Approximation of Nash Equilibria in Bayesian Games

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Abstract

We define a new concept of Constrained Strategic Equilibrium (CSE) for Bayesian games. We show that a sequence of CSEs approximates an equilibrium under standard conditions. We also provide an algorithm to implement the CSE approximation method numerically in a broad class of Bayesian games, including games without analytically tractable solutions. Finally, we illustrate the flexibility of the CSE approximation with a series of auction examples, including a complex multi-unit auction.

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1. Introduction

Nash equilibrium (hereafter NE) is the most widely accepted game theoretic solution concept in economics. It is not only used by theorists to solve non-cooperative games, but it also provides practitioners with a benchmark to explain behavior observed in the field. As we shall see however, the extension of the NE concept to Bayesian games is often analytically intractable. In the present paper, we propose an alternative equilibrium concept that enables one to approximate an analytically intractable NE in a broad class of Bayesian games.

The use of numerical techniques to solve increasingly complex models (e.g. dynamic or general equilibrium models) has now become widespread in economics. These techniques have not only been employed in empirical applications, but they have also been used both as complement and substitute to economic theory.[1] Likewise, numerical approximations have been adopted in the analysis of Bayesian games, and most notably, auctions. Indeed, as discussed later, auction models typically do not possess a closed form solution, except under simplifying but often empirically questionable assumptions.[2] Numerical methods have been used in the analysis of auctions i) to gain insights about the properties of an equilibrium strategy, ii) to estimate complex structural models, and iii) to conduct counterfactual analyses based on structurally estimated parameters.[3] Essentially, two approaches have been proposed to approximate NEs in auction games. The first consists in discretizing the action space (e.g. Athey 2001), while the second consists in solving the set of differential equations generated by the first order conditions of the problem (e.g. Marshall et al. 1994, Li and Riley 1999, Bajari 2001).[4] In the present paper, we propose a new method consisting in finding a solution in

[1] See the numerous examples in Amman, Kendrick and Rust (1996), Judd (1998), Miranda and Fackler (2002), and Tesfatsion and Judd (2006). See also Judd (1997).

[2] The class of Bayesian games with intractable NEs extends well beyond auctions. It includes in particular, Cournot and Bertrand oligopolies with incomplete information on cost and/or demand, non-linear pricing models, search models, principal-agents models, and noisy signaling models.

[3] For examples of i) see e.g. Engelbrecht-Wiggans and Kahn (1998), Fibich and Gavious (2003), or Gallien and Gupta (2007). For examples of ii) see e.g. Bajari and Hortaçsu (2003), Eklöf (2005), or Marshall et al. (2006). For examples of iii) see e.g. Eklöf (2005), Krasnokutskaya and Seim (2006), or Armantier and Sbaï (2007).

[4] To the best of our knowledge, the first approach has only been adopted in practice as a mean to prove the existence of an equilibrium (e.g. Athey 2001, McAdams 2003). In the remainder of the paper, we will therefore concentrate on the second approach when comparing numerical techniques.

a simplified strategy space.[5] As we shall see, two of the most notable advantages of this approach, are that i) beyond auctions, it is applicable to a broad class of Bayesian games, and ii) it enables one to solve even the most complex models.

We define a Constrained Strategic Equilibrium (hereafter CSE) as a NE of a modified game in which strategies are constrained to belong to an appropriate subset typically indexed by an auxiliary parameter vector. We show that any sequence of CSEs has a subsequence that converges toward a NE when the strategy space is compact. The compacity condition is standard in the Bayesian games literature, and it is satisfied in the class of models for which existence of an equilibrium has been established. The CSE approximation method is therefore relevant in a large number for games of interest to economists.

We also show how the approximation principle may be implemented in practical applications. In particular, the parametrization of the constrained strategies enables one to calculate the CSEs numerically in a broad class of Bayesian games, including complex games without NE in closed form. In addition, we develop several criteria to evaluate in practice the quality of the CSE approximation. Beyond the traditional measures used in numerical analysis, we propose two original criteria with game theoretic interpretation. The first criterion compares the CSE to its unconstrained best-response, either in terms of distance or payoffs. In the second criterion, the CSE is reinterpreted as a NE in a slightly perturbed game. The approximation quality is then evaluated by a measure of the distance between the original and the perturbed games.

Although the CSE approximation may be applied to most Bayesian games, we illustrate its advantages with a series of private-values auction examples. In the first example, an independent private-values auction, the NE may be calculated analytically. Therefore, we can illustrate the accuracy of the method by comparing the CSE approximation with the actual NE. In the second example, an asymmetric first-price auction, the NE cannot be expressed in closed form, but it may be approximated by other numerical techniques developed specifically for such a situation. This asymmetric example therefore gives us the opportunity to compare the CSE approach to existing approximation methods. Finally, we consider a complex multi-unit auction model which, to the best of our knowledge, has never been solved, either analytically or numerically. This last example therefore illustrates how the CSE approximation provides a new tool to analyze games with considerable, but yet still poorly understood, economic implications.

[5] In a sense, our approach could be considered as a generalization of Athey (2001). A notable difference, however, is that we constrain the strategy space, not the action space.

We believe our paper contributes to the existing literature in at least three ways. First, we define formally the CSE concept, and we propose sufficient conditions under which a sequence of CSEs converges toward a NE (Section 2). Second, we propose an algorithm to calculate the CSE approximation and evaluate its precision (Section 3). Third, we illustrate the practical relevance of the CSE approach with a series of auction examples, including a complex multi-unit model that cannot be solved with existing numerical techniques (Section 4).

2. Approximation with Constrained Strategic Equilibria

Before we introduce the model, let us briefly summarize the basic notations used throughout the paper. Sets are denoted with capital letters, while a generic element of a set is denoted with the same lower case letter (e.g. $t \in T$). The subscript i denotes a specific player i, while the subscript $-i$ refers to the set of all players except i (e.g. $t_{-i} = (t_1, .., t_{i-1}, t_{i+1}, .., t_N)$). Unless mentioned otherwise, a letter without subscript represents the vector or the Cartesian product across all players of the corresponding individual variable (e.g. $t = (t_1, .., t_N)$ or $T = \mathbf{\Pi}_{i=1}^{N} T_i$). Finally, the set $\{1, ..., N\}$ is denoted by N.

2.1. The Model and the Constrained Strategic Equilibrium

We consider a single play of a N-person simultaneous move game.[6] Each player is endowed with a privately known "type" t_i, with $T_i \subset \mathbb{R}^p$ compact. The vector of types t is drawn from a joint distribution with cumulative distribution function (hereafter c.d.f.) $F(t)$.[7] Player i selects an action a_i, where $A_i \subset \mathbb{R}^{p'}$ denotes the set of possible actions that player i can take. Players are endowed with individual Von Neuman-Morgenstern utility functions $U_i(a; t)$. A strategy profile s consists in N measurable functions transforming signals into actions ($a_i = s_i(t_i)$, $\forall i \in N$), and it is said to be feasible if $U(s(t); t)$ is integrable with respect to F.[8] We assume in the remainder that $S = \mathbf{\Pi}_{i=1}^{N} S_i$ is a subset of all feasible strategy profiles. Finally, $\{N, F, U, S\}$ is assumed to be common knowledge.[9]

[6] The methodology developed below extends naturally to a broader class of games including sequential moves or repeated games.

[7] This general framework includes as special cases of interest i.i.d., exchangeable (or affiliated), asymmetrically distributed, and multi-dimensional types.

[8] Although our approach generalizes to mixed-strategies, this paper concentrates exclusively on pure strategies.

[9] In standard symmetric games, $(S_i, T_i) = (S_j, T_j)$ $\forall i, j \in N$, and (U, F) are exchangeable.

Following (e.g.) Reny and Zamir (2004), we adopt the ex-ante formulation of the Bayesian game. The set of NEs ($S^* \subset S$) is then composed of strategy profiles s^* such that

$$\widetilde{U}_i\left(s_i^*, s_{-i}^*\right) \geq \widetilde{U}_i\left(s_i, s_{-i}^*\right)\ \ \forall s_i \in S_i \text{ and } \forall i \in N\ , \tag{2.1}$$

where $\widetilde{U}_i(s) = E_t\left[U_i(s(t); t)\right]$ is the expected utility of player i.[10] In practice, a NE is typically calculated under the interim formulation of the Bayesian game as a solution of the following fixed point problem,

$$s_i^*(t_i) \in \underset{a_i \in A_i}{ArgMax}\ \widehat{U}_i\left(a_i, s_{-i}^*; t_i\right)\ \ \forall t_i \in T_i \text{ and } \forall i \in N\ , \tag{2.2}$$

where $\widehat{U}_i(s; t_i) = E_{t_{-i}|t_i}\left[U_i(s_i(t_i), s_{-i}(t_{-i}); t_i, t_{-i})\right]$ is the expected utility of player i conditional on his type t_i.[11] When possible, the corresponding First Order Conditions (hereafter FOC) are reformulated for each $i \in N$ as

$$B_i[s^*](t_i) = 0, \text{ where } B_i[s](t_i) = \frac{d}{da_i}\widehat{U}_i(a_i, s_{-i}; t_i)_{|a_i = s_i(t_i)}\ \ \forall t_i \in T_i\ , \tag{2.3}$$

which typically produces a set of differential equations characterizing the solution. Except under fairly restrictive assumptions whose empirical validity is often questionable (e.g. symmetry, independence, risk neutrality, linearity of the demand or cost functions), it is in general impossible to solve (2.2) analytically. Numerical methods have been proposed in the specific context of the first-price single-unit asymmetric auction (e.g. Marshall et al. 1994, Li and Riley 1999, Bajari 2001). These methods consist essentially in finding a solution to the set of differential equations produced by the FOC (2.3). In many complex games however, such as the multi-unit auction example presented in Section 4, these methods are not applicable because the FOC (2.3) cannot be expressed in closed form, or they are too complex to be solved numerically.

We now consider an alternative equilibrium concept that enables the approximation of NEs even in the most complex games. The definition of a CSE parallels

[10] There is now an extensive literature providing sufficient conditions under which a NE exists (i.e. $S^* \neq \varnothing$) in a variety of Bayesian games. See e.g. Milgrom and Roberts (1990), Vives (1990), Lebrun (1996), Reny (1999), Maskin and Riley (2000), Athey (2001), Reny and Zamir (2004), McAdams (2006), Van Zandt and Vives (2007), and Reny (2007).

[11] As demonstrated by (e.g.) Schlaifer (1959), the ex-ante and interim formulations of a Bayesian game yield the same set of NEs under standard assumptions. This is the case in particular when S_i consists of non strictly dominated strategies. For more general conditions see Proposition (2.3) in Armantier, Florens and Richard (2004).

that of a NE in the ex-ante game, except that the strategies are now restricted to constrained sets $S_i^k \subset S_i$. More formally, $S^{k*} \subset S^k$, the set of CSEs in S^k, is composed of strategy profiles s^{k*} such that

$$\widetilde{U}_i\left(s_i^{k*}, s_{-i}^{k*}\right) \geq \widetilde{U}_i\left(s_i^k, s_{-i}^{k*}\right) \;\; \forall s_i^k \in S_i^k \text{ and } \forall i \in N \; . \tag{2.4}$$

The CSEs may also be expressed as a fixed point solution of the constrained best-response correspondence

$$s_i^{k*} \in \underset{s_i^k \in S_i^k}{ArgMax} \; \widetilde{U}_i\left(s_i^k, s_{-i}^{k*}\right) \quad \forall i \in N \; . \tag{2.5}$$

As we shall see in the next section, the determination of this fixed point is greatly simplified under a parametrization of the strategies in S_i^k. As a result, a CSE may be determined in the most complex Bayesian games, even when the FOC (2.3) characterizing the NEs do not have an explicit expression.

2.2. Approximation of Nash Equilibria

We now identify conditions under which a sequence of CSEs converges toward a NE. To this end, we assume in the remainder that S is endowed with an appropriate topology, and we consider a family of constrained sets $\left\{S^k\right\}_{k=1\to\infty}$ such that i) $S^k \subset S^{k+1}$ $\forall k > 0$, and ii) $\underset{k\geq 1}{\cup} S^k$ is dense in S.[12]

Proposition 2.1. *If $\widetilde{U}$ is continuous and if a sequence of CSEs $\left\{s^{k*}\right\}_{k=1\to\infty}$ has a subsequence with limit $\overline{s} \in S$, then $\overline{s} \in S^*$.*

Proof: see Appendix 1.

Corollary 2.2. *If S is compact, $\widetilde{U}$ is continuous and there exists a CSE s^{k*} $\forall k > 0$, then there exists a NE in S, and any sequence of CSEs $\left\{s^{k*}\right\}_{k=1\to\infty}$ has a subsequence that converges toward a NE.*

[12] Observe that if a NE s^* belongs to a constrained set S^k, then it also belongs to the set of CSEs $S^{k'*}$, $\forall k' \geq k$. The traditional independent private-values auction with types uniformly distributed on $[0,1]$ provides an example of such a situation. Indeed, the unique NE $s^*(t) = t/(N+1)$ belongs to any constrained set of polynomial or piecewise linear strategies for $k \geq 1$. As a result, s^* is also a CSE in any of these constrained sets.

Proof: see Appendix 2.

The compacity of the strategy space is standard in Bayesian games. In particular, this condition is verified in all Bayesian games for which existence has been established (e.g. Lebrun 1996, Reny 1999, Athey 2001, McAdams 2006). As a result, Proposition 2.1 applies to a large class of games including several auction models (e.g. first-price, asymmetric, all-pay), different forms of Cournot and Bertrand oligopolies with incomplete information on cost and/or demand, noisy signaling games or search models with incomplete information, and some models with multi-dimensional types and/or actions (e.g. multi-markets oligopoly competition, and multi-units auctions).[13] In practical applications of these games analytical tractability is often obtained at the expense of more realistic assumptions. By offering the possibility to approximate with arbitrary precision intractable NEs, the CSE approximation technique therefore enables one to analyze these models under richer and empirically more relevant assumptions.

As an example of a general strategy space for which the compacity condition is satisfied, consider the set of functions of bounded variation.[14] To keep the notations from obscuring the point, consider a symmetric Bayesian game in which $T_i = [0,1]$ and $A_i = \mathbb{R}$.[15] Let us denote $V(s) = \sup_{0=t_1<....<t_{J+1}=1} \sum_{j=1}^{J} |s(t_j) - s(t_{j+1})|$, the total variation of a function s, and consider the BV norm $\|s\|_{BV} = |s(0)| + V(s)$. Then, $\overline{S}_v$, the closure of $S_v = \{s \mid \|s\|_{BV} \leq v\}$ is compact under the norm L_1 for any $v \in \mathbb{R}$.[16] This set of functions with uniformly bounded variation includes most well defined bounded functions such as the continuous monotonic functions over $[0,1]$, the bounded functions with a countable number of discontinuity points, or the differentiable functions with bounded first derivative. In other words, $\overline{S}_v$ includes the set of monotonic bounded strategies considered in many Bayesian

[13] Observe that several of the games mentioned (e.g. the first price auction) are often considered "discontinuous", as the utility function U_i is not continuous. Nevertheless, Proposition 2.1 may be applied to those games as long as the (unconditional) expected utility function $\widetilde{U}$ is continuous.

[14] In Armantier et al. (2004) we propose an even more general strategy space endowed with the Sobolev norm for which we show that the compacity condition is satisfied. For compactness criteria in different functional spaces see also Dörfler, Feichtinger, and Gröchenig (2002).

[15] To simplify, we omit the player's subscript in the remainder of this section.

[16] A proof of this statement is given by (e.g.) Ziemer (1989) in Corollary 5.3.4 (p 227). Note also that the Rellich-Kondrachov compact embedding theorem generalizes this results to the case $T_i \subset \mathbb{R}^p$ and $A_i \subset \mathbb{R}^{p'}$ (see Theorem 2.5.1, p 62 in Ziemer 1989).

games, such as auctions, non-linear pricing models, or Cournot oligopolies.

Finally, we propose a family of constrained strategies which is dense in $\overline{S}_v$. For a given $\alpha \in \mathbb{N}$, consider $\overline{S}_v^k$, the constrained set of piecewise polynomial strategies $s^k \in \overline{S}_v$ of highest degree α defined on a partition Ξ^k of $[0,1]$ in k disjoint intervals Δ_j $(j = 1, .., k)$.[17]

Proposition 2.3. $\underset{k\geq 1}{\cup}\overline{S}_v^k$ *is dense in* $\overline{S}_v$ *with respect to* L_1.

Proof: see Appendix 3.

3. Numerical Implementation

Although the optimal implementation of the CSE approach is context specific, we show in this section how one may evaluate the CSEs in a wide range of games, including games without closed form NE. We also develop a set of criterion to evaluate the approximation quality in practical applications.

3.1. Numerical Determination of the CSEs

To start, consider a family of parametrized constrained strategies: $s_i^k(t_i) = s_i\left(d_i^k, t_i\right) \in S_i^k$, with $d_i^k \in D_i^k \subset \mathbb{R}^{\gamma(k)}$ where $\gamma(k)$ is a function of final dimension. This parametrization provides a major computational advantage as the determination of a CSE reduces to finding $d^{k*} \in D^k$ solving the system of non-linear equations

$$\frac{\partial}{\partial d_i^k}\widetilde{U}_i\left(d_i^k, d_{-i}^k\right) = 0 \qquad \forall i \in N \ . \tag{3.1}$$

The expected utility functions in (3.1) are often difficult (if not impossible) to express analytically even when $s_i\left(d_i^k, t_i\right)$ has a simple functional form. In other words, the system of non-linear equations (3.1) must typically be solved numerically. We differentiate two cases.

Continuous Games: when $U_i\left(d^k; t\right)$ is C_1 in d_i^k, we can approximate the integrals in (3.1) with standard Monte Carlo techniques. For instance, one can simply replace the expected utility by its empirical analog $\widetilde{U}_i^M\left(d^k\right) = \frac{1}{M}\sum_{m=1}^{M} U_i\left(d^k; \widetilde{t}_m\right)$

[17] To guarantee that $\overline{S}_v^k \subset \overline{S}_v^{k+1}$ we implicitly assume that Ξ^k is a thinner partition than $\Xi^{k'}$ $\forall k > k'$.

where $\widetilde{t}_m$ denotes hereafter a vector of N random types generated from F, and M is the size of the Monte Carlo approximation.[18] In practice, one may reduce considerably the computational burden by selecting a family of constrained strategies such that $\frac{\partial}{\partial d_i^k} U_i\left(d^k; t\right)$ may be expressed analytically.

Discontinuous Games : Consider the class of games in which actions may be ranked according to a scoring rule $\nu\left(a_i\right) \in \mathbb{R}$, and the highest score wins and takes all.[19] Such games include auctions, Bertrand oligopolies, or patent race models. The utility functions may then be written

$$U_i\left(d^k; t\right) = V_i\left(d^k, t\right) \mathbb{I}_{\left\{\nu\left(s_i\left(d_i^k, t_i\right)\right) \geq \overline{\nu}_{-i}\right\}} \,, \tag{3.2}$$

where $\overline{\nu}_{-i} = \underset{j \neq i}{Max}\ \nu\left(s_j\left(d_j^k, t_j\right)\right)$, $\mathbb{I}$ is the indicator function, and V_i is the utility function of player i when she wins the game. A general approach to evaluate the CSE in discontinuous games consists in approximating the expected utility with

$$\widetilde{U}_i^M\left(d^k\right) = \frac{1}{M} \sum_{m=1}^{M} V_i\left(d^k, \widetilde{t}_m\right) G_i\left[\nu\left(s_i\left(d_i^k, \widetilde{t}_{i,m}\right)\right)\right] \,, \tag{3.3}$$

where G_i, the c.d.f. of $\overline{\nu}_{-i}$, represents the probability that player i wins the game. In most applications G_i cannot be calculated analytically and needs to be approximated by simulations. For instance, G_i may be replaced by a nonparametric approximation of the form

$$\widehat{G}_i\left[\nu\left(a_i\right)\right] = \frac{1}{\boldsymbol{M}} \sum_{m=1}^{\boldsymbol{M}} K\left[\frac{\nu\left(a_i\right) - \overline{\nu}_{-i}^m}{h}\right], \tag{3.4}$$

where $\overline{\nu}_{-i}^m = \underset{j \neq i}{Max}\ \nu\left(s_j\left(d_j^k, \widetilde{t}_{j,m}\right)\right)$, K denotes an arbitrary c.d.f., $\boldsymbol{M}$ is the Monte Carlo size, and h is a "bandwidth" controlling the smoothness of the kernel. Horowitz (1992) shows that when the derivative of K is a second order kernel, and $h \propto \boldsymbol{M}^{-\frac{1}{5}}$, one can make $\widehat{G}_i$ arbitrarily close to G_i by selecting $\boldsymbol{M}$ sufficiently large. Therefore, an accurate approximation of a NE may be systematically

[18] Note that acceleration techniques such as Quasi Monte Carlo methods should typically be used to speed-up computation (see Press, Flannery, Teukolsky and Vetterling 1992).

[19] In some discontinuous games the losers receive a compensation, or an outside option. The payoff allocated to the losers is normalized here to zero.

achieved since we fully control the Monte Carlo sizes M and $\boldsymbol{M}$.[20]

The computational burden may be greatly reduced when the scoring rule and the strategies are monotonic. Indeed, the system of FOC in (3.1) may then be written for all $i \in N$

$$E\left[\frac{\partial}{\partial d_i^k} V_i\left(d^k, t\right) \mathbb{I}_{\left\{s_i\left(d_i^k, t_i\right) \geq \overline{s}_{-i}\right\}}\right] + E_{t_{-i}|t_i}\left[\frac{\frac{\partial}{\partial d_i^k} s_i\left(d_i^k, t_i\right)}{\frac{\partial}{\partial t_i} s_i\left(d_i^k, t_i\right)} V_i\left(d^k, t\right) f_i\left(t_i\right)\right]_{|t_i = s_i^{-1}\left(d_i^k, \overline{s}_{-i}\right)} \tag{3.5}$$

where $\overline{s}_{-i} = \underset{j \neq i}{ArgMax}\ \nu\left(s_j\left(d_j^k, t_j\right)\right)$, $E_{t_{-i}|t_i}$ is the expectation with respect to t_{-i} conditional on t_i, and f_i is the marginal distribution of t_i.[21] Once again, the expectations in (3.5) may be approximated with arbitrary precision by simulations, and the computational burden may be reduced considerably by choosing an appropriate family of constrained strategies.[22]

3.2. Approximation Criteria

In this section, we propose several criteria to evaluate how well a CSE in S^k approximates a NE. Our objective is not to study the exact theoretical properties of each criterion. Instead, we intend to provide practitioners with a set of tools that will allow them to decide in practical applications when to stop the approximation procedure consisting in calculating CSEs in expanding constrained sets S^k. These

[20] In the nonparametric estimation of an econometric model, $\boldsymbol{M}$ cannot exceed the size of the sample available. Accurate estimates are therefore not necessarily guaranteed in small samples, even when an optimal bandwidth is selected. This issue is not relevant here since we can select an arbitrary large value for $\boldsymbol{M}$, and therefore, reach any precision level desired. Note, that this remains true when the CSE approximation technique is used to estimate a structural model without an analytically tractable NE (see e.g. Eklöf 2005, or Armantier and Sbaï 2006). Indeed, the size of the Monte Carlo simulation $\boldsymbol{M}$, remains independent of the size of the sample used to estimate the structural model.

[21] The result is a direct application of the formulas: $\frac{\partial}{\partial x} \int_{g_1(x)}^{\overline{x}} \varphi(x,y)\, dy = -g_1'(x)\, \varphi(x, g_1(x)) + \int_{g_1(x)}^{\overline{x}} \frac{\partial}{\partial x} \varphi(x,y)\, dy$ and $\frac{\partial}{\partial x} \varphi^{-1}(x,z) = -\frac{\frac{\partial}{\partial x}\varphi(x,y)}{\frac{\partial}{\partial y}\varphi(x,y)}$, where $\varphi(x,y) = z$.

[22] In particular, the computational burden is reduced considerably when the constrained strategies are easily invertible, and when $\frac{\frac{\partial}{\partial d_i^k} s_i\left(d_i^k, t_i\right)}{\frac{\partial}{\partial t_i} s_i\left(d_i^k, t_i\right)}$ can be expressed analytically.

approximation criteria therefore complement the algorithm just presented to form a fully implementable numerical approximation technique.

In numerical analysis, an approximation procedure typically stops when the approximate solution or the objective function does not change significantly from an iteration to the next. In the present context, these stopping rules are defined as

$$C_1(k) = \left\| s^{k*} - s^{k-1*} \right\| \quad \text{and} \quad C_2(k) = \left\| B\left[s^{k*} \right] \right\| \ , \tag{3.6}$$

where B is the FOC operator defined in (2.3). Note that $C_1(k)$ is directly related to Proposition 2.1 stating that when a sequence of CSEs converges, its limit is a NE. We now propose two additional classes of criteria with game theoretic interpretation. Although not implementable in all games, these criteria are particularly relevant as they represent natural measures from a game theoretic perspective of the quality of a NE approximation.

3.2.1. The Unconstrained Best-Response to a CSE

To keep the notations from obscuring the point, suppose that the game under consideration is such that the unconstrained best-response operator is a continuous function.[23] Let us also denote $BR(s)$ the vector of dimension N, whose i^{th} component represents player i ex-ante unconstrained best-response strategy in S_i, when her opponents play s_{-i}: $BR_i(s_{-i}) = Arg \max_{s_i \in S_i} \widetilde{U}(s_i, s_{-i})$. The approximation quality may then be evaluated by $C_3(k) = \left\| s^{k*} - BR\left(s^{k*}\right) \right\|$, the distance between the CSE and its unconstrained best-response in S.[24] Moreover, $\Delta\widetilde{U}_i^k = \widetilde{U}_i\left(BR_i\left(s_{-i}^{k*}\right), s_{-i}^{k*}\right) - \widetilde{U}_i\left(s^{k*}\right)$ is the highest benefit player i can expect when she deviates from her CSE strategy s_i^{k*}. An alternative measure of the CSE approximation quality is then given by $C_4(k) = \left\| \Delta\widetilde{U}^k \right\|$.[25]

In most games, the ex-ante best-response does not have an explicit solution. Nevertheless, the criteria $C_3(k)$ and $C_4(k)$ may be approximated in general situ-

[23] The criteria proposed below have been generalized in Armantier et al. (2004) to the case of multiple best-responses. In this case, the method consists in measuring the distance between the CSE strategies and the corresponding sets of unconstrained best-responses.

[24] Note that unconstrained best-responses are considerably easier to calculate than actual NEs. Indeed, a player's unconstrained best-response is characterized by an independent maximization problem since the strategies of the other players are known. NEs on the other hand, require solving a multidimensional fixed point problem involving N maximizations.

[25] It is trivial to generalize this criterion to measure the gains from joint deviations by subsets of players.

ations by:

$$C_3^M(k) = \frac{1}{NM}\sum_{i=1}^{N}\sum_{m=1}^{M}\left[s_i^{k*}\left(\widetilde{t}_{i,m}\right) - a_{i,m}^*\right]^2 , \tag{3.7}$$

$$C_4^M(k) = \frac{1}{NM}\sum_{i=1}^{N}\sum_{m=1}^{M}\left[U_i\left(a_{i,m}^*, s_{-i}^{k*}; \widetilde{t}_m\right) - U_i\left(s^{k*}; \widetilde{t}_m\right)\right]^2 ,$$

where the action $a_{i,m}^* = \underset{a_i \in A_i}{ArgMax} \sum_{m' \neq m}^{M} U_i\left(a_i, s_{-i}^{k*}; \widetilde{t}_{i,m}, \widetilde{t}_{-i,m'}\right)$ is player i's interim best-response when she receives the private signal $\widetilde{t}_{i,m}$ and her opponents use the strategy profile s_{-i}^{k*}.[26]

3.2.2. The CSE as a NE of a Neighboring Game

Before presenting the criterion, let us introduce some necessary notations. To simplify, let us denote $\Gamma(F)$, the Bayesian game characterized by $\{N, F, U, S\}$. Let $\mathbb{F}$ be the set of c.d.f. with common support T. Let us also generalize the notation $S^*(F)$ to denote the set of NEs in the game $\Gamma(F)$. The inverse correspondence $S^{*^{-1}}$ is then such that s^* is a NE in the game $\Gamma(F_{s^*})$ for any c.d.f. $F_{s^*} \in S^{*^{-1}}(s^*)$. Finally, let us assume that the game under consideration is such that $S^*(F)$ is a continuous function with respect to a given topology (e.g. L^p or Sobolev).[27] Note that the continuity assumption may be interpreted as a stability condition imposed on the game in the sense that slight perturbations of the types' distribution generate neighboring NEs.[28]

We can now characterize the last approximation criterion. Consider a game $\Gamma(F)$ where $F \in \mathbb{F}$, and s^{k*} a CSE in S^k. An alternative measure of the distance between the CSE and a NE is given by

$$C_5(k) = \|F - F_{s^{k*}}\| \quad \text{where} \quad F_{s^{k*}} = S^{*^{-1}}\left(s^{k*}\right) . \tag{3.8}$$

In other words, a CSE may be considered a good approximation of a NE when $C_5(k)$ is close to zero, since in that case s^{k*} is not only a CSE in $\Gamma(F)$, but it is also a NE in a virtually identical game $\Gamma(F_{s^{k*}})$.

[26] Since s_{-i}^{k*} is given, the determination of $a_{i,m}^*$ is straightforward with any numerical optimizer.

[27] See Armantier et al. (2004) for a generalization of the criterion to the case where $S^*(F)$ is a correspondence.

[28] The continuity of $S^*(F)$ is verified in several Bayesian games. In particular, Lebrun (2002) shows that the equilibrium bid function is continuous in the types's distribution in the symmetric first-price private-values auction. Lebrun (2002) also shows that this result extend to asymmetric first-price private-values auction when the players' types have common support.

The computation of $S^{*^{-1}}$ might be far from trivial in practice. One may however apply standard econometric techniques to approximate $F_{s^{k*}}$ in a given parametric class.[29] For instance, consider a symmetric game and a family of distributions $F(. \mid \theta)$ parametrized by a vector $\theta \in \Theta \subset \mathbb{R}^q$. Let us denote θ_0 the true parameter characterizing the game's distribution $F(. \mid \theta_0)$, and $s^*_{\theta_0}$ the corresponding NE we wish to approximate. The FOC characterizing a NE may be used as moment conditions to characterize the parameter θ_1 that makes the CSE s^{k*} a NE in the game with distribution $F(. \mid \theta_1)$. Indeed, observe that the FOC in (2.3) are verified for any $t_i \in T_i$. Therefore, we can write the moment conditions $E_{\theta_1}\left[B\left[s^{k*}\right](t_i)\right] = 0$, where E_{θ_1} denotes the unconditional expectation when the types' distribution is set to $F(. \mid \theta_1)$.[30] The method of simulated moments estimator of θ_1 is based upon the empirical counterpart of this orthogonality condition:

$$\widehat{\theta}_1 = Arg\min_{\theta\in\Theta}\ [C'\Omega C] \text{ where } C = \sum_{m=1}^{M}\sum_{i=1}^{N} B_i\left[s^{k*}\right]\left(\widetilde{t}^{\theta}_{i,m}\right)\ , \tag{3.9}$$

where $\widetilde{t}^{\theta}_m$ is a vector of N types simulated from the distribution $F(. \mid \theta)$, and Ω is a symmetric definite positive matrix that may be chosen in order to improve the quality of the estimation. The criterion $C_5(k)$ may then be replaced by a similar measure $C_5^M(k) = \left\|\theta_0 - \widehat{\theta}_1\right\|$.

4. Examples

The object of this section is to illustrate the relevance of the CSE approximation method through different examples. To do so, we consider an auction model proposed by Reny (1999). There are four risk-neutral bidders competing for L units of a homogenous good. Bidder i receives a bi-dimensional type $t_i = (t_{i,1}, t_{i,2})$ according to a continuous distribution function F_i with support $[0,1]^2$. Each bidder is assumed to have strictly decreasing marginal values for the successive units. More specifically, bidder i marginal valuation for the l^{th} unit of the good

[29] It is important to understand that the objective here is not to estimate the unknown structural parameters of the game (as in e.g. Guerre, Perrigne and Vuong 2000), but rather to apply econometric techniques to address a game theoretic problem. Note in particular that no sample of observations is involved in the calculation of the criterion, as it relies only on simulated types.

[30] To simplify, we assume here that $q < N$ and the N FOC are not redundant. Otherwise, to generate the appropriate number of moment conditions, one could nonlinearly transform the FOC (which are verified for any $t_i \in T_i$), or interact them with an appropriate set of instruments.

is $t_{i,1} \cdot (t_{i,2})^{l-1}$.[31] The type $t_{i,1}$ may be interpreted as the value allocated by bidder i to the first unit of the good, while $t_{i,2}$ represents a discount factor applied to additional units. Knowing only his own type, bidder i submits a nonnegative sequence of bids $s_i(t_i) = (s_{i,1}, .., s_{i,L})$, verifying $s_{i,1} \geq ... \geq s_{i,L}$. The L highest bids among the LN submitted are winning bids, with ties broken equiprobably. The auction belongs to the class of discriminatory payment mechanisms, as winning bidders pay the seller their winning bid for each unit won.

Reny (1999) proves that this Bayesian game possesses a NE in pure and non-decreasing strategy. The set of admissible strategy profiles S may then be restricted to the non-decreasing strategies defined over the support $[0,1]^2$ and verifying $s(0) = 0$. As explained in Section 2, S is then compact, and any strategy profile in S may be approximated with arbitrary precision by a piecewise linear function. In other words, we know that any converging sequence of piecewise linear CSEs we can construct, converges toward a NE of the game.[32]

4.1. First-Price Independent Private-Values Model

Consider first the case $L = 1$, and let us assume that the private-values $t_i = t_{i,1}$ are scalars independently drawn from a Beta distribution with parameters $(3,3)$. In this traditional first-price independent private-values model, the NE has a well-known closed form solution. We are therefore in a position to compare the actual NE, with its CSE approximation. We consider here a piecewise linear family of constrained strategies of the form:[33]

$$s_i\left(d_i^k, t_i\right) = \sum_{j=1}^{2^{k-1}} \left[d_{i,2j-1}^k + d_{i,2j}^k \cdot t_i\right] \mathbb{I}_{\left\{t_i \in \left[\bar{t}_{j-1}, \bar{t}_j\right]\right\}}, \tag{4.1}$$

[31] Reny's model is more general as bidder i's marginal valuation for the l^{th} unit of the good, denoted in Reny (1999) by $v_i^l(t_i)$, is a continuous and strictly increasing function of a multi-dimensional type t_i , verifying $v_i^1(.) \geq ... \geq v_i^L(.)$, and $v_i^1(0) = ... = v_i^L(0)$. The CSE approximation approach can obviously accommodate different specifications of types and valuations from the one adopted in this section.

[32] In Armantier et al. (2004) we discuss in more details how the conditions necessary for the application of Proposition 2.1 and the convergence criteria are satisfied in each of the examples below.

[33] Although more sophisticated constrained strategies may be considered (e.g. spline or higher piecewise polynomials, neural networks, wavelets or Fourier transforms), we find that simple piecewise linear functions provide accurate approximations, while being computationally less demanding. We also refer the reader to Armantier et al. (2004) where the NE in the examples given in Section 4.1 have been approximated by polynomials.

where $d_{i,1}^k$ is set equal to zero to satisfy the constraint $s_i\left(d_i^k,0\right) = 0$, and $\bar{t}_j$ $\left(j = 0, ..., 2^{k-1}\right)$ are knots corresponding to (e.g.) fractiles of the types' marginal distributions $\left(\text{i.e. } \bar{t}_j = F_i^{-1}\left(\frac{j}{2^{k-1}}\right)\right)$.[34]

The actual NE as well as the different piecewise linear CSEs for k varying from one to five are plotted in Figure 1.[35] This figure illustrates the rapid convergence of the sequence of CSE as k increases. Observe in particular that the approximations for $k \geq 4$ are virtually indistinguishable from the NE. This observation is confirmed by the different approximation criteria reported in Table 1.[36] In particular, the criteria $C_1(k)$ and $C_2(k)$ attest to the rapid convergence of the CSE as k increases. Criteria $C_3(k)$ and $C_4(k)$ indicate that, with k as low as three, there is very little difference between the CSE and its unconstrained best-response, both in terms of distance and expected profit.[37] Finally, criterion $C_5(k)$ confirms that a CSE rapidly becomes a NE in a virtually identical game (i.e. with an almost identical distribution).[38]

[34] Observe that the $2^{k-1}+1$ knots are defined such that $S^k \subset S^{k+1}$. Moreover, note that for a given k, more accurate approximations than those presented here may be obtained with an adaptive algorithm that selects a finer partition of knots where the approximated NE is not smooth, and a coarser mesh where it is smooth (see DeVore 1998). An example of such an adaptive method, as well as other illustrative programs may be found on one of the authors' website at http://www.sceco.umontreal.ca/liste_personnel/armantier/index.htm.

[35] To solve the system of non-linear equations, we rely on Fortran subroutines developed in the field of "Interval Arithmetic" (see e.g. Neumaier 1990 and Kearfott 1996). We refer the reader to the GLOBSOL project website (http://www.mscs.mu.edu/~globsol/) for additional information.

[36] The approximation criteria are calculated here in percentage in order to facilitate the comparison between the different auction examples. Criteria $C_1(k)$ to $C_4(k)$ are calculated by Monte Carlo simulations with the norm L_2. For instance, $C_1(k) = \frac{1}{M}\sum_{m=1}^{M}\left(\frac{s^{k*}\left(\widetilde{t}_m\right) - s^{k+1*}\left(\widetilde{t}_m\right)}{s^{k+1*}\left(\widetilde{t}_m\right)}\right)^2$ where the $\widetilde{t}_m$ are generated with the Common Random Number technique. Finally, $C_5(k) = \left(\frac{\widehat{\alpha}_1-\alpha_0}{\alpha_0}\right)^2 + \left(\frac{\widehat{\beta}_1-\beta_0}{\beta_0}\right)^2$, where $(\alpha_0, \beta_0) = (3,3)$ are the true parameters of the types' distribution, and $\left(\widehat{\alpha}_1, \widehat{\beta}_1\right)$ are the GMM estimates of the parameters of a Beta distribution when one assumes that the CSE in S^k is an actual NE.

[37] In particular, criterion $C_4(k)$ attests to the robustness of the CSE as an equilibrium concept, as players have little incentives to deviate. Indeed, with k as low as 2, a player can only increase her expected profit by 0.10% when she deviates from her piecewise linear CSE strategy to select an unconstrained strategy.

[38] In practice, these five criteria should be used to decide the number of knots that provides a sufficiently accurate approximation. Here for instance, one may find it appropriate to consider $k = 4$, as the approximation quality does not improve substantially with higher values of k.

Since we know the actual NE in this example, we can also calculate the mean square error, as well as the difference in expected profit between the NE and the different CSEs. As indicated at the bottom of Table 1, these two additional criteria confirm that the CSE approach provides accurate approximations.[39] Moreover, we report in Table 2 the expected profit and the probability of winning of a representative bidder when using the NE, or the polynomial CSE with $k = 5$. As expected, the CSE approximation is so accurate, that the economic outcomes of the auction are virtually indistinguishable.

4.2. Asymmetric First-Price Auction

We now turn to a single-unit asymmetric first-price private-values auction. As in the previous example, the first two bidders draw their values from a $Beta\,(3,3)$. Bidders 3 and 4 on the other hand, receive their types from a more favorable distribution, a $Beta\,(5,3)$.[40] This asymmetric auction model does not possess a NE in closed form. As previously mentioned, the FOC in this model produce an explicit set of differential equations that may be solved numerically. This example therefore gives us the opportunity to compare the CSE approach with the differential equations method proposed by e.g. Marshall et al. (1994), Li and Riley (1999), and Bajari (2001).

We plot in Figure 2 the piecewise linear CSE for $k = 5$, as well as the NE approximation obtained with the differential equations method.[41] The shapes of the bid functions indicate that the bidders with the less favorable type distribution must compensate by bidding slightly more aggressively. Although subtle, this difference in behavior has serious economic implications. Indeed, we can see in Table 2 that bidders 3 and 4 are twice as likely to win the auction, and their profits are nearly 2.4 time larger.

Figure 2 also indicates that the CSE and the differential equations methods produce nearly identical results, except possibly for very low types. The approximation quality may be compared more formally in Table 1 where the approximation criteria, although all very close to zero, are slightly larger for the differential

[39] In Table 1, both criteria are expressed in percentage.

[40] The mean and standard deviation of the values assigned to bidders 1 and 2 are respectively 0.5 and approximately 0.19, while the mean and standard deviation of bidders 3 and 4's values are approximately 0.625 and 0.16.

[41] Since the CSE approximations for different values of k are difficult to distinguish, we only plot in the remainder of this section the approximation for $k = 5$.

equations approximation.[42] In other words, the piecewise linear CSEs appear to provide a slightly more accurate approximation in the sense that i) they are closer to their unconstrained best-responses, and ii) they are NEs in slightly less perturbed games.[43] Precision however, is only one of the components that one may take into consideration when selecting an approximation technique. In many empirical applications, computational speed is also a key concern.[44] As indicated in the last column of Table 1, we find that the piecewise linear CSE is nearly 6 time faster than the competing differential equations method.[45] In other words, the CSE approach appears to perform better in this example, not only in terms of accuracy, but also in terms of computational speed.[46]

4.3. Multi-Unit Private-Values Auction

We now consider the case $N = L = 4$. In other words, we have four bidders competing for four units of a homogenous good in a first-price discriminatory private-values auction. We consider here a symmetric case in which the value of the first unit is independently distributed across bidders from a $Beta(3,3)$.[47] The discount factor $t_{i,2}$ is assumed to be independently drawn across bidders

[42] The approximation criteria in Table 1 barely differ across both types of players. Therefore, we only present their average in Table 1.

[43] This result is consistent with Bajari (1999) and Turocy (2001) who suggest that the differential equations method may be prone to inaccuracies for types with low probability of winning. The CSE on the other hand, appears to be less prone to this type of inaccuracies, as it is not calculated for a given type but rather across all types.

[44] For instance, the structural estimation of a treasury auction model in Armantier and Sbaï (2006) required to calculate different CSE approximations for a total of nearly 10^9 different types. Such an estimation is therefore only possible if the approximation technique is sufficiently fast.

[45] The comparison consisted in calculating the approximated asymmetric equilibrium for 10^6 different private-values. Note also that once the CSE has been evaluated, the marginal cost of computing the approximated NE for an additional vector of private types is virtually zero. Indeed, it only requires calculating a simple piecewise linear function.

[46] This result is consistent with similar comparisons conducted in Armantier et al. (2004), as well as Eklöf (2005). As noted by Marshall et al. (1994), the algorithms developed to solve the two points boundary value problem in the differential equations method, typically rely upon slow numerical processes, and often suffer from pathologies at the origin. In contrast, the boundary conditions reduce to trivial constraints on the parameters under the CSE approach. For instance, the constant in the first portion of the piecewise linear constrained strategies is set equal to zero to satisfy $s^k(0) = 0$. Likewise, we can easily parametrize the piecewise strategies to guarantee that all bidders submit the same bid when receiving the highest type $t_{i,1} = 1$.

[47] See Armantier et al. (2004) for an asymmetric example.

from a normal distribution $N(0.75, 0.15)$ truncated on $[0, 1]$.[48] The difficulties in solving such a multi-unit auction model emerge not only from the fact that types and bids are multi-dimensional, but also from the constraint that bids for successive units must be non-increasing. In fact, to the best of our knowledge, this model has not previously been solved in the literature.[49] In particular, the differential equations approach cannot be implemented here, as the FOC of the model cannot be expressed explicitly. The practical relevance of such a multi-unit auction model however, is undeniable. Indeed, it provides the means to analyze some of the auctions with the most considerable, but yet still poorly understood, economic implications such as treasury, electricity or spectrum auctions.[50]

A strategy in this multi-unit example is a function of three variables l, $t_{i,1}$, and $t_{i,2}$, where $l = 1, ..., 4$ represents the successive bids submitted by bidder i for each of the four units for sale. Therefore, to approximate the intractable NE, we generalize the constrained strategy in (4.1) to three dimensional piecewise linear functions in $(l, t_{i,1}, t_{i,2})$.[51] The symmetric bid functions calculated for the mean discount factor $t_{i,2}$ are plotted in Figures 3.[52] This figure provides us with a unique insight into equilibrium behavior in such multi-unit environments. Indeed, Figure 3 indicates that, although the shape of the bid functions are somewhat similar to the single unit case (Figure 1), participants tend to bid less aggressively when given the possibility to win several units. Finally, note that the approximation criteria in Table 1 suggest that the CSE approach still provides an excellent approximation even in this complex multi-unit auction.

5. Discussion

Because it enables the economist to solve rich and empirically relevant models without consideration for analytical tractability, the CSE approach has recently

[48]The distributions of types have not been selected for tractability, but rather to illustrate that the CSE method can accommodate any class of distributions.

[49]Engelbrecht-Wiggans and Kahn (1998) propose an algorithm in the specific case of a symmetric discriminatory auction with only two units, and implement it with uniformly distributed types.

[50]Recent econometric analyses of multi-unit auctions include Hortaçsu (2002), Fevrier, Preguet and Visser (2004), Armantier and Sbaï (2006), and Chapman, McAdams and Paarsch (2006).

[51]The theoretic constraints $s_i^k(l, 0, 0) = 0$, $s_i^k(l, 1, 1) = s_i^k(l', 1, 1)$ and $\partial s_i^k / \partial l \leq 0$ considerably reduce the dimension of the problem.

[52]See Armantier et al. (2004) for additional plots of bid functions calculated for different values of the discount factor $t_{i,2}$.

been adopted in the theoretical and empirical analysis of several complex Bayesian games. In particular, Eklöf (2005) applies the CSE approach to estimate the social cost implied by the inefficient allocation of contracts in first-price sealed-bid procurement auctions with asymmetric bidders. Likewise, Armantier and Sbaï (2006, 2007), and Armantier and Lafhel (2007) apply the CSE approach to study Treasury auction models accounting for supply uncertainty, as well as possible informational and risk aversion asymmetries across bidders.[53] The CSE approach has also generated interest in the field of artificial intelligence. For instance, Arunachalam and Sadeh (2003), and Estelle et al. (2005) use a CSE approximation to study strategic interactions in a supply chain game. Likewise, Reeves and Wellman (2004), as well as Mackie-Mason and Wellman (2006), advocate the use of the CSE approach to solve infinite Bayesian games.

In contrast with the applications just mentioned, the contribution of the present paper is to provide the methodological foundations of the CSE approximation method. More specifically, we defined formally a constrained strategic equilibrium, and we established sufficient conditions under which a sequence of CSEs converges toward a NE. In addition, we provided a fully implementable algorithm to calculate the CSE approximations numerically in a broad class of Bayesian games. Finally, we illustrated the advantages of the CSE approach with different auction examples, including a multi-unit auction which, to the best of our knowledge, has not been previously solved. It has to be noted, however, that the range of applications of the CSE method is not limited to auctions. Indeed, the approximation theorem and the algorithm introduce in the paper are directly applicable in a wide range of Bayesian games relevant to economists (e.g. principal-agent, non-linear pricing, adverse selection or moral hazard models) for which closed form NE may be obtained at best under strong simplifying assumptions.

[53] There has been several additional applications of the CSE approach. For instance, Armantier et al. (1998) analyze a procurement from the French aerospace industry, in which the good for sale is allocated to the player bidding the highest ratio of quality over price. Armantier et al. (2004) approximate a similar model under the additional assumption of collusion, or informational asymmetry. Armantier and Florens (2003) evaluate the welfare implications associated with the use of a complex redistributive allocation mechanism (known as the "Juste Retour") at European Spatial Agency procurements.

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Appendix 1: Proof of Proposition 2.1.

Consider a strategy profile $s \in S$. Since $\underset{k\geq 1}{\cup} S^k$ is dense in S, then there exists $\left\{s^k\right\}_{k=1\to\infty}$ $\left(\text{with } s^k \in S^k\right)$ such that $s^k \overset{k\to\infty}{\to} s$. Then, since $s^{k*} \in S^{k*}$ we have $\forall i \in N$ and $\forall k = 1 \to \infty$

$$\widetilde{U}_i\left(s_i^{k*}, s_{-i}^{k*}\right) \geq \widetilde{U}_i\left(s_i^{k}, s_{-i}^{k*}\right) \ . \tag{6.1}$$

If the sequence $\left\{s^{k*}\right\}_{k=1\to\infty}$ has a subsequence with limit $\overline{s} \in S$, then there exists $\{k_m\}_{m=1\to\infty}$ such that $\left(s_i^{k_m*}, s_{-i}^{k_m*}\right) \overset{m\to\infty}{\to} \left(\overline{s}_i, \overline{s}_{-i}\right)$ and we still have, $\forall i \in N$, and $\forall k = 1 \to \infty$

$$\widetilde{U}_i\left(s_i^{k_m*}, s_{-i}^{k_m*}\right) \geq \widetilde{U}_i\left(s_i^{k_m}, s_{-i}^{k_m*}\right) \ . \tag{6.2}$$

Finally, since $\widetilde{U}_i$ is continuous in s, we can write the previous equation as m tends toward infinity,

$$\widetilde{U}_i\left(\overline{s}_i, \overline{s}_{-i}\right) \geq \widetilde{U}_i\left(s_i, \overline{s}_{-i}\right) \ \forall i \in N \ . \tag{6.3}$$

Therefore $\overline{s} \in S^*$.

Appendix 2: Proof of Corollary 2.2.

The proof is trivial: If S is compact, then any sequence of CSEs $\left\{s^{k*}\right\}_{k=1\to\infty}$ has a subsequence with limit $\overline{s} \in S$, and $\overline{s}$ is a NE from Proposition 2.1.

Appendix 3: Proof of Proposition 2.3.

The proof consists in showing that for any $s \in \overline{S}_v$ and any $\varepsilon > 0$, we can find a scalar k and a function $s^k \in \overline{S}_v^k$ such that $\left\|s - s^k\right\|_{L_1} < \varepsilon$.

For any $s \in \overline{S}_v$ and any $\varepsilon > 0$, let us start by considering a function $\widetilde{s} \in S_v$ satisfying $\left\|s - \widetilde{s}\right\|_{L_1} < \frac{\varepsilon}{2}$. Let us also denote $\widetilde{s}_j$ the restriction of $\widetilde{s}$ to the interval Δ_j, and $\widetilde{s}_j^k$ the projection of $\widetilde{s}_j$ onto $\overline{S}_v^k$ with respect to the BV norm. Finally, let us define s^k as the union of the projections $\widetilde{s}_j^k$ over every interval Δ_j. By property of the norm, we have $\left\|\widetilde{s}_j^k\right\|_{BV} \leq \left\|\widetilde{s}_j\right\|_{BV}$, which implies that $\left\|s^k\right\|_{BV} \leq \left\|\widetilde{s}\right\|_{BV}$, and consequently $s^k \in \overline{S}_v^k$.

Moreover, from Theorem 3.2 in Birman and Solomjak (1967), we know that there exists a constant c such that $\left\|\widetilde{s} - s^k\right\|_{L_1} < c\frac{1}{k}$, and consequently we can find k such that $\left\|\widetilde{s} - s^k\right\|_{L_1} < \frac{\varepsilon}{2}$. Finally, $\left\|s - s^k\right\|_{L_1} \leq \|s - \widetilde{s}\|_{L_1} + \left\|\widetilde{s} - s^k\right\|_{L_1} \leq \varepsilon$, which completes the proof.

Table 1

Approximation Criteria

	Single Unit							Multi Unit
	Symmetric					Asymmetric		
Criteria	k=1	k=2	k=3	k=4	k=5	Differential Equation	CSE (k=5)	CSE (k=5)
$C_1(k)$	—	4.625E-3	6.053E-4	8.398E-5	3.486E-6	—	8.872E-7	4.741E-6
$C_2(k)$	7.119E-2	5.292E-3	4.706E-4	3.531E-5	1.643E-6	9.652E-7	8.607E-8	1.491E-6
$C_3(k)$	1.045E-3	5.585E-5	4.558E-6	2.145E-7	3.533E-8	6.245E-7	9.261E-9	2.889E-8
$C_4(k)$	8.384E-3	9.988E-4	9.683E-5	7.854E-6	5.762E-7	9.622E-8	9.610E-9	1.068E-8
$C_5(k)$	3.133E-3	5.895E-5	8.191E-7	7.450E-8	1.423E-8	3.550E-7	1.699E-9	5.482E-8
MSE	5.794E-3	5.655E-4	6.717E-5	8.204E-6	9.851E-7	—	—	—
$\Delta \tilde{U}$	3.638E-3	6.088E-4	9.668E-5	6.184E-6	7.274E-7	—	—	—
Time	3s	5s	9s	17s	28s	394s	67s	241s

Table 2

Auctions Outcomes

	Single Unit						Multi Unit			
	Symmetric		Asymmetric							
			Differential Equation		CSE (k=5)					
	NE	CSE (k=5)	Player 1 or 2	Player 3 or 4	Player 1 or 2	Player 3 or 4	Unit 1	Unit 2	Unit 3	Unit 4
Probability of Winning	0.250	0.250	0.167	0.333	0.167	0.333	0.540	0.279	0.128	5.826E-2
Expected Profit	3.961E-2	3.961E-2	1.802E-2	4.274E-2	1.802E-2	4.274E-2	0.106	3.517E-2	1.301E-2	5.702E-3

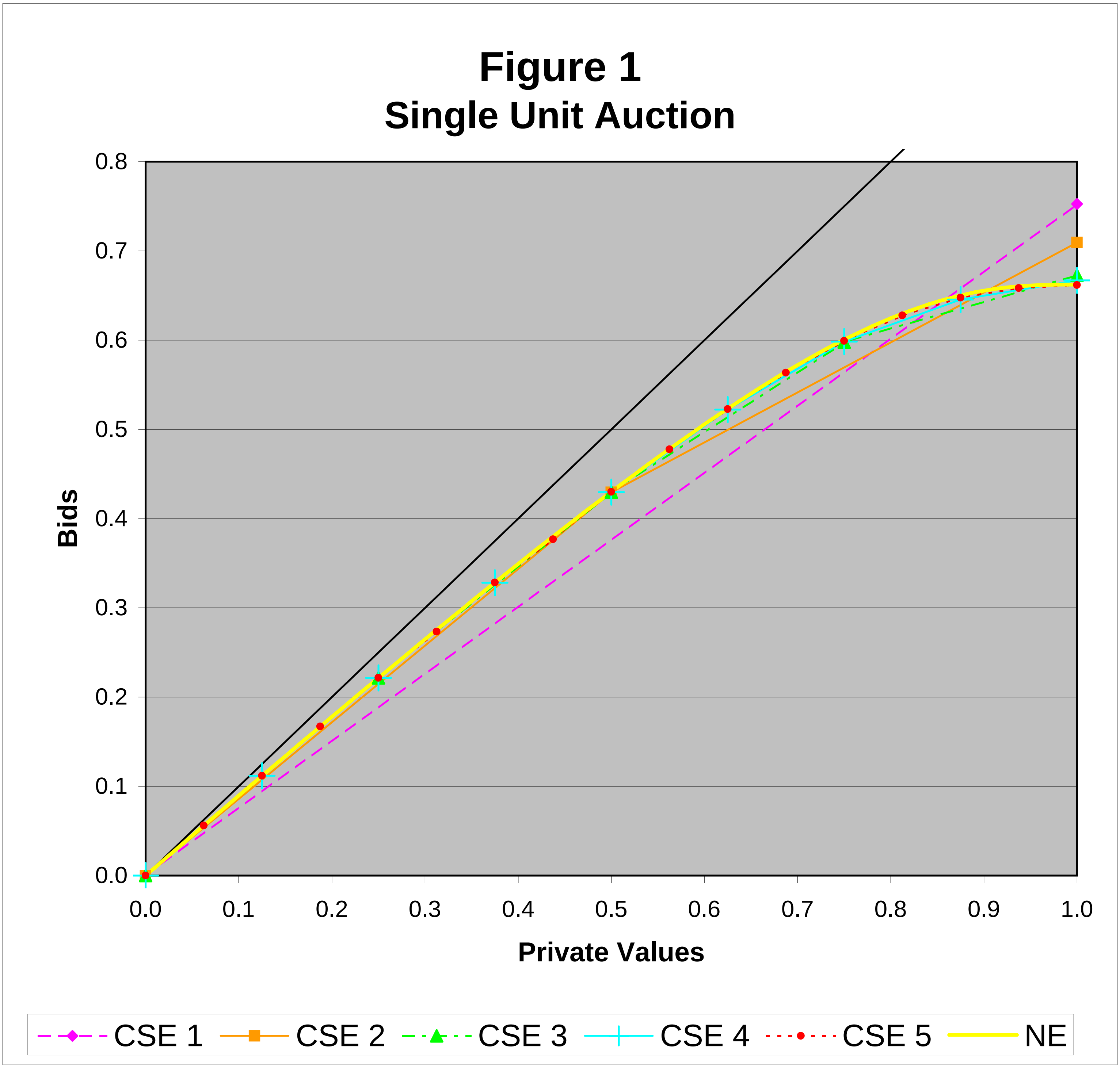

Figure 1
Single Unit Auction
Bids
Private Values
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
CSE 1
CSE 2
CSE 3
CSE 4
CSE 5
NE

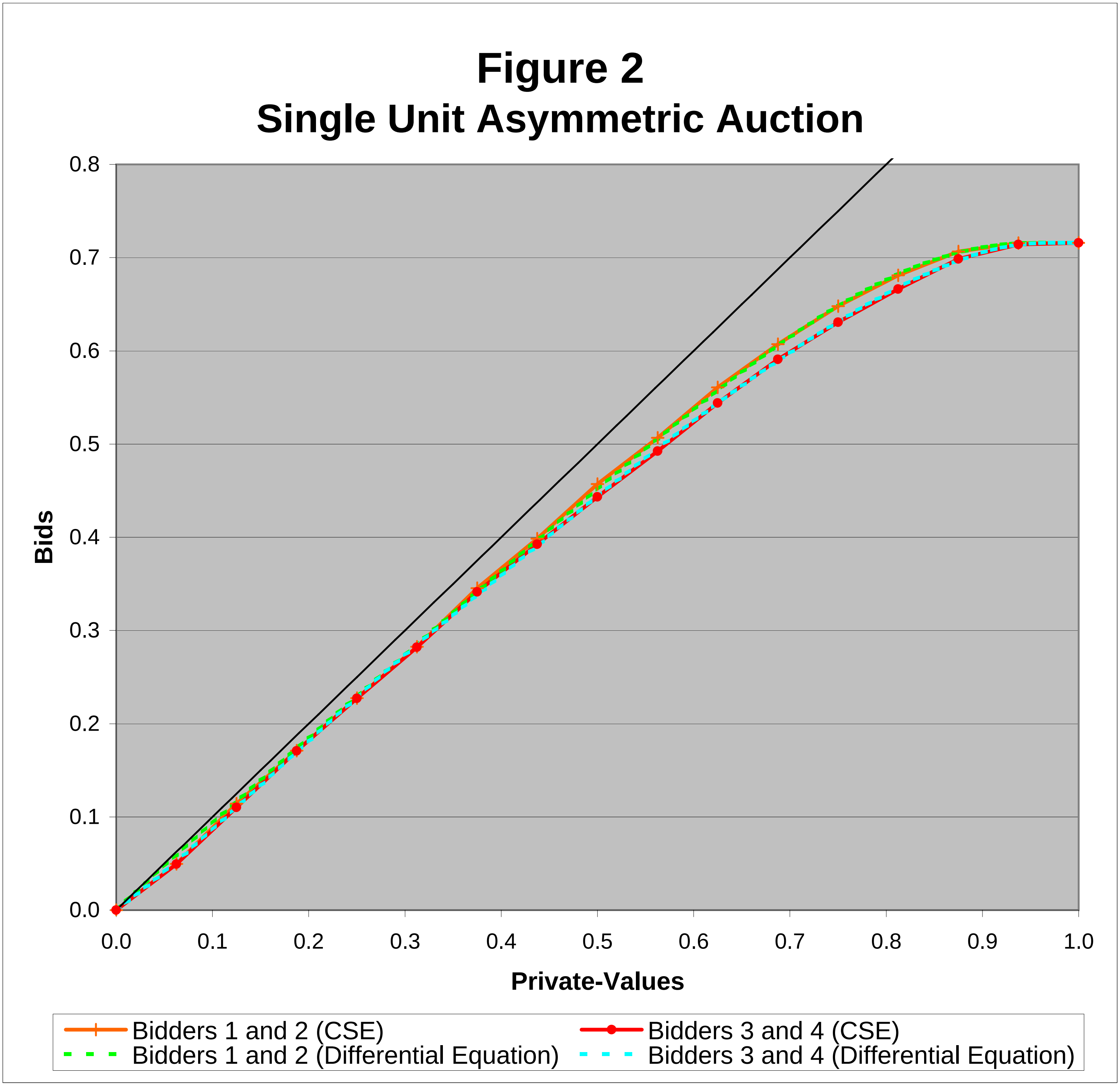
Figure 2
Single Unit Asymmetric Auction
Bids
Private-Values
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
Bidders 1 and 2 (CSE)
Bidders 1 and 2 (Differential Equation)
Bidders 3 and 4 (CSE)
Bidders 3 and 4 (Differential Equation)

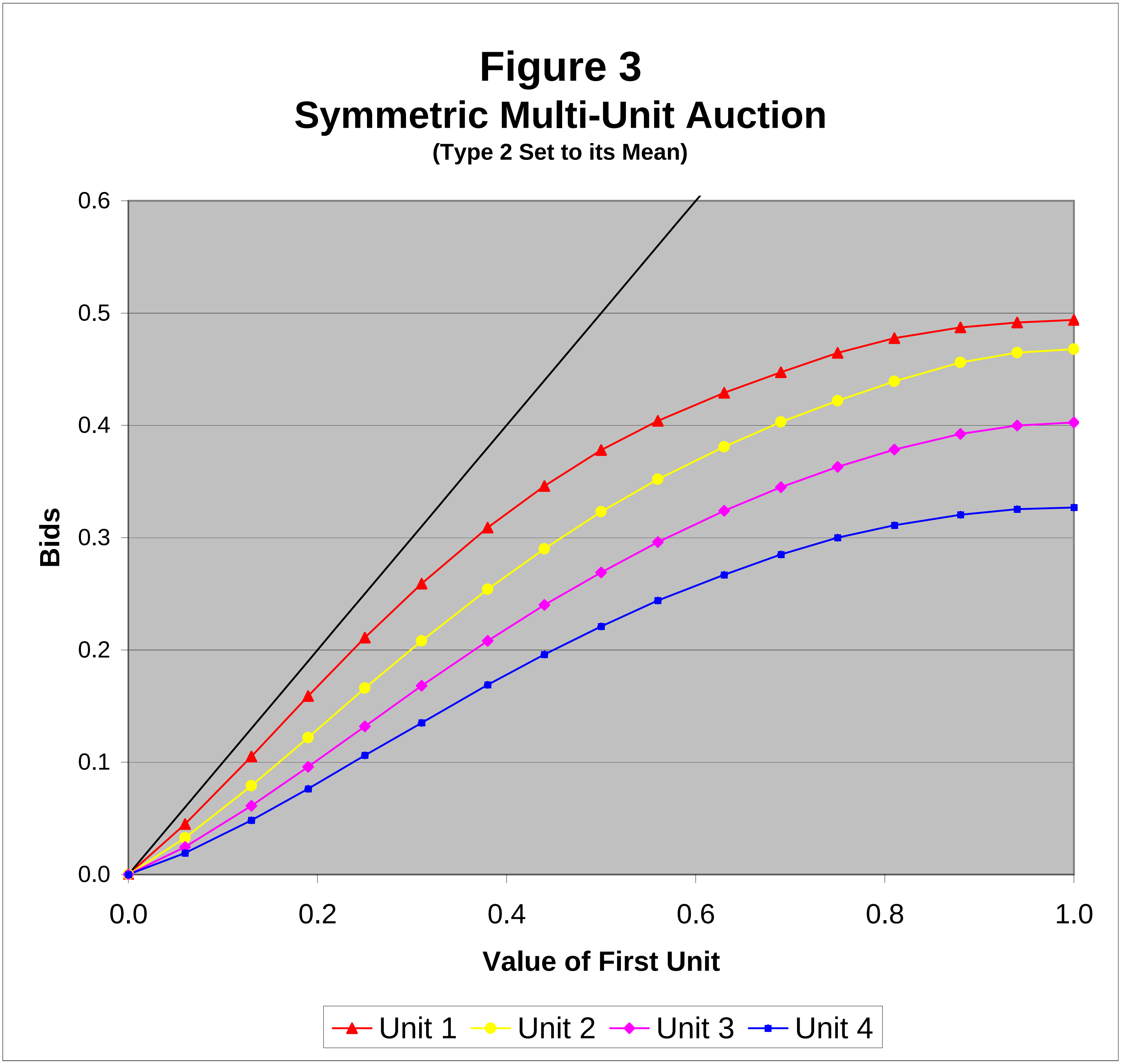
Figure 3
Symmetric Multi-Unit Auction
(Type 2 Set to its Mean)
Bids
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.0
0.2
0.4
0.6
0.8
1.0
Value of First Unit
Unit 1
Unit 2
Unit 3
Unit 4