Efficiency and Equilibrium when Preferences are Time-Inconsistent*

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Abstract

We consider an exchange economy with time-inconsistent consumers whose preferences are additively separable. If consumers have identical discount factors, then allocations that are Pareto efficient at the initial date are also renegotiation-proof. In an economy with a sequence of markets, competitive equilibria are Pareto efficient in this sense, and for generic endowments, only if preferences are locally homothetic.

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1. Introduction

There has been a recent upsurge of interest in models in which consumers have present-biased time-inconsistent preferences. This interest is motivated in part by introspection, by experiments, and by the possibility that certain types of behavior can be more easily understood using such preferences.[1]

Much of the literature has relied on additively separable preferences with identical subjective discount factors and homothetic utility functions.[2] As is the case when preferences are time consistent, this means that the distribution of wealth does not affect equilibrium prices when markets are complete. This paper points out that there is an additional implication that is not expected, not robust, and therefore potentially misleading. The implication is that the competitive equilibrium of an exchange economy with a sequence of markets is Pareto efficient from the perspective of consumers making decisions at any given point in time.

This efficiency result is unexpected because the classic proof of the First Welfare Theorem fails. In an economy with a sequence of markets, consumer choices are taken to be the outcome of an intrapersonal game. A decision maker at a point in time is not in full control of the consumption sequence selected in equilibrium from the budget set, and so a Pareto improvement may well be budget feasible at equilibrium prices, for every consumer in the economy. It is easy to construct explicit examples in which the efficiency result fails when consumers discount future utilities differently.[3] We show that, even when consumers discount future utilities in the same way, homotheticity is also in essence necessary for the efficiency result. When preferences are not locally homothetic, competitive equilibria are inefficient for generic endowments.

The underlying reason for the special role of homotheticity stems from the fact that time-inconsistency distorts intertemporal marginal rates of substitution by a factor that depends on marginal propensities to consume out of next-period wealth. The linear consumption function implied by identical homothetic preferences ensures that this distortion is the same across consumers. This guarantees efficiency. But when

[1]Strotz [16] and Phelps and Pollak [13] initiated the study of additively separable preferences that exhibit time-inconsistency. Laibson [8] pointed to the role of partially illiquid assets in providing commitment to consumers with time-inconsistent preferences.

[2]Equilibrium models that make essential use of homotheticity include Barro [2], Kocherlakota [6], Krusell, Kuruşçu and Smith [7], and Luttmer and Mariotti [9].

[3]Consider for example an economy with a time-consistent consumer and a time-inconsistent consumer who both have log utility.

consumers do not have the same homothetic preferences, marginal propensities to consume out of next-period wealth typically differ across consumers. In turn this causes marginal rates of consumption between current and next-period consumption to differ across consumers. The resulting allocation of resources will be inefficient.

2. Efficiency in an exchange economy

2.1. The economy

We consider a three-period exchange economy with a finite number I of consumer types. There is a continuum of consumers of each type, and for notational simplicity we take each of these continua to be of unit measure. A single good is available for consumption in every period. A consumer of type i has positive endowments e_t^i of this good in period t, and aggregate endowments in this period are denoted by e_t. A consumer of type i has preferences over non-negative consumption sequences $c^i = (c_1^i, c_2^i, c_3^i)$ given by:

$$U_1^i(c^i) = u^i(c_1^i) + \delta_1 u^i(c_2^i) + \delta_2 u^i(c_3^i)$$

in period 1, and by:

$$U_2^i(c^i) = u^i(c_2^i) + \delta_1 u^i(c_3^i)$$

in period 2. The subjective discount factors δ_1 and δ_2 are positive, and the period utility functions $u^i : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ are assumed to be strictly increasing, continuous, and strictly concave. These preferences are time-inconsistent whenever $\delta_1^2 \neq \delta_2$, with a bias toward the present if $\delta_1^2 < \delta_2$. Note that, although the period utility functions u^i may vary across consumer types, we take the discount factors δ_1 and δ_2 to be the same for all consumer types.

2.2. Efficient allocations

A symmetric allocation in this economy is a vector $c \in \mathbb{R}_+^{3I}$ of consumption sequences, one for each consumer type.[4] An allocation is feasible if aggregate consumption in every period does not exceed aggregate endowments. Because preferences may change over time, several notions of efficiency can be useful.

Definition 1. *A feasible allocation c is*:

[4] Our efficiency concepts can be easily extended to asymmetric allocations. However, because the u^i are strictly concave, such allocations can never be efficient in any of the senses defined below.

(i) Date-t Pareto efficient if there is no other feasible allocation $\tilde{c}$ such that $U_t^i(\tilde{c}^i) \geq U_t^i(c^i)$ for all i, with a strict inequality for at least one i.

(ii) Renegotiation-proof if it is date-2 Pareto efficient and there is no other date-2 Pareto efficient allocation $\tilde{c}$ such that $U_1^i(\tilde{c}^i) \geq U_1^i(c^i)$ for all i, with a strict inequality for at least one i.

Date-1 efficiency is the natural notion of efficiency when consumers can commit ex ante to a sequence of consumption choices. When this is not the case, renegotiation-proof allocations correspond to a notion of constrained efficiency: these allocations are efficient when evaluated using date-1 preferences, subject to the constraint that the implied date-2 allocations are efficient when evaluated using date-2 preferences. It turns out that this constraint does not bind when all consumers discount future utilities in the same way.

Proposition 1. *When consumers have identical discount factors, the sets of date-1 Pareto efficient and of renegotiation-proof allocations coincide.*

Proof. Let $e = (e_1, e_2, e_3)$, and consider the set $\mathcal{U}_1$,

$$\mathcal{U}_1 = \left\{ U_1 \in \mathbb{R}^I : U_1 \leq (U_1^1(c^1), \ldots, U_1^I(c^I)) \text{ for some } c \in \mathbb{R}_+^{3I} \text{ such that } \sum_{i=1}^I c^i \leq e \right\}.$$

Because the aggregate resource constraint is convex and the utility functions U_1^i are concave and continuous, $\mathcal{U}_1$ is a closed and convex set. Date-1 Pareto efficiency of an allocation $c \in \mathbb{R}_+^{3I}$ implies that c belongs to the boundary of $\mathcal{U}_1$. By the Separating Hyperplane Theorem, there exists a $\lambda \in \mathbb{R}^I \setminus \{0\}$ such that $\lambda \cdot U_1(c) \geq \lambda \cdot U_1$ for all $U_1 \in \mathcal{U}_1$, and, since $\mathcal{U}_1 - \mathbb{R}_+^I \subset \mathcal{U}_1$, we must have $\lambda \geq 0$. In particular, c solves:

$$\max_{c \in \mathbb{R}_+^{3I}} \left\{ \sum_{i=1}^I \lambda^i U_1^i(c^i) : \sum_{i=1}^I c^i \leq e \right\}. \tag{1}$$

Since the resource constraints are independent across time and preferences are additively separable, and since the discount factors δ_1 and δ_2 are the same across consumers, the solution to (1) can be obtained by solving:

$$\max_{c_t \in \mathbb{R}_+^I} \left\{ \sum_{i=1}^I \lambda^i u^i(c_t^i) : \sum_{i=1}^I c_t^i \leq e_t \right\}$$

for all t. This in turn implies that c solves:

$$\max_{c\in\mathbb{R}^{3I}_+}\left\{\sum_{i=1}^{I}\lambda^i U_2^i(c^i):\sum_{i=1}^{I}c^i\le e\right\}.$$

Therefore, since $\lambda\ge 0$ and $\lambda\ne 0$, there exists no feasible allocation $\tilde{c}$ such that $U_2^i(\tilde{c}^i)>U_2^i(c^i)$ for all i. Using the fact that the u^i are continuous and strictly increasing, one can verify that this implies that c is date-2 Pareto efficient. □

This result says that full efficiency and constrained efficiency impose the same restrictions on allocations, provided consumers have identical discount factors. This justifies our use of date-1 Pareto efficiency as our efficiency concept in Section 4. It is straightforward to extend this result to multi-period economies in which consumers of type i have preferences in period t given by $\sum_{n=0}^{T-t}\delta_n u^i(c^i_{t+n})$, with T possibly infinite. Proposition 1 also holds under uncertainty if preferences after every history can be represented by an expected utility function using subjective probabilities that are updated using Bayes' rule.[5]

2.3. Different discount factors

Things change when consumers have discount factors δ_1^i and δ_2^i that are not the same for all i. Let $c(\lambda)$ be a date-1 Pareto efficient allocation given a vector of Pareto weights λ for date-1 utilities. That is, $c(\lambda)$ solves:

$$\max_{c\in\mathbb{R}^{3I}_+}\left\{\sum_{i=1}^{I}\lambda^i\big[u^i(c_1^i)+\delta_1^i u^i(c_2^i)+\delta_2^i u^i(c_3^i)\big]:\sum_{i=1}^{I}c^i\le e\right\}. \tag{2}$$

For $c(\lambda)$ to remain Pareto efficient at date 2, it must be that $(c_2(\lambda),c_3(\lambda))$ solves:

$$\max_{(c_2,c_3)\in\mathbb{R}^{2I}_{++}}\left\{\sum_{i=1}^{I}\mu^i\big[u^i(c_2^i)+\delta_1^i u^i(c_3^i)\big]:\sum_{i=1}^{I}(c_2^i,c_3^i)\le(e_2,e_3)\right\} \tag{3}$$

for some vector of Pareto weights μ. Using (2)–(3) together with the fact that there is a one-to-one relationship between efficient allocations and Pareto weights, it is not difficult to check that the only circumstance in which this will be the case is when the ratio $\delta_2^i/(\delta_1^i)^2$ is constant across consumers. This is automatically satisfied if consumers have

[5] An axiomatic foundation of such preferences can proceed mostly along the usual lines. To allow for time-inconsistency and still obtain subjective probabilities that satisfy Bayes' rule, one has to assume that preferences are consistent across information sets.

time-consistent preferences. When this is not the case, this ratio can be interpreted as a measure of the consumers' time-inconsistency. Thus date-1 Pareto efficient allocations are renegotiation-proof if and only if all consumers exhibit the same degree of time-inconsistency: when their discount factors are given by $\beta\delta^i$ and $\beta(\delta^i)^2$ for some common time-inconsistency parameter β. In multi-period economies, this generalizes to the requirement that the discount factors of a type-i consumer are given by $\beta_1\delta^i$, $\beta_2(\delta^i)^2$, $\beta_3(\delta^i)^3$, ..., for all i.

3. Competitive equilibria in economies with a sequence of markets

The Second Welfare Theorem implies that date-1 efficient allocations can be implemented using competitive markets in which trade in one- and two-period bonds takes place only at date 1. When preferences are time-consistent, one can use this to construct an equivalent equilibrium for an economy with a sequence of markets in which consumers can trade in one-period bonds (Arrow [1]). A consumption plan that is feasible in one economy is feasible in the other, and time-consistency ensures that consumers who make plans at one date will not want to revise them at a later date. This last observation is no longer true when preferences are time-inconsistent, and we therefore need to study economies with a sequence of markets separately.

3.1. Markets

We consider the following market structure. Consumers trade in markets for one-period discount bonds at dates 1 and 2. They face no constraints on borrowing, other than that they must be able to pay off their debts at date 3. The sequence of bond markets allows consumers to exchange consumption at any one date for consumption at any other date. The price of date-t consumption in terms of some numeraire is denoted by p_t and thus the price in terms of date-t consumption of a bond that pays one unit of consumption at date $t+1$ is simply p_{t+1}/p_t. We let $p = (p_1, p_2, p_3)$ and normalize p so that it belongs to the unit simplex Δ_3 of $\mathbb{R}^3_{++}$.

Following Pollak [14] and Peleg and Yaari [12], we view each individual consumer as composed of a sequence of autonomous decision makers, indexed by time. We refer to the decision maker at date t as the "date-t consumer." Taking prices as given, a trading strategy for the date-t consumer is a decision how much to consume and save given any history at date t. For any given individual consumer, we require these trading strategies to form a subgame-perfect equilibrium of the intrapersonal game played between the corresponding date-1, -2 and -3 consumers.

3.2. The date-2 exchange economy

At date 3, a typical consumer simply consumes his or her wealth, which consists of endowments and maturing bonds. At date 2, the same consumer chooses how much to consume and how many bonds to buy. Given wealth $w_2 \geq 0$ and prices $p \in \Delta_3$, a date-2 consumer of type i solves:

$$\max_{(c_2,c_3)\in\mathbb{R}^2_+} \big\{u^i(c_2) + \delta_1 u^i(c_3) : p_2 c_2 + p_3 c_3 \leq p_2 w_2\big\}. \tag{4}$$

Let $c^i_2(p, w_2)$ and $c^i_3(p, w_2)$ be the decision rules that solve (4) for various prices p and wealth levels w_2. The utility perceived by the date-1 consumer from these choices is captured by a value function V^i defined by:

$$V^i(p, w_2) = \delta_1 u^i(c^i_2(p, w_2)) + \delta_2 u^i(c^i_3(p, w_2)).$$

For given prices p, there is no guarantee that this value function will be concave in date-2 wealth w_2 if preferences are not time-consistent.[6] This may give a date-1 consumer an incentive to trade in lotteries (Luttmer and Mariotti [10]). In the absence of lottery markets, the non-concavity of $V^i(p, \cdot)$ can cause the set of optimal consumption and savings choices of a date-1 consumer to be non-convex.

3.3. Competitive equilibrium

By trading in one-period bonds, a date-1 consumer of type i with wealth w_1 can choose levels of date-1 consumption and date-2 wealth that solve:

$$\max_{(c_1,w_2)\in\mathbb{R}^2_+} \big\{u^i(c_1) + V^i(p, w_2) : p_1 c_1 + p_2 w_2 \leq p_1 w_1\big\}. \tag{5}$$

The set of solutions to this decision problem is denoted by $[c^i_1, w^i_2](p, w_1)$. For any price vector $p \in \Delta_3$, let $w^i_1(p)$ denote date-1 wealth of a consumer of type i:

$$w^i_1(p) = \frac{1}{p_1}\sum_{t=1}^{3} p_t e^i_t. \tag{6}$$

[6]Morris [11] shows that in the case of present bias ($\delta_2 > \delta_1^2$) the value function $V^i(p, \cdot)$ is concave if $\delta_1 p_2 \leq (\geq) p_3$ and the absolute risk tolerance of u^i is convex (concave).

Given prices p and date-1 choices (c_1, w_2), the consumption allocation of a consumer of type i is given by $d^i(p, c_1, w_2) = (c_1, c_2^i(p, w_2), c_3^i(p, w_2))'$. By combining the solution to (5) with (6) one can construct the demand correspondence of a consumer of type i:

$$D^i(p) = d^i(p, [c_1^i, w_2^i](p, w_1^i(p))).$$

By construction, a date-1 consumer of type i will be indifferent between all points in $D^i(p)$. Because there is a continuum of unit measure of such consumers, their aggregate demand is given by the convex hull $\text{co}D^i(p)$. A point in $\text{co}D^i(p)$ that is not an extreme point is obtained by having appropriate fractions of type-i consumers choose points in $D^i(p)$. A competitive equilibrium is given by prices p such that:

$$0 \in \sum_{i=1}^{I} \big[\text{co}D^i(p) - e^i\big].$$

We can now prove the following result.

Proposition 2. *A competitive equilibrium exists.*

The proof can be constructed using Debreu's [3] excess demand approach. The main difficulty consists in showing that if $\{p_n\}$ is a sequence of prices that converges to the boundary of Δ_3, then, for each i, the sequence $\{\inf_{z \in D^i(p_n)} \|z\|\}$ goes to infinity. That this boundary property holds is not a priori obvious because D^i is not the outcome of a decision problem when consumers of type i have time-inconsistent preferences. We refer to Luttmer and Mariotti [10] for details.

4. Smooth preferences

Under what circumstances will the competitive equilibria shown to exist in Proposition 2 be Pareto efficient at the initial date? Clearly, date-1 Pareto efficiency does not hold for asymmetric equilibria in which the non-convexity of the demand correspondence means that consumers of the same type must choose different allocations. If the utility functions u^i are sufficiently smooth, then more can be said about the efficiency properties of symmetric competitive equilibria. We therefore introduce the following assumption.

Assumption S. *The utility functions u^i have continuous derivatives up to any order on $\mathbb{R}_{++}$, and $\lim_{c \downarrow 0} \mathrm{D}u^i(c) = +\infty$.*

4.1. *Efficient allocations*

Suppose Assumption S holds. Given a vector of aggregate endowments e, the set of interior date-1 Pareto efficient allocations is then given by those $c \in \mathbb{R}^{3I}_{++}$ that for some $\lambda \in \mathbb{R}^{I}_{++}$ and $p \in \mathbb{R}^{3}_{++}$ satisfy the marginal conditions:

$$\lambda^i \mathrm{D}u^i(c^i_t) = p_t \tag{7}$$

and feasibility conditions:

$$\sum_{i=1}^{I} c^i_t = e_t, \tag{8}$$

for all i and t. We can take λ to be in the unit simplex Δ_I of $\mathbb{R}^I_{++}$. Let $\mathcal{P}$ be the set of pairs (e, c) of aggregate endowments and consumption allocations that satisfy (7)–(8). In the Appendix we show that $\mathcal{P}$ is a $(I+2)$-dimensional manifold. Thus, given aggregate endowments, the manifold of Pareto efficient allocations is, as expected, of dimension $I - 1$.

4.2. *Equilibrium allocations*

Fix some price vector $p \in \Delta_3$. Under Assumption S, the decisions of a date-2 consumer of type i with positive wealth w_2 are fully characterized by the date-2 budget constraint and the usual first-order condition:

$$\frac{p_3}{p_2} = \frac{\delta_1 \mathrm{D}u^i(c^i_3(p, w_2))}{\mathrm{D}u^i(c^i_2(p, w_2))}. \tag{9}$$

Moreover, for a fixed p, the decision rules $c^i_2(p, \cdot)$ and $c^i_3(p, \cdot)$ are differentiable functions of wealth. By differentiating (9) and the date-2 budget constraint with respect to w_2 one can verify that $\mathrm{D}_w V^i(p, w_2)$ must be given by:

$$\mathrm{D}_w V^i(p, w_2) = f^i(c^i_2(p, w_2), c^i_3(p, w_2))\, \mathrm{D}u^i(c^i_2(p, w_2)), \tag{10}$$

where the function f_i is defined by:

$$f^i(x, y) = \frac{\delta_1 \frac{[\mathrm{D}u^i(x)]^2}{\mathrm{D}^2 u^i(x)} + \delta_2 \frac{[\mathrm{D}u^i(y)]^2}{\mathrm{D}^2 u^i(y)}}{\frac{[\mathrm{D}u^i(x)]^2}{\mathrm{D}^2 u^i(x)} + \delta_1 \frac{[\mathrm{D}u^i(y)]^2}{\mathrm{D}^2 u^i(y)}}.$$

Note that in the case of time-consistent preferences, this expression reduces to δ_1, as expected from (10) and the envelope condition for (4). The consumption and wealth choices of a date-1 consumer of type i with positive wealth w_1 must satisfy the date-1 budget constraint and the usual first-order condition:

$$\frac{p_2}{p_1} = \frac{\mathrm{D}_w V^i(p, w_2)}{\mathrm{D}u^i(c_1^i(p, w_1))}. \tag{11}$$

Any feasible consumption allocation that satisfies (9)–(11) for all consumers i and some prices p is a candidate for a symmetric competitive equilibrium allocation. Alternatively, given aggregate endowments e, a feasible allocation c that is part of a competitive equilibrium must for some $\lambda \in \Delta_I$ and $p \in \mathbb{R}^3_{++}$ satisfy:

$$\lambda^i \begin{bmatrix} \mathrm{D}u^i(c_1^i) \\ f^i(c_2^i, c_3^i)\,\mathrm{D}u^i(c_2^i) \\ f^i(c_2^i, c_3^i)\,\mathrm{D}u^i(c_3^i) \end{bmatrix} = p \tag{12}$$

for all i. Because the first-order conditions (11) need not be sufficient, some of the feasible allocations admitted by (12) may not correspond to an equilibrium. The collection of (e, c) such that c is a symmetric competitive equilibrium allocation given aggregate endowments e is contained in a manifold of dimension $I + 2$.

4.3. Efficient equilibria are non-generic

A comparison of (7) and (12) shows that competitive equilibria are efficient if and only if the $f^i(c_2^i, c_3^i)$ are the same across consumers. By adding this restriction to the conditions (7)–(8) for efficiency, we can determine which of the efficient allocations could potentially be decentralized as equilibrium allocations. From now on, we shall assume that preferences are time-inconsistent. It then follows from (7) and the definition of f^i that adding the restriction that the $f^i(c_2^i, c_3^i)$ coincide to the definition of an efficient allocation is equivalent to adding the requirement that for some $\xi > 0$ and all i:

$$\frac{\mathrm{D}^2 u^i(c_3^i)}{\mathrm{D}^2 u^i(c_2^i)} = \xi. \tag{13}$$

Relative to the definition of $\mathcal{P}$, this adds I additional restrictions and the new variable ξ. Since $\mathcal{P}$ is an $(I + 2)$-dimensional manifold, this suggests that the set of aggregate endowments and efficient equilibrium allocations is 3-dimensional. For given aggregate endowments, this would imply that there are only isolated points at which the equilibrium and efficient allocations coincide.

Whether or not this is indeed the case depends on whether the equations (13) are locally independent of the efficiency conditions (7)–(8). The following three examples show why this need not be true.

Example 1. If $u^i(c) = (c^{1-\rho} - 1)/(1-\rho)$ for some $\rho > 0$ and all i, then (13) is implied by the efficiency conditions (7)–(8). This implies that competitive equilibria are in fact efficient. For these preferences, the fact that the $\mathrm{D}u^i(c_3^i)/\mathrm{D}u^i(c_2^i)$ are the same across consumers implies that consumption growth between dates 2 and 3 is the same for all consumers. In turn, this implies that the $\mathrm{D}^2u^i(c_3^i)/\mathrm{D}^2u^i(c_2^i)$ are also the same across consumers, which makes (13) redundant. Thus, in particular, the linear competitive equilibrium studied in Luttmer and Mariotti [9] is efficient.

Example 2. Consider arbitrary utility functions u^i but suppose that $e_2 = e_3$. Then efficiency in the 2-period exchange economy that starts at date 2 requires that $c_2^i = c_3^i$ for all consumers. Constant consumption across dates 2 and 3 for all consumers again makes (13) redundant.

Example 3. Suppose consumers are identical and have identical endowments. Then any symmetric equilibrium would clearly be efficient, although one need not necessarily exist.

Our main result shows that these examples of efficiency are special, either because of homotheticity, or because of non-generic endowments. Specifically, if preferences are nowhere locally homothetic, then condition (13) will, for generic endowments, be independent of the efficiency conditions (7)–(8), and a symmetric competitive equilibrium will not be date-1 Pareto efficient.

We will say that preferences are locally homothetic if the ratio:

$$s^i(x) = \frac{\mathrm{D}^3u^i(x)\mathrm{D}u^i(x)}{[\mathrm{D}^2u^i(x)]^2}$$

is constant over some range. That is, preferences exhibit locally linear risk tolerance $\mathrm{D}u^i(x)/\mathrm{D}^2u^i(x)$. Our next assumption precisely rules out this case.[7]

Assumption Z. *The utility functions u^i are such that $\mathrm{D}s^i$ is zero on a closed set of measure zero.*

This gives rise to the following result.

[7] A weaker version of Assumption Z would require that the set of points at which $\mathrm{D}s^i$ vanishes is nowhere dense in $\mathbb{R}_{++}$. The results derived below hold under this alternative assumption provided "measure zero" is replaced by "nowhere dense and closed" in all the statements below.

Proposition 3. *Under Assumptions S and Z, the set of date-1 Pareto efficient allocations and the set of equilibrium allocations intersect only at isolated points, except for economies with aggregate endowments in a closed set of measure zero.*

Because the set of date-1 Pareto efficient allocations and the set of renegotiation-proof allocations coincide when consumers have identical discount factors, Proposition 3 can be interpreted as a constrained inefficiency result. The importance of homotheticity is easy to understand. Specifically, (11) can be expressed using the marginal propensity to consume out of date-2 wealth as:

$$\frac{p_2}{p_1} = \left\{ \mathrm{D}_w c_2^i(p, w_2)\, \delta_1 + \left[1 - \mathrm{D}_w c_2^i(p, w_2)\right] \frac{\delta_2}{\delta_1} \right\} \frac{\mathrm{D}u^i(c_2^i(p, w_2))}{\mathrm{D}u^i(c_1^i(p, w_1))}. \tag{14}$$

This is the generalized Euler equation of Harris and Laibson [5]. It follows from (14) that date-1 efficiency requires all consumers to have the same marginal propensity to consume out of date-2 wealth in equilibrium. Proposition 3 shows that this can hold generically only for the linear consumption functions implied by identical homothetic preferences. In analogy with the incomplete markets literature (Stiglitz [15], Geanakoplos and Polemarchakis [4]), the intuition is that competitive date-1 consumers do not internalize the impact of their savings decisions on date-2 and -3 prices, which in turn affect the decisions of date-2 consumers and therefore the welfare of date-1 consumers. Identical homothetic preferences ensure that equilibrium prices do not depend on the distribution of wealth across consumers, and this is what leads to a (constrained) efficient allocation in equilibrium.

5. Concluding remarks

Renegotiation-proofness is a benchmark for efficiency in an economy in which it is not possible to commit not to renegotiate. One would expect renegotiation-proof allocations to arise in an environment in which a contract is enforced unless all parties to the contract agree to re-write it, and in which bargaining is efficient. Our results show that a sequence of competitive markets need not achieve this benchmark of efficiency. An interesting open question is whether there are decentralized mechanisms, other than a complete set of date-1 markets, that do.

We have focussed on exchange economies. The example of Krusell, Kuruşçu, and Smith [7] shows that the competitive equilibrium in a production economy with identical consumers and homothetic preferences can yield a higher level of utility to consumers at the initial date than does any renegotiation-proof allocation. Thus renegotiation-proof allocations need no longer be Pareto efficient from the perspective of consumers at the initial date. Instead, competitive markets generate a form of commitment that makes these consumers better off than when they have access to efficient centralized bargaining procedures. However, the use of homothetic preferences in Krusell, Kuruşçu, and Smith [7] rules out the sort of inefficiency of competitive markets that can occur even in an exchange economy.

A Proof of Proposition 3

Step 1. As defined in (7)–(8), $\mathcal{P}$ is parameterized by pairs (e, λ) of aggregate endowments and Pareto weights. It will be more convenient to parameterize $\mathcal{P}$ instead using the vector of aggregate endowments e, together with a feasible allocation c_t at one particular date t. To construct such a parameterization, consider any (e_t, λ) in $\mathbb{R}_{++} \times \Delta_I$ and solve the date-t version of (7)–(8) for (e_t, c_t). This defines a function g that maps $\mathbb{R}_{++} \times \Delta_I$ onto the set F of strictly positive (e_t, c_t) that satisfy the feasibility constraint (8). The inverse of this function is given by $(e_t, l(c_t))$, where:

$$l^i(c_t) = \left[\sum_{j=1}^{I} \frac{1}{\mathrm{D}u^j(c_t^j)}\right]^{-1} \frac{1}{\mathrm{D}u^i(c_t^i)}$$

for each i. One can show that g is a diffeomorphism (see Luttmer and Mariotti [10]).

Fix any t, take a vector (e, c_t) such that $(e_t, c_t) \in F$, and define:

$$(e_s, c_s) = g(e_s, l(c_t))$$

for $s = 1, 2, 3$. This defines a map φ_t that takes any point (e_t, c_t, e_{-t}) from $\Theta = F \times \mathbb{R}^2_{++}$ and maps it into $\mathcal{P}$. The fact that g is a diffeomorphism implies that $\varphi_t : \Theta \to \mathcal{P}$ is a diffeomorphism as well. Clearly, Θ is a $(I+2)$-dimensional manifold, and so $\mathcal{P}$ must be too. Given aggregate endowments, the manifold of Pareto efficient allocations is, as expected, of dimension $I - 1$.

Step 2. Define, for every t:

$$A_t = \left\{ (e_t, c_t, e_{-t}) \in \Theta : \prod_{i=1}^{I} \mathrm{D}s^i(c_t^i) = 0 \right\},$$

$$B_t = \left\{ (e, c) \in \mathcal{P} : \prod_{i=1}^{I} \mathrm{D}s^i(c_t^i) = 0 \right\}.$$

For every t, we have $\varphi_t^{-1}(B_t) \subset A_t$. Assumption Z implies that A_t has measure zero in Θ. Since φ_t is a diffeomorphism, it then follows that B_t has measure zero in $\mathcal{P}$, for every t. Thus, leaving out points from the efficient manifold at which some $\mathrm{D}s^i(c_t^i)$ vanishes amounts to leaving out a set of points that is of measure zero in the efficient manifold. Intuitively, the fact that φ_t is a diffeomorphism implies that the efficient manifold has no tangent spaces of the form $\{(e, c) : c_t^i = 0\}$. The fact that B_t has measure zero in $\mathcal{P}$ follows naturally from this and Assumption Z.

For each i, let C^i be the set of points where $\mathrm{D}s^i$ is not equal to zero, and let $C = \prod_{i=1}^{I} C^i$. Write $\mathcal{P}^* = \mathcal{P} \cap (\mathbb{R}^3_{++} \times C^3)$, and $\Theta^* = \Theta \cap (\mathbb{R}_{++} \times C \times \mathbb{R}^2_{++})$. Since the $\mathrm{D}s^i$ are continuous it follows that C is an open subset of $\mathbb{R}^I_{++}$. Similarly, $\mathcal{P}^*$ and Θ^* are relatively open subsets of $\mathcal{P}$ and Θ, respectively. As a result of Assumption Z, $\mathcal{P}^*$ differs from $\mathcal{P}$ by a closed set of measure zero. For every t, $\varphi_t : \Theta^* \to \mathcal{P}^*$ is again a diffeomorphism.

Step 3. It turns out that eliminating points from the consumption spaces where some $\mathrm{D}s^i$ vanishes is not enough to prove our genericity result. By focusing on points in C we can eliminate some additional critical points from the commodity space without eliminating non-negligible pieces from $\mathcal{P}$. For each i, define $r^i : \mathbb{R}^3_{++} \times C^3 \to \mathbb{R}$ by $r^i(e, c) = s^i(c_2^i) - s^i(c_3^i)$, and consider the function $R^i : \Theta^* \to \mathbb{R}$ defined as $R^i(\theta) = r^i(\varphi_1(\theta))$. We then obtain the following result.

Lemma 1. *The function $R^i : \Theta^* \to \mathbb{R}$ only has regular values.*

Proof. Note that $\mathrm{D}R^i(\theta) = \mathrm{D}r^i(e, c)\,\mathrm{D}\varphi_1(\theta)$ for $(e, c) = \varphi_1(\theta)$ and $\theta = (e_1, c_1, e_{-1})$. Since $c \in C$ whenever $\theta \in \Theta^*$, $\mathrm{D}r^i(e, c) \neq 0$. Consider varying the e_{-1}-component of θ. Since (e_1, c_1) is fixed, $\lambda = l(c_1)$ must be fixed. Thus we are to investigate changes in (c_2^i, c_3^i) as (e_2, e_3) varies for fixed λ. Efficiency requires that consumption of all consumers co-moves strictly with the aggregate. Thus, by varying (e_2, e_3) in arbitrary directions, one can vary (c_2^i, c_3^i) in arbitrary directions. This means that one can find a linear combination of the columns of $\mathrm{D}\varphi_1(\theta)$ that is not orthogonal to $\mathrm{D}r^i(e, c)$. It follows that $\mathrm{D}R^i(\theta) \neq 0$. $\square$

Recall that Θ^* is a relatively open subset of Θ, and thus an $(I+2)$-dimensional manifold. Lemma 1 together with the Preimage Theorem implies that the zero set of R^i is a submanifold of Θ^* of dimension $I+1$. The fact that φ_1 is a diffeomorphism implies that the image under φ_1 of this submanifold is a submanifold of $\mathcal{P}^*$ of lower dimension than $\mathcal{P}^*$. For every i, we can therefore eliminate from $\mathcal{P}^*$ the points $(e, c) = \varphi_1(\theta)$ that satisfy $R^i(\theta) = 0$ for some $\theta \in \Theta^*$. Write $\mathcal{P}^{**}$ for the resulting open subset of $\mathcal{P}$. By construction, $\mathcal{P}^{**}$ differs from $\mathcal{P}$ by a closed set of measure zero, and $s^i(c_2^i)$ and $s^i(c_3^i)$ never coincide for any i on $\mathcal{P}^{**}$.

Step 4. We are now ready to complete the proof of Proposition 3. A convenient way to describe the set of efficient equilibrium allocations defined by (7)–(8) and (12) is obtained by eliminating the Pareto weights and shadow prices. This gives:

$$\begin{bmatrix} \mathrm{D}u^i(c_2^i) - \phi \mathrm{D}u^i(c_1^i) \\ \mathrm{D}u^i(c_3^i) - \psi \mathrm{D}u^i(c_2^i) \\ \mathrm{D}^2u^i(c_3^i) - \xi \mathrm{D}^2u^i(c_2^i) \end{bmatrix} = 0 \tag{15}$$

for all i, and some $(\phi, \psi, \xi) \in \mathbb{R}^3_{++}$, together with the feasibility conditions:

$$e_t - \sum_{i=1}^{I} c_t^i = 0 \tag{16}$$

for all t. Given a vector of aggregate endowments e, we have to solve for the consumption allocation c and (ϕ, ψ, ξ). Note that (15)–(16) is a system of $3(I+1)$ equations and $3(I+1)$ unknowns. Differentiating the left-hand side of (15) with respect to (c^i, ϕ, ψ, ξ) and scaling the t^{th} row of the derivative by $\mathrm{D}u^i(c_t^i)$ yields:

$$\begin{bmatrix} A^i & B \end{bmatrix} = \begin{bmatrix} -\frac{\mathrm{D}^2u^i(c_1^i)}{\mathrm{D}u^i(c_1^i)} & \frac{\mathrm{D}^2u^i(c_2^i)}{\mathrm{D}u^i(c_2^i)} & 0 & -\phi^{-1} & 0 & 0 \\ 0 & -\frac{\mathrm{D}^2u^i(c_2^i)}{\mathrm{D}u^i(c_2^i)} & \frac{\mathrm{D}^2u^i(c_3^i)}{\mathrm{D}u^i(c_3^i)} & 0 & -\psi^{-1} & 0 \\ 0 & -\frac{\mathrm{D}^3u^i(c_2^i)}{\mathrm{D}^2u^i(c_2^i)} & \frac{\mathrm{D}^3u^i(c_3^i)}{\mathrm{D}^2u^i(c_3^i)} & 0 & 0 & -\xi^{-1} \end{bmatrix},$$

with obvious notation. The derivative of the left-hand side of (15) and (16) therefore has the same rank as:

$$\begin{bmatrix} 0 & A^1 & 0 & \cdots & 0 & B \\ 0 & 0 & \ddots & 0 & \vdots & \vdots \\ \vdots & \vdots & \ddots & A^{I-1} & 0 & B \\ 0 & 0 & \cdots & 0 & A^I & B \\ I_3 & -I_3 & \cdots & -I_3 & -I_3 & 0 \end{bmatrix}. \tag{17}$$

Suppose now that we restrict attention to aggregate endowments and allocations in:

$$W = \mathbb{R}^{3(I+1)}_{++} \backslash (\mathcal{P} \backslash \mathcal{P}^{**}).$$

W is an open subset of $\mathbb{R}^{3(I+1)}_{++}$ that differs from $\mathbb{R}^{3(I+1)}_{++}$ by a closed set of measure zero. The determinant of A^i is given by:

$$\frac{\mathrm{D}^2 u^i(c^i_1)}{\mathrm{D}u^i(c^i_1)} \frac{\mathrm{D}^2 u^i(c^i_2)}{\mathrm{D}u^i(c^i_2)} \frac{\mathrm{D}^2 u^i(c^i_3)}{\mathrm{D}u^i(c^i_3)} \left[s^i(c^i_3) - s^i(c^i_2) \right].$$

The strict concavity of u^i implies that this is zero if and only if $s^i(c^i_3) = s^i(c^i_2)$. But this cannot happen on W for any i. Thus all the A^i are non-singular on W, implying that (17) has full rank. Therefore zero is a regular value of the map defined by the left-hand sides of (15)–(16). The Transversality Theorem implies that for generic endowments e, efficient equilibrium allocations c are isolated. This proves Proposition 3.

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