosing a particular functional form for the discounting function it is useful to distinguish three properties.

Certainty. The one period discounts $D_s(1)$ for all $s \geq 0$ are anticipated with certainty at time 0, and at every time $0 \leq t \leq s$.

Strict stationarity. The discounting function $D_s(\tau)$ is independent of the time s when the evaluation is made, i.e. $D_s(\tau) = D(\tau)$.

Time consistency. The utility at one time s and the utility at another time s' are consistent with the discounting function. For successive time periods the utility U_s from contemplating consumption stream x at time s should be equal to the utility of consuming at time s plus the utility of contemplating that same stream at time $s + 1$, discounted by one time period, i.e.

$$(3) \qquad\qquad U_s(x) = u(x_s) + D_s(1)U_{s+1}(x).$$

Time consistency is often viewed as a necessary condition for rationality.

As originally shown by Samuelson and Koopmans, the only discounting function that satisfies all of these properties is the exponential. It is easy to show by backward induction

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[1]See (Ainslie and Herrnstein 1981, Ainslie 1992, Loewenstein and Prelec 1992, Camerer, Loewenstein and Rabin 2003, Thaler 2005, Berns, Laibson and Loewenstein 2007).

[2]The term "hyperbolic discounting" is used somewhat loosely. Some use it to refer to any discounting function that is not exponential, while others use it to refer specifically to functions of the form of Eq. (2). We adopt the latter usage.

that certainty and time consistency are equivalent to

(4) $$D_{s+\tau}(1) = D_s(\tau+1)/D_s(\tau) \quad \text{or}$$

(5) $$D_s(\tau) = D_s(1)D_{s+1}(1)\ldots D_{s+\tau-1}(1)$$

Strict stationarity requires that $D_s(1) = K = $ constant, which then gives the exponential discounting function $D_s(\tau) = \exp(-r\tau)$, where $K = e^{-r}$.

Suppose Suzy is contemplating how painful it will be to clean her room (O'Donaghue and Rabin 1999). If she uses an exponential discounting function with constant discount rate r then she is time consistent, regardless of how far ahead she is thinking. The ratio of the pain of cleaning her room today to the pain of cleaning it tomorrow is always e^r, and is always anticipated to be e^r. Contemplating cleaning the room a year in advance, under exponential discounting $D(365)/D(366) = D(0)/D(1) = e^r$.

In contrast, under the certainty assumption, hyperbolic discounting necessarily violates time consistency or else strict stationarity (see Strotz (1956) and Laibson (1997)). To see why this is true, suppose Suzy uses hyperbolic discounting as in Eq. (2), and assume $\alpha = \beta = 1$. This implies that $D_0(1) = (1+1)^{-1} = 1/2$. Since $1/D_0(1) = 2$, she will feel that it is twice as bad to clean her room today as opposed to postponing it until tomorrow. However, she views it as only $1.003 = D_0(365)/D_0(366)$ times as bad to clean it in $t = 365$ days vs. $t+1 = 366$ days. The time inconsistency comes because when she is asked one year later about cleaning her room on those same two days, she will give a different answer: Just as before, she will say it is twice as bad to clean her room immediately rather than do it the next day: $1/D_{365}(1) = 1/D_0(1) = 2$. The overwhelming empirical evidence that real people like Suzy use hyperbolic discounting is thus often viewed as contradicting rationality.

It is possible, however, to take the view that Suzy's hyperbolic discounting is time consistent, but just not strictly stationary. In particular, let us modify Suzy's utility so that

$$D_s(\tau) = (1+\alpha s)^\beta (1+\alpha(s+\tau))^{-\beta}$$

For any fixed s, $D_s(\tau) \to \tau^{-\beta}$ as $\tau \to \infty$, so she is asymptotically hyperbolic. Moreover, noting that $D_s(1) = (1+\alpha s)^\beta(1+\alpha(s+1))^{-\beta}$, the time consistency conditions are satisfied. From the point of view of time 0 (but not any other time), her attitude toward the future is given by Eq. (2). The trouble is, in Fisher's terms, that she is growing steadily more patient, in the sense that her one period discounts are increasing. For example, when $\alpha = \beta = 1$, today $D_0(1) = 1/2$, but a year ahead $D_{365}(1) = D_0(366)/D_0(365) = 366/367 = 1/1.003$. Thus under this interpretation she satisfies the time consistency condition of Eq. (3) but does not satisfy the strict stationarity condition. We take it as axiomatic that people do not systematically grow more patient with time; for one thing death hazard rates on average grow with age. Thus certainty, strict stationarity (or more generally weakly growing impatience) and time consistency rule out (asymptotic) hyperbolic discounting. In short, if $D_0(\tau) \sim \tau^{-\beta}$, then $D_s(1) = D_0(s)/D_0(s+1) \sim (s/(s+1))^\beta$ which is strictly increasing toward 1 as $s \to \infty$.

This conclusion changes dramatically, however, if we assume a stochastic world in which the one period discounts are uncertain. We can alway write the one-period discount factors in our certainty model in the form $D_t(1) = e^{-r_s}$, where r_s is the one-period *discount rate* at time s. The time consistency condition of Eq. (4) can trivially be written in terms of the discount rate as

(6) $$D_s(\tau) = e^{-r_s}e^{-r_{s+1}}\ldots e^{-r_{s+\tau-1}}.$$

The discount rate r_t can be thought of as an interest rate, or it can be thought of as a psychological state representing the attitude of an agent on day t about receiving utility on day $t+1$. For example, in the Thaler experiment an agent's patience might vary from day to day, or she might update her probability that Thaler will flee to Brazil that night, or that she might die that night, in which case r_t can be interpreted as a hazard rate. In any case the uncertainty in r_t can be used to represent her changing view of the future. There are many reasons why it is natural to let discount rates vary – interest rates, for example, vary all the time, and attitudes about the future vary, often for good reason.

It is therefore natural to let the one period discount rates be stochastic. We suppose that at every time s, Suzy knows her one-period discount rate r_s, but is uncertain about her future one-period discount rates. We also suppose that Suzy knows the stationary Markov process $P(r_{s+1}|r_s)$ these discount rates follow, where $P(r_{s+1}|r_s)$ is the probability of discount rate r_{s+1} given discount rate r_s. When Suzy's psychological state r_s is stochastic, her utility $U_{s,r_s}(x)$ and her discounting $D_{s,r_s}(\tau)$ both become state dependent as well as time dependent. The time consistency condition of Eq. (3) now takes the more general form

(7) $$U_{s,r_s}(x) = u(x_s) + e^{-r_s} \sum_{r_{s+1}} P(r_{s+1}|r_s)U_{s+1,r_{s+1}}(x).$$

Define a feasible path of length $\tau-1$ from r_s as any sequence of possible future discount rates $\vec{r} = \{r_{s+1}, r_{s+2}, \ldots, r_{s+\tau-1}\}$. Assuming that P gives equal probability for each of the $N(\tau)$ feasible paths $\vec{r}$, define the *certainty equivalent discount function* $\bar{D}_{s,r_s}(\tau)$ as the average over all possible paths, i.e.

$$\bar{D}_{s,r_s}(1) = e^{-r_s}$$

(8) $$\bar{D}_{s,r_s}(\tau) = \frac{1}{N(\tau)} \sum_{\vec{r}} e^{-r_s} e^{-r_{s+1}} \ldots e^{-r_{s+\tau-1}}.$$

By applying the time consistency equation recursively from period $s+\tau$ back to the beginning of period s, one can derive the standard theorem in finance that time consistency implies

$$U_{s,r_s}(x) = u(x_s) + \sum_{\tau=1}^{\infty} \bar{D}_{s,r_s}(\tau)u(x_{s+\tau})$$

This is equivalent to the definition of utility discounting given in Eq. (1), except that the utility is now allowed to depend on the variable one period discount rate r_s, and it is written in terms of the certainty equivalent discount function $\bar{D}_{s,r_s}(\tau)$, which also depends on r_s. Note that r_s defines the state of the system.

Suzy's discount rates are *stochastically stationary,* since they follow a stationary Markov process. In particular, the discount factors $\bar{D}_{s,r} = \bar{D}_r$ depend on time s only through the one period discount rate r_s that then prevails. Furthermore, we can say that Suzy is growing stochastically more impatient if everywhere $E[r_{s+\tau}|r_s] > r_s$, and stochastically more patient if the reverse inequality holds.

We will now show that Suzy's dilemma of cleaning her room can change dramatically, once we let her psychological state be stochastic. If Suzy's discount rates behave according to the stochastic model we are about to give, asymptotic hyperbolic discounting is both time consistent and stochastically stationary, and thus Suzy is rational; exponential discounting, by contrast, fails to be time consistent, and so should be regarded as irrational. In addition we will see that on average Suzy's impatience increases with time.

Consider the case where the one period discount rates change according to a standard finance model for interest rates called the *geometric random walk* (Ho and Lee 1986). At each time step the current discount rate is either multiplied by a volatility factor e^v, yielding $r_{t+1} = r_t e^v$, or divided by the same factor, yielding $r_{t+1} = r_t e^{-v}$. The two choices have equal probability. If the initial discount rate r_0 is positive, r_t is always positive. The geometric mean of r_t is constant and the arithmetic mean is an increasing function of time. Thus if Suzy's discount rates follow a geometric random walk, she will on average have discount rate $r_{365} > r_0$. Thus on average she gets more impatient with time – yet as we will show, she discounts the far future much less than an agent with a constant discount rate.

The intuition behind this puzzle lies in the fact that in the geometric random walk, discount rates are serially correlated: a very low r_t will necessarily be followed by another low r_{t+1}, and a high r_t will be followed by another high r_t, at least if volatility is not too large. Following a long tradition in finance (Litterman, Scheinkman and Weiss 1991), Weitzman (1998) observed that serially correlated uncertainty in interest rates leads to less discounting in the long run than when interest rates are certain at the mean level. To take the simplest example, suppose that today the one period discount rate is r_0, and that in the future there are two possible feasible paths, each of which has constant interest rate, either r or r', with $r < r_0 < r'$. For concreteness, suppose that $\frac{1}{2}r + \frac{1}{2}r' > r_0$, so that impatience is stochastically growing or staying constant, and even further, that $\frac{1}{2}e^{-r} + \frac{1}{2}e^{-r'} < e^{-r_0}$. Then the certainty equivalent discount function is

$$\bar{D}_{0,r_0}(2) = e^{-r_0}(e^{-r} + e^{-r'})/2 < e^{-2r_0}$$
$$\bar{D}_{0,r_0}(\tau) = e^{-r_0}(e^{-r(\tau-1)} + e^{-r'(\tau-1)})/2.$$

For sufficiently large τ, $\bar{D}_{0,r_0}(\tau) \sim \exp(-r\tau) > e^{-r_0\tau}$, i.e. the smaller interest rate dominates. This is why Suzy can have time consistent utility and put much more weight on the distant future than an exponential discounter with constant rate r_0, even though her discount rate rate begins at r_0 and on average increases.

Under the geometric random walk the one-period discount rate r_t at any given time t is a stationary Markov process, so the certainty equivalent discount function $\bar{D}_{s,r}(\tau) = \bar{D}_r(\tau)$ is time stationary. Each r_t is determined randomly by the number of increases minus the number of decreases along the path leading up to r_t. One can therefore visualize the possible states (t, r) of the system as the nodes in a recombining binary tree in which each interior node at a given level comes from a pair of nodes at the previous level, as shown in Figure 1. (Note the tree is turned on its side). Within the tree discount rates are constant along horizontal lines and increase exponentially along vertical lines.

The certainty equivalent discount function $\bar{D}_r(\tau)$ can be computed numerically for each τ using Eq. 8. An example is shown in Figure 2. The parameters are chosen so that a single time step of the simulation corresponds to a year, with an initial interest rate of $r_0 = 4\%$ and v chosen so that $e^v = 1.5$, corresponding to an annual volatility of 50%. For roughly the first eighty years the certainty equivalent discount function for the geometric random walk stays fairly close to the exponential, but afterward the two diverge substantially, with the geometric random walk giving a much larger weight to the future. A comparison using more realistic parameters is given in Table 1. For large times the difference is dramatic.

In Figure 2 we plotted the result in double logarithmic scale to highlight that for large times the discounting function approaches a power law, corresponding to a straight line on double logarithmic scale. In this case the exponent is -0.507. (This was measured by making a least squares fit to the indicated points in the tail). We have performed many
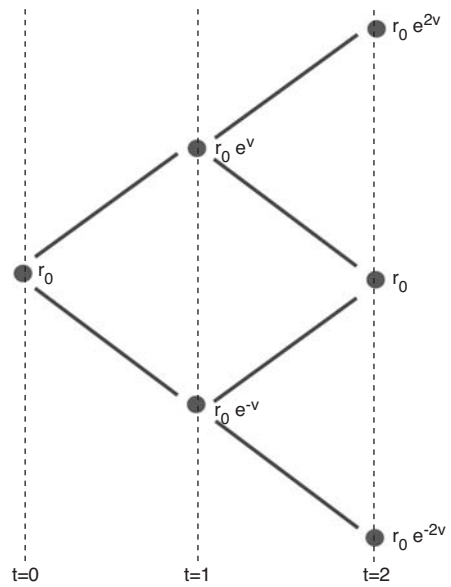
FIGURE 1: A SCHEMATIC REPRESENTATION OF POSSIBLE INTEREST RATES. EACH LEVEL OF THE TREE CORRESPONDS TO THE TIME $t = 0, \ldots, T$, WITH TIME INCREASING FROM LEFT TO RIGHT. THE POSSIBLE INTEREST RATES AT ANY TIME t INCREASE ALONG VERTICAL LINES, AND ARE $r_0 e^{-vt}, r_0 e^{-v(t-2)}, \ldots, r_0 e^{vt}$.
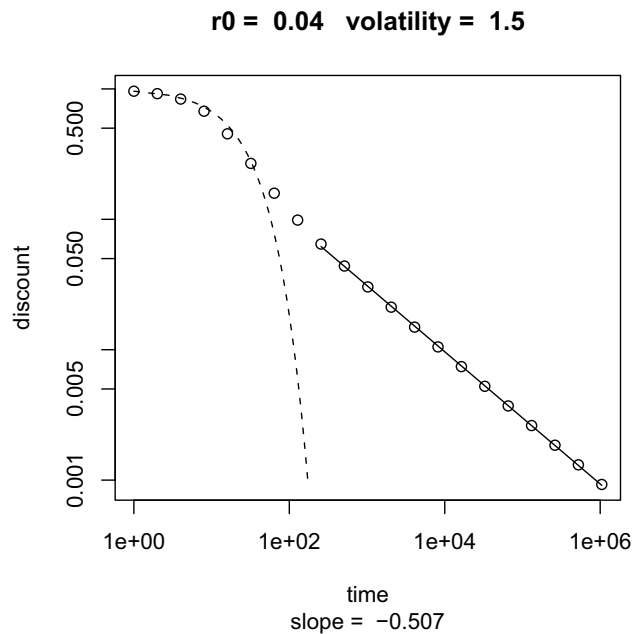
FIGURE 2: CERTAINTY EQUIVALENT DISCOUNT FUNCTION VS. TIME FOR THE GEOMETRIC RANDOM WALK, PLOTTED IN DOUBLE LOGARITHMIC SCALE. THE PARAMETERS CORRESPOND TO AN ANNUAL INTEREST RATE OF 4% AND AN ANNUAL VOLATILITY OF 50%. THE DASHED LINE CORRESPONDS TO EXPONENTIAL DISCOUNTING, AND THE SOLID LINE IS A LEAST SQUARES FIT TO THE INDICATED PART OF THE TAIL.

year	GRW	exponential
20	0.462	0.456
60	0.125	0.095
100	0.051	0.020
500	0.008	2×10^{-9}
1000	0.005	4×10^{-18}

TABLE 1: COMPARISON OF EFFECTIVE DISCOUNTING FUNCTIONS $D_{r_0}(\tau)$ ON DIFFERENT TIME HORIZONS τ. THE FIRST COLUMN IS THE TIME IN YEARS; THE SECOND COLUMN IS THE CERTAINTY EQUIVALENT DISCOUNTING FUNCTION FOR THE GEOMETRIC RANDOM WALK WITH AN INITIAL INTEREST RATE OF 4% AND A VOLATILITY OF 15%; THE THIRD COLUMN IS FOR EXPONENTIAL DISCOUNTING AT 4% PER YEAR.
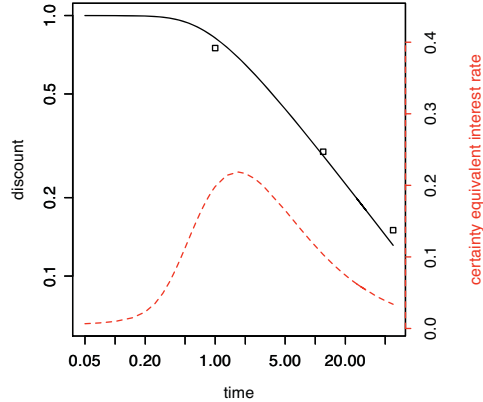
FIGURE 3: A COMPARISON OF THALER'S DATA ON DISCOUNTING (SQUARES) TO A FIT USING THE GEOMETRIC RANDOM WALK WITH THE INDICATED PARAMETERS. THE DASHED PLOT SHOWS THE CERTAINTY EQUIVALENT DISCOUNT RATE $\bar{r}(\tau)$, WHICH FIRST INCREASES AND THEN DECREASES. TIME IS MEASURED IN MONTHS.

simulations with different parameters, and providing we simulate the discounting out to a sufficiently large horizon, we always observe power law tails with exponents near $-1/2$.

In the limit as $\tau \to \infty$ it is possible to prove that for all $r_0 > 0$, $\bar{D}_{r_0}(\tau)$ satisfies asymptotic hyperbolic discounting with $\beta = 1/2$. The proof is given in the appendix. The simplest case is when the volatility v is so large that it can be considered infinite. In this case the tree can be divided into two regions: In the region below the median the interest rate is $r = r_0 e^{-\infty} = 0$, and in the region above the median it is $r = r_0 e^{\infty} = \infty$. The interest rate paths can be divided into three groups: (1) Paths that remain entirely below the median, which experience no discounting beyond the initial e^{-r_0}. (2) Paths that at any point go above the median; these experience infinite rates and thus contribute nothing. (3) Paths that hit the median exactly k times but never cross above it. This is thus a classic barrier crossing problem (Feller 1950). As shown in the appendix, by making repeated use of the reflection principle it is possible to place accurate bounds on $\bar{D}_{r_0}(t)$ and show that in the large time limit it goes as $t^{-1/2}$.

For volatilities that are not infinite the analysis becomes more complicated but the behavior remains essentially the same. The difference is that the dividing line between the two regions is no longer sharp because there is a band down the center where the interest rate can no longer be considered to be either zero or infinity. Nonetheless, it is still possible to show that the asymptotic scaling goes as $t^{-1/2}$ up to logarithmic corrections.

To demonstrate that the generalized random walk might generally apply outside the realm of interest rate modeling, where the discount rates r_t really do represent psychological states, we fit the experimental results obtained by Thaler (2005) as mentioned earlier[3]. The result is shown in Figure 3. This also illustrates that the geometric random walk has the interesting property that, depending on the parameters, the certainty equivalent discount rate $\bar{r}(\tau) \equiv \log \bar{D}_{r_0}(\tau)/\tau$ first increases and then decreases as a function of time, as we saw in the Weitzman example (see also Litterman, Scheinkman and Weiss (1991)).

How well does the geometric random walk model real interest rates? During the nine-

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[3]The simulation was done by dividing each month up into 100 subintervals. The parameters in the title are on a monthly scale. The fit was done by trial and error.

teenth and twentieth centuries the real interest rate for United States long bonds has varied from roughly 2% to 8%. Newell and Pizer (2003) compared the geometric random walk model to several other stochastic interest rate models, including those with mean reversion, and found that it provided the best fit. The constant interest rate model, in contrast, is obviously a much poorer approximation. The current economic crisis has reminded us that it is possible for real interest rates to go to zero or even become negative.

The difference between the geometric random walk model and exponential discounting becomes stark when one considers really long horizons. Suppose we compare the cumulative weight that the two models give to the future beyond a given time t' by taking the integral of $\bar{D}_{r_0}(t)$ from t' to ∞. For $t' = 100$ years and an annual interest rate of 4%, under exponential discounting the far future only gets a weight of 2%. In contrast, when the same calculation is made for the geometric random walk model for interest rates, the contribution for the far future is always infinite, due to the fact that the integral to infinity of the function $t^{-1/2}$ is infinite.

We do not assert that the geometric random walk is necessarily the best model for interest rates. Another alternative is the so-called square root process, in which $\sqrt{r}$ follows a geometric random walk. One could presumably use the same techniques we have used here to investigate whether that gives rise to an asymptotic power law for the discount function. Allowing for strong mean reversion would destroy the hyperbolic discounting. What this analysis makes clear, however, is that the long term behavior of valuations depends extremely sensitively on the interest rate model. The fact that the present value of actions that affect the far future can shift from a few percent to infinity when we move from a constant interest rate to a geometric random walk calls seriously into question many well regarded analyses of the economic consequences of global warming. For example, Nordhaus (1994, 2008) has used exponential discounting with a time discount rate[4] of 3% to evaluate long term climate effects, while the Stern report has used a time discount rate of .1%. As Dasgupta (2008) has pointed out, the dramatic differences these two analyses suggest about the urgency with which we need to respond to global warming depend sensitively on the discount rate. Our results here, like Weitzman's, suggest that the smaller discount is closer to the truth. But no fixed discount rate is really adequate – as our analysis makes abundantly clear, the proper discounting function is not an exponential.

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[4]Dasgupta (2008) distinguishes the time discount rate δ from the consumption discount rate ρ; when a power law utility function with exponent η is used, they are not the same. Nordhaus (1994) uses $\delta = 3\%$ per year and $\eta = 1$, giving $\rho = 4.3\%$, while Stern uses $\delta = 0.1\%$ per year and $\eta = 1$, resulting in the values for the consumption discount rate $\rho 1.4\%$.

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Appendix: Proof of asymptotic hyperbolic (power law) behavior

We now prove that the asymptotic behavior is always described by a power law with $\beta = 1/2$, regardless of the volatility $v > 0$ and the initial interest rate $r_0 > 0$. For any fixed horizon t, we need to compute the product of the 2^{t-1} one-period discounts along each path, and then average this number over the total 2^{t-1} paths.

We divide the proof into two cases, an easy case where v is infinite, and a slightly more delicate case where v is finite.

In the case v is infinite, the one-period discount factor is either 0 whenever a node above the median is hit, in which the whole path contributes nothing, or 1 if a node below the median is hit, or e^{-r_0}, each time the median is hit. It follows that the heart of the calculation is the computation of the number of paths in a random walk that remain strictly below the median. This is familiar to probabilists as the ballot theorem, and it is usually derived from the reflection principle. We also need to compute how many paths hit the median without going above it, and how many times each of these hit the median. Since these things are unfamiliar to most economists, we derive the calculations from first principles. These techniques can be found in Feller (Feller 1950).

When v is finite we show that we can describe a band around the median such that the one-period discount factor is effectively 0 once we go above the upper band, and is effectively 1 below the lower band. We then must show that not much time is spent inside the band. In fact we must derive the fraction of time the typical path spends at each node inside the band. But that turns out to be a feasible computation related to the famous gambler's ruin problem for finite Markov processes. Again we give the derivation from first principles.

A. Infinite volatility

First consider the case where the volatility parameter v is very large; to keep things simple let it be infinite. In this case the tree can be divided into two regions. In the region below the median the interest rate is $r = r_0 e^{-\infty} = 0$, and in the region above the median it is $r = r_0 e^{\infty} = \infty$. The interest rate paths can be divided into three groups: (1) those that from time 1 onward always remain strictly below the median, π^* in number, which will turn out to be the good paths (2) those that remain below or on the median, and touch the median at least once, π in number, which will turn out to be mediocre paths, and (3) those that have at least one state above the median, $2^{t-1} - \pi - \pi^*$ in number, which are the irrelevant paths.

We will now show that it is possible to count the number of paths in each category above. In a binomial random walk, let i be the net number more downs than ups, so that paths that end above the median have $i < 0$, paths that land on the median have $i = 0$, and paths that end below the median have $i > 0$. Let $\left[\begin{array}{c} n \\ i \end{array} \right]$ denote the number of paths of length $n = t - 1$ which have i more downs than ups, in contrast to rounded brackets $\left(\begin{array}{c} n \\ j \end{array} \right)$, which denote "n choose j". Using the binomial formula we can express

$$\left[\begin{array}{c} n \\ i \end{array} \right] = \left(\begin{array}{c} n \\ (n+i)/2 \end{array} \right) = \frac{n!}{((n-i)/2)!((n+i)/2)!}.$$

We now use the reflection principle (Feller 1950) to compute π and π^*. For convenience assume n is even. Let $i \geq 0$. By the reflection principle the number of paths that eventually end up at (n, i) but somewhere along the way hit $(t, -1)$ is equal to the *total*

number of paths that begin at $(0, -2)$ and reach (n, i). (Note that we are imagining that paths might be able to start at a point that is off the tree in Figure 1, and we are reflecting across the horizontal line $i = -1$). The number of paths that cross into the "dead zone" above the median and end up at i is therefore $\begin{bmatrix} n \\ i+2 \end{bmatrix}$. The number of paths that end up at (n, i) and never go above the median is just the total number of paths to (n, i) minus those that cross the median at least once, i.e. $\begin{bmatrix} n \\ i \end{bmatrix} - \begin{bmatrix} n \\ i+2 \end{bmatrix}$. The number of paths that always remain on or below the median is therefore

$$
(9) \qquad \pi^* + \pi = \sum_{i \geq 0} \left(\begin{bmatrix} n \\ i \end{bmatrix} - \begin{bmatrix} n \\ i+2 \end{bmatrix} \right) = \begin{bmatrix} n \\ 0 \end{bmatrix}.
$$

Letting $n = t - 1$ gives

$$
(10) \qquad \pi^* + \pi = \begin{bmatrix} t-1 \\ 0 \end{bmatrix} = \binom{t-1}{(t-1)/2}.
$$

Similarly, to compute π, paths that remain strictly below the median must go down on the first step to $i = 1$, and then never go above that level for the next $n - 1$ steps, ending up at $i \geq 2$, one step still lower. Hence by the same logic used in Eq. 9 we have

$$
\pi = \sum_{i \geq 1} \left(\begin{bmatrix} n-1 \\ i \end{bmatrix} - \begin{bmatrix} n-1 \\ i+2 \end{bmatrix} \right) =
$$

$$
\begin{bmatrix} n-1 \\ 1 \end{bmatrix} = \binom{n-1}{n/2} = \frac{1}{2} \binom{n}{n/2} = \frac{1}{2} \binom{t-1}{(t-1)/2}.
$$

Comparing to Eq. 10 we see that $\pi^* = \pi$.

Paths that at any point go above the median experience infinite rates and thus contribute nothing to the valuation. Paths that remain entirely below the median experience no discounting beyond the initial e^{-r_0}. Paths that hit the median exactly $k \geq 1$ times but never cross above the median make a contribution $e^{-r_0} e^{-k r_0}$. Since $\pi = \pi^*$, the number of paths that remain below the median is equal to the number that touch but do not cross the median. However, for those that touch but do not cross the discounting is more severe. Thus we know that their contribution is strictly less than than those that do not cross. Using Eq. 8, when $n = t - 1$ is even the discount $D(t)$ can therefore be bounded between

$$
e^{-r_0} (\pi + \pi^*) / 2^{t-1} \geq D(t) \geq e^{-r_0} \pi / 2^{t-1}
$$

$$
e^{-r_0} \binom{t-1}{(t-1)/2} / 2^{t-1} \geq D(t) \geq e^{-r_0} \binom{t-1}{(t-1)/2} / 2^t
$$

$$
\frac{e^{-r_0}}{\sqrt{\pi/2}} (t-1)^{-1/2} \geq D(t) \geq \frac{e^{-r_0}}{\sqrt{\pi/2}} (t-1)^{-1/2} / 2,
$$

where the last line results from applying Stirling's approximation, $n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n}$

and the following standard calculation:

$$\binom{n}{n/2}/2^n = \frac{n!}{(n/2)!(n/2)!}/2^n =$$

$$\frac{\sqrt{2\pi}n^{n+1/2}e^{-n}}{[\sqrt{2\pi}(\frac{1}{2}n)^{\frac{1}{2}n+1/2}e^{-\frac{1}{2}n}]^2}/2^n = \frac{1}{\sqrt{\pi/2}}\frac{1}{n^{1/2}}$$

For $t-1$ odd the formula is similar. Thus for large t the discount $D(t)$ decreases as a power law with exponent $\beta = 1/2$.

Actually we can give an exact formula for $D(t)$, asymptotically. By the argument just given to show $\pi^* = \pi$, it follows that among the paths that always stay on or below the median, those that hit the median once before the very end and never hit it again must be equal to those that hit more than once and stay below the median. This ignores the paths that stay below until the very last step. Hence π_1 is just barely more than $\pi/2$ for large t. The same argument shows that π_2 is just barely more than $\pi_1/2$. Thus we see that asymptotically

$$D(t) \approx \frac{e^{-r_0}}{2^t}\binom{t-1}{(t-1)/2}[1 + \frac{1}{2}e^{-r_0} + \frac{1}{4}e^{-2r_0} + ...]$$

$$= \frac{e^{-r_0}}{2^t}\binom{t-1}{(t-1)/2}/(1 - \frac{1}{2}e^{-r_0})$$

$$= \frac{e^{-r_0}}{(1 - \frac{1}{2}e^{-r_0})}(t-1)^{-1/2}$$

B. Finite volatility

We shall show that there are constants K and C such that when t is sufficiently large,

$$K\log(\log(t))\frac{1}{\sqrt{t}} \geq D(t) \geq \frac{C}{\log(t)}\frac{1}{\sqrt{t}}.$$

We shall begin by arguing the second inequality, which places a lower bound on the discounting. To understand the proof it helps to first realize that because the discount rate decreases exponentially as one moves below the median, there is a lower boundary such that any path that reaches this boundary and stays below it effectively experiences no further discounting. This occurs for the horizontal line defined by $i = N = N(t)$ the closest integer greater or equal to

$$N^*(t) = \frac{1}{v}\log(t).$$

The first thing to observe is that although $N(t) \to \infty$,

$$\frac{N(t)}{t} \to 0$$

Hence for large t, the fraction of paths that remain inside the strip, defined by the median and the horizonal line N steps below the median, goes to zero. Nevertheless, for large t, $N(t)$ is large enough that interest rates with $i > N$ are effectively 0 (no discounting). A

path that experienced discounting at rate $r = \exp(-vN)$ at every node for all $t - 1$ steps would accumulate a total discount factor of

$$(e^{-e^{-vN}})^t \geq (e^{-e^{-\log(t)}})^{t-1} = (e^{-1/t})^{t-1} \geq 1/e$$

which is a constant bounded away from 0, independent of t.

It is possible to conceptually divide the discounting tree into three regions: (1) The region above the median (2) the region more than N below the median, i.e. below the horizontal line $i = N$, where the discounting is effectively zero, and (3) the *strip* between the median and the line $i = N$, where the discounting must be taken into account. The strategy of the proof is to compute the lower bound by estimating the discounting experienced by the paths that start at the median, exit the lower boundary of the strip, and then never cross back into it. For convenience we will call these the *important paths*. We will compute the lower bound by first computing the fraction of all paths that are important important paths and then placing an upper bound on the amount of discounting they experience.

We shall first estimate the fraction $f(t)$ of important paths. An important path must satisfy three criteria: (1) At $t = 1$ the interest rate decreases. (2) The path crosses though the lower boundary of the strip without ever hitting the median. (3) Once below the lower boundary of the strip they never cross back into it. The probability of step (1) is trivially $1/2$. Similarly, we have already computed the probability of step (3), which is just the barrier crossing problem again; it has probability at least $B/\sqrt{t}$, where B is a constant. (At the worst the path exits the median immediately). Step (2) reduces to the gambler's ruin problem, and has probability $1/N$. Although this is a standard result (Feller 1950), since the explanation is fairly simple we recapitulate it here.

When considering the probability that a path inside the strip exits at N without hitting the median first, we can think in terms of a Markov process whose states are horizontal lines at positions $\{0, 1, ..., N\}$, with 0 and N absorbing states. Let $\Pi(i)$ denote the probability a path starting from i hits N before it hits 0. Since the probability of going either up or down is $1/2$, it is automatically true that inside the strip

$$\Pi(i) = \frac{1}{2}\Pi(i-1) + \frac{1}{2}\Pi(i+1).$$

This is a linear relation, so $\Pi(i)$ must vary linearly from one boundary to the other. By definition $\Pi(0) = 0$ and $\Pi(N) = 1$, and for $0 < i < N$,

$$\Pi(i) = \frac{i}{N}.$$

This shows that if we waited infinitely long, the proportion of paths starting at i that hit N before 0 is i/N. But for large t, almost all the paths do hit either N or 0. Putting in $i = 1$ gives us the result. Combining the results for steps (1 - 3) gives

$$f(t) \geq \frac{1}{2N}\frac{B}{\sqrt{t}} = \frac{A}{\log(t)\sqrt{t}}.$$

Now the question remains how much discounting is done on these paths before they cross below N? We count how many times a typical path hits each level between 0 and N. The computation must be exact, for if the paths hit each level equally often, there would be too much discounting and our result would fail. Fortunately we can show

that the frequency distribution is Λ-shaped, with the peak in the middle, where there is already not much discunting. We derive this answer by again exploiting the gambler's ruin solution.

For $0 < i \leq k \leq N$, denote by $E(i, k)$ the expected number of times a path in the Markov process starting from i hits k before exiting. Clearly $E(i, N) = \Pi(i)$. Moreover, for any k with $0 < i \leq k < N$,

$$\Pi(i) = E(i, k) \frac{1}{2} \frac{1}{(N - k)},$$

since in order to get from i to N the path must go through k (or start at k), from which it has probability $1/2$ of going further down to $k+1$, from which it has probability $1/(N-k)$ of hitting N before returning to k. Otherwise it is absorbed by 0, or else it returns again to k, from which again it has probablity $(1/2)(1/(N - k))$ of hitting N before returning to k.

Plugging in $\Pi(i) = i/N$ gives

$$E(i, k) = 2\Pi(i)(N - k) = 2i(N - k)/N.$$

Many of these hits are from paths that will exit at 0, whereas we are only concerned with the hits $W(i, k)$ lying on paths that exit at N. But from the gambler's ruin problem, we know that exactly the fraction k/N of the paths that hit k go on to exit at N before exiting at 0, hence the expected number of hits at k from a path that starts at i and exits at N is

$$W(i, k) = \frac{k}{N} E(i, k) = i \frac{2k(N - k)}{N^2}$$

Only the fraction $1/N$ of the paths starting at $i = 1$ exit at N. Thus if we condition only on the paths that hit N, and put in $i = 1$, the expected number of times a path starting at $i = 1$ hits k, is N times as large, i.e.

$$W(1, k | \text{end at } N) = NW(1, k) = \frac{2k(N - k)}{N}.$$

This concludes our calculation of the expected number of times important paths hit each level k. Notice that this frequency is indeed Λ-shaped, hitting its maximum around $k = N/2$.

To compute the discount we take the product of the discount rates, weighted by the number of times each discount rate occurs. Hence for a path that hit each level k exactly as often as expected, the total discount would be

$$e^{-r_0} \prod_{k=1}^{N} (e^{-r_0 e^{-vk}})^{\frac{2k(N-k)}{N}} \geq e^{-r_0} \prod_{k=1}^{N} (e^{-r_0 e^{-vk}})^{2k}$$

$$= e^{-r_0} \prod_{k=1}^{N} e^{-2k r_0 e^{-vk}}$$

$$\geq e^{-r_0} e^{-2r_0 \sum_{k=1}^{\infty} k e^{-vk}}$$

$$= e^{-r_0} e^{-2r_0 e^{-v}/(1-e^{-v})^2} = L$$

To get this result we have used the fact that the infinite series $\sum kx^k$ sums to $x/(1 - x)^2$,

as well as the fact that $(N - k)/N \leq 1$, and replacing a finite sum by an infinite sum.

Of course the typical path does not hit each of these levels k the average number of times. But it is evident that the most discounting occurs precisely when all of these hits are indeed distributed so that every path has the average amount of discounting. Hence in the worst case each of the N paths is discounted by L.

Thus we have derived a lower bound on the certainty equivalent discount,

$$D(t) \geq e^{-1} L \frac{A}{\log(t)\sqrt{t}} = \frac{C}{\log(t)} \frac{1}{\sqrt{t}}.$$

We now compute an upper bound, proving that

$$K \log(\log(t)) \frac{1}{\sqrt{t}} \geq D(t).$$

The strategy is to count all the paths that could conceivably make a non-zero contribution to the valuation, and derive an upper bound by assuming that these paths experience no discounting at all. Let $n = n(t) = \log(N(t)) = \log(\log(t))$. The one-period discount at a node n levels *above* the median is

$$e^{-e^{n(t)}} = e^{-e^{\log(\log(t))}} = e^{-\log(t)} = 1/t.$$

Thus an interest rate path that ever experiences gets above level $i = n$ during its history will make a contribution that is negligible compared with $1/\sqrt{t}$ as $t \to \infty$, even if it had probability 1. We can thus get an upper bound on the discount factor by simply counting the number of paths that never go above n.

The strategy for counting the number of paths is to count the total number of paths (including those that go above n) and then use the reflection principle to subtract those that go above n, leaving just the number of paths that never go above n. The number of paths that start at $t = 0$ and end up at $0 \leq k \leq n(t)$ net up moves at time t are by definition $\begin{bmatrix} n \\ -k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}$ in number. Hence the number of paths that end up with less than or equal to n net up moves, regardless of how they get there, are

$$\sum_{k=1}^{n} \begin{bmatrix} t \\ -k \end{bmatrix} + \sum_{k=0}^{\infty} \begin{bmatrix} t \\ k \end{bmatrix}.$$

Now we calculate the number of paths that end up at k, but pass beyond n at some point along the way. From the reflection principle using the horizontal line $i = n$, the number of paths that end up below n, having at some point been above n, are

$$\sum_{k=n+1}^{\infty} \begin{bmatrix} t \\ k \end{bmatrix}.$$

Subtracting this from the expression above and making use of the fact that $\begin{bmatrix} t \\ k \end{bmatrix} \leq$

$\begin{bmatrix} t \\ 0 \end{bmatrix}$, the total number of paths that never go above n are

$$\sum_{k=-n}^{n+1} \begin{bmatrix} t \\ k \end{bmatrix} \leq (2n+1) \begin{bmatrix} t \\ 0 \end{bmatrix}.$$

But as we saw before, $\begin{bmatrix} t \\ 0 \end{bmatrix}/2^t \leq \frac{C}{\sqrt{t}}$. Thus the fraction of paths that never go above $n(t) = log(log(t))$ is at most $(2\log(\log(t)+1)\frac{C}{\sqrt{t}}$. Assuming that these experience no discounting at all gives us our conclusion,

$$K \log(\log((t))\frac{1}{\sqrt{t}} \geq D(t).$$

These bounds contain logarithmic corrections that are not pure power laws But the logarithmic corrections are relatively so slow that they are not important. The logarithm is a classic example of a slowly varying function, and the bounds we have derived satisfy the definition of a power law.