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ADVERTISING IN A DIFFERENTIATED DUOPOLY AND ITS POLICY IMPLICATIONS FOR AN OPEN ECONOMY

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Advertising in a Differentiated Duopoly and Its Policy Implications for an Open Economy*

Jie Ma† and Alistair M. Ulph‡

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Abstract

In this paper, we develop a model of advertising in a differentiated duopoly in which firms first decide how much to invest in cooperative or predatory advertising and then engage in product market competition (Cournot or Bertrand). We then use this model, with endogenously determined type of advertising, to explore the policy implications in the context of a Brander-Spencer third-country model of strategic trade. We first analyse optimal policies when governments use both trade and industrial policies and show that these policies are substitutes. We then study optimal policy when governments can use only one policy instrument and show that industrial policy is robust, i.e., governments will always use an advertising subsidy irrespective of the type of advertising and form of market competition. More interestingly we show that for a range of parameter values we also get robust trade policy in which governments always use a trade subsidy irrespective of the type of advertising or form of market competition.

Key Words: Cooperative advertising, Predatory advertising, First best policy combination, Robust industrial policy, Robust trade policy

JEL Classification: F13, L13

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1 Introduction

Strategic trade policy has become a core part of international trade policy analysis since the seminal paper Brander and Spencer (1985) was published. However, despite a voluminous literature since then, the policy implications remain controversial, mainly because the *ex ante* trade policy recommendation is very sensitive to the *ex post* market conduct.[1] Recent studies, such as Bagwell and Staiger (1994), Maggi (1996) and Leahy and Neary (2001) show that if firms engage in strategic investment competition (e.g., for R&D or capacity) prior to product market competition, then industrial policy, in the form of an investment subsidy, would be more robust than trade policy. Neary and Leahy (2000) develop a general framework to analyse optimal intervention towards dynamic oligopoly, emphasizing the implications of different kinds of government commitment, and point out that when firms make strategic investments prior to product market competition, the first best policy combination should be designed for both profit-shifting and correcting domestic firm's strategic behaviour to influence rival's decision and domestic government's decision (if possible), which is socially wasteful. They also argue that a general model may not be useful in providing a general guide to policy making, and that it might be better to conduct case studies of particular policy combinations. Advertising is a fruitful field for such a case study, since its policy implications in the context of strategic trade policy have not been much explored.

In this paper, we first construct a model of advertising in a differentiated duopoly. This is modelled as a two-stage game, in which at the first stage firms decide how much to invest in cooperative advertising, or predatory advertising or both. According to Church and Ware (2000 p. 566):

> "One of the more important distinctions in the study of advertising is between *cooperative advertising*, which increases demand for rival firms' products as well as those of the advertising firm, and *predatory advertising*, which increases demand for the advertising firm only by attracting customers away from its rivals."

At the second stage they engage in product market competition.

We then analyse policy setting in the context of a Brander-Spencer third-country model of strategic trade, beginning with the case where governments can set trade and industrial policies, and then considering the cases where they can set only industrial policy or only trade policy.

The main results of this paper are as follows.

First, firms will invest only in one type of advertising, which is determined by the relative effectiveness of the two types of advertising and the degree of product differentiation. Second, when governments use both trade and industrial policies,

[1] Brander (1995) provides a comprehensive discussion on strategic trade policy models.

these policies are substitutes. Third, new evidence is found to support trade policy. When governments can use only trade policy, for a range of parameters, which can be wide, trade policy in the form of a trade subsidy is similarly robust, i.e., governments always use that irrespective of the type of advertising or form of market competition.[2] Fourth, further evidence is found to support industrial policy. When governments can use only industrial policy, it is robust, i.e., governments will always use an advertising subsidy irrespective of the type of advertising and form of market competition.

An obvious question is how these results for advertising relate to results for other forms of strategic investment, such as R&D. Cooperative advertising and R&D with spillovers are similar in that they raise rival's profit, i.e., they both have positive externality effects. However, predatory advertising decreases rival's demand hence profits and so has a negative externality effect, which has no analogue in R&D.[3]

How do our results relate to other studies of advertising?[4] In the classic paper Dixit and Norman (1978) advertising shifts utility and demand, which raises problems of evaluating welfare effects.[5] Becker and Murphy (1993) try to solve this problem by treating advertising as a good that consumers purchase. They point out that advertising can increase demand of a product because the advertised good and advertising are complements. This allows conventional welfare analysis of advertising. In this paper we follow the approach used by Dixit and Norman (1978), and can use this to analyse government policy, because our use of the Brander-Spencer third-country model means we can ignore the welfare analysis of the effects of advertising on consumers.

Mantovani and Mion (2002) use a similar two-stage game analysis of advertising as in the basic model of this paper. However, our paper differs from theirs in two respects. First this paper uses the basic model to examine the policy implications for an open economy, whereas they use theirs to study the effect of entry deterrence and endogenous exit.[6] Second, this paper considers both quantity and price competition, whereas they consider only price competition.

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[2] Neary and Leahy (2000) argue that export subsidy may be a practical policy option: "...it may be possible to evade the WTO prohibition on export subsidies (e.g., by providing export credits) but budgetary constraints may preclude direct assistance to investment." Moreover, in practice, WTO does not prohibit the use of an export tax rebate policy, which is equivalent to the effect of an export subsidy.

[3] Intuitively, a firm's production cost could not be increased by rival's R&D investment. Moreover, studies of R&D and strategic trade have not explored the implications for robustness of strategic trade policy. It is an interesting question whether the robustness result obtained in the advertising case carries over to the R&D case.

[4] Bagwell (2001) is a good introduction to the literature on economics of advertising.

[5] The representative criticisms can be found in Schmalensee (1986).

[6] There are some detailed differences. Mantovani and Mion (2002) treat advertising as a discrete variable, while we treat it as a continuous variable. While we distinguish between cooperative and predatory advertising they distinguish between the market enlargement and predatory effects of advertising. But if market enlargement dominates predatory effects in terms of impact on rivals, we call this cooperative, and *vice versa* for predatory.

The paper is organized as follows. The basic model of advertising in a differentiated duopoly is presented in Section 2 and analysed in Section 3. In Section 4 we examine the policy implications of the basic model for an open economy. Section 5 concludes the paper and points out further extensions. All of the proofs, and the design of simulation, and the discussion on the second order condition of welfare maximization are presented in Appendix.

2 Basic Model

The basic model is characterized as a two-stage game. In the first stage, two firms, firm 1 and firm 2, which produce a differentiated product respectively, decide simultaneously how much to invest in cooperative advertising, or predatory advertising, or both. In the second stage they engage in product market competition.

Consumers

Assume that the representative consumer's preferences are given by the quasilinear utility function

$$U(x_1, x_2, m) = u(x_1, x_2) + m,$$

where x_1 and x_2 are the outputs of the two firms respectively and m is a numéraire good. In addition,

$$u(x_1, x_2) = a_1 x_1 + a_2 x_2 - \frac{1}{2}\left[b(x_1)^2 + 2x_1x_2 + b(x_2)^2\right],$$

where

$$a_i = a\left[1 + \mu(m_i + m_j) + \nu(n_i - n_j)\right], \tag{1}$$
$$a > 0,\ b > 1,\ \mu > 0,\ \nu > 0,\ i = 1, 2,\ j = 1, 2,\ i \neq j.$$

a_i measures the market scale for firm i and b represents the sensitivity of demand to a firm's own product. For simplicity, we assume that the sensitivity of demand to the rival firm's product is equal to 1. m_i and n_i are the respective cooperative and predatory advertising levels of firm i. μ and ν evaluate the effectiveness of the two types of advertising respectively.

Denote by p_i the price for each firm's product. Then the indirect demand system is given by

$$p_i = a_i - bx_i - x_j.$$

The corresponding direct demand system is given by

$$x_i = \alpha_i - \beta p_i + \gamma p_j,$$

where

$$\alpha_i = \alpha\left[1 + \mu(m_i + m_j) + \left(\frac{b+1}{b-1}\right)\nu(n_i - n_j)\right], \tag{2}$$

and

$$\alpha = \frac{a}{b+1}, \quad \beta = \frac{b}{b^2-1}, \quad \gamma = \frac{1}{b^2-1}.$$

This formulation implies that one unit of a firm's cooperative advertising investment will increase own and rival market scale by $a\mu$ units when firms play Cournot and by $\alpha\mu = (\frac{a}{b+1})\mu$ units when firms play Bertrand; one unit of a firm's predatory advertising investment will make own market scale increase and rival market scale decrease by $a\nu$ units when firms play Cournot and make own market scale increase and rival market scale decrease by $\alpha(\frac{b+1}{b-1})\nu = (\frac{a}{b-1})\nu$ units when firms play Bertrand.

To simplify notation, without loss of generality, we normalise $\nu = 1$, and henceforth interpret μ as a measure of the **relative** effectiveness of cooperative advertising.

Firms

Firm i maximizes profit Π_i. We assume that firms have the same CRS production technologies, with cost function: $C_i(x_i) = cx_i$, $i = 1, 2$, where, for the usual reason, $a > c$. The investment cost function of each firm is given by $c_i(m_i, n_i) = \frac{1}{2}k(m_i + n_i)^2$, i.e., we suppose that there exists joint investment diseconomy for each firm,[7] where

$$\frac{\partial c_i}{\partial m_i} = \frac{\partial c_i}{\partial n_i} = k(m_i + n_i), \; \frac{\partial^2 c_i}{\partial m_i^2} = \frac{\partial^2 c_i}{\partial n_i^2} = \frac{\partial^2 c_i}{\partial m_i \partial n_i} = k.$$

We make the following assumption on k.

Assumption 1 $\underline{k} < k < \overline{k}$, where

$$\underline{k} = \max\{2k_1, 2k_2, 2k_3, 2k_4\},$$

and

$$k_1 = \frac{2a^2b\mu^2}{(2b+1)^2}, \; k_2 = \frac{2a^2b}{(2b-1)^2}, \; k_3 = \frac{2a^2b(b-1)\mu^2}{(b+1)(2b-1)^2}, \; k_4 = \frac{2a^2b(b+1)}{(b-1)(2b+1)^2},$$

and

$$\overline{k} < \infty.$$

It can be easily shown that $k_1 > k_3$, $k_2 < k_4$. Hence $\underline{k} = \max\{2k_1, 2k_4\}$.

The first inequality in Assumption 1 sets the greatest lower bound on k and ensures that in the investment stage of the game:

[7] It might be argued that such a formulation cannot capture the potential increasing-return-to-scale effects of advertising. However, the problem might not be so serious as it seems to be. If advertising investment will incur a fixed cost, then there will be an increasing-return-to-scale range for investment. Clearly a firm will invest in advertising only within this range. If we make an appropriate additional assumption on the fixed cost appropriately, our analysis will still hold. Of course, fixed costs are one of the important factors determining market structure. However, it is not the focus of this paper and we assume that the market structure is given.

1. the profit function of each firm will be a concave function in its own choice,
2. the own effect of any type of advertising will be greater than the corresponding cross effect.

The second inequality sets an upper bound on k to ensure that firms' advertising investments are not so low as to be negligible.

We now solve for the sub-game perfect Nash equilibrium (SPNE) of the basic model.

3 Analysis of Basic Model

We first discuss the case where firms in the product market play Cournot and then turn to the case where firms play Bertrand.

3.1 The Cournot case

In the last stage of the game, firm i maximizes its profit function:

$$\Pi_i^C = (a_i - bx_i - x_j - c)x_i.$$

Note that at this stage the investment costs are sunk and as usual quantities are strategic substitutes. The Nash Equilibrium is:

$$x_i^{*C} = \frac{a\left[1 + \mu\left(m_i + m_j\right) + \left(\frac{2b+1}{2b-1}\right)\left(n_i - n_j\right)\right] - c}{2b+1}. \tag{3}$$

Moreover, we have $p_i^{*C} - c = bx_i^{*C}$. The effects of the different types of advertising on the equilibrium quantity x_i^{*C} and equilibrium price p_i^{*C} are as follows.

$$\frac{\partial x_i^{*C}}{\partial m_i} = \frac{\partial x_i^{*C}}{\partial m_j} = \frac{a\mu}{2b+1} > 0,\ \frac{\partial x_i^{*C}}{\partial n_i} = \frac{a}{2b-1},\ \frac{\partial x_i^{*C}}{\partial n_j} = -\frac{\partial x_i^{*C}}{\partial n_i} < 0. \tag{4}$$

$$\frac{\partial p_i^{*C}}{\partial m_i} = \frac{\partial p_i^{*C}}{\partial m_j} = \frac{ab\mu}{2b+1} > 0,\ \frac{\partial p_i^{*C}}{\partial n_i} = \frac{ab}{2b-1},\ \frac{\partial p_i^{*C}}{\partial n_j} = -\frac{\partial p_i^{*C}}{\partial n_i} < 0. \tag{5}$$

The equilibrium profit of firm i is:

$$\Pi_i^{*C} = b\left(x_i^{*C}\right)^2.$$

We use that to replace the product market competition stage and get the reduced extensive form game.

In the first stage of the game, firm i maximizes its profit function in the reduced extensive form game:

$$\pi_i^C = \Pi_i^{*C} - c_i\left(m_i, n_i\right).$$

By Assumption 1, we ensure that the profit function of each firm is concave with respect to its own choice and then there exists a pure strategy Nash Equilibrium, which is unique and stable.[8]

Lemma 1

1. If $\mu > \frac{2b+1}{2b-1}$, there exists a symmetric equilibrium, in which both firms invest in cooperative advertising. Cooperative advertising is a strategic complement and makes rival's profit increase.

2. If $\mu < \frac{2b+1}{2b-1}$, there exists a symmetric equilibrium, in which both firms invest in predatory advertising. Predatory advertising is a strategic substitute and makes rival's profits decrease.[9]

From the specification of the investment cost function, the marginal costs of increasing any type of advertising by one unit are the same. Therefore, in order to make the optimal investment decision, each firm compares the marginal revenues from the two types of advertising and chooses the larger one. Given the equilibrium of the subsequent competition, when $\mu > \frac{2b+1}{2b-1}$, the marginal revenue from cooperative advertising is greater than that from predatory advertising, and vice versa.

When firms invest in cooperative advertising, the equilibrium value of cooperative advertising is

$$m_i^{*C} = m^{*C} = \frac{2a(a-c)b\mu}{(2b+1)^2 k - 4a^2 b\mu^2}.$$

It is easy to show that:

$$\frac{\partial m^{*C}}{\partial a} > 0, \ \frac{\partial m^{*C}}{\partial b} < 0, \ \frac{\partial m^{*C}}{\partial c} < 0, \ \frac{\partial m^{*C}}{\partial \mu} > 0, \ \frac{\partial m^{*C}}{\partial k} < 0.$$

Substituting the expression for m^{*C} into the expressions for the equilibrium quantity x_i^{*C} and equilibrium price p_i^{*C}, we get their equilibrium values respectively,

$$x_i^{*C} = x^{*C}(m) = \frac{(a-c)(2b+1)k}{(2b+1)^2 k - 4a^2 b\mu^2},$$

$$p_i^{*C} = p^{*C}(m) = \frac{ab+(b+1)c}{2b+1} + \frac{4(a-c)a^2b^2\mu^2}{(2b+1)\left[(2b+1)^2 k - 4a^2 b\mu^2\right]}.$$

When firms invest in predatory advertising, the equilibrium value of predatory advertising is

$$n_i^{*C} = n^{*C} = \frac{2a(a-c)b}{(4b^2-1)k}.$$

[8] Fudenberg and Tirole (1991) and Nikaido (1968).

[9] Of course, if $\mu = \frac{2b+1}{2b-1}$, there exists a symmetric equilibrium, in which both firms invest in both cooperative advertising and predatory advertising. We omit this case.

It is easy to show that:

$$\frac{\partial n^{*C}}{\partial a} > 0, \ \frac{\partial n^{*C}}{\partial b} < 0, \ \frac{\partial n^{*C}}{\partial c} < 0, \ \frac{\partial n^{*C}}{\partial k} < 0.$$

Substituting the expression for n^{*C} into the expressions for the equilibrium quantity x_i^{*C} and equilibrium price p_i^{*C}, we get their equilibrium values respectively,

$$x_i^{*C} = x^{*C}(n) = \frac{a-c}{2b+1}$$
$$p_i^{*C} = p^{*C}(n) = \frac{ab+(b+1)c}{2b+1}.$$

3.2 The Bertrand case

In the last stage of the game, firm i maximizes its profit function:

$$\Pi_i^B = (p_i - c)(\alpha_i - \beta p_i + \gamma p_j).$$

Note that at this stage the investment costs are sunk and as usual prices are strategic complements. The Nash Equilibrium is:

$$p_i^{*B} = \frac{\alpha \left[1 + \mu(m_i + m_j) + \left(\frac{b+1}{b-1}\right)\left(\frac{2\beta-\gamma}{2\beta+\gamma}\right)(n_i - n_j)\right] + \beta c}{2\beta - \gamma}. \tag{6}$$

Moreover, we have $x_i^{*B} = \beta\left(p_i^{*B} - c\right)$. The effects of the different types of advertising on the equilibrium price p_i^{*B} and equilibrium quantity x_i^{*B} are as follows.

$$\frac{\partial p_i^{*B}}{\partial m_i} = \frac{\partial p_i^{*B}}{\partial m_j} = \frac{\alpha\mu}{2\beta-\gamma} > 0, \ \frac{\partial p_i^{*B}}{\partial n_i} = \frac{\alpha\left(\frac{b+1}{b-1}\right)}{2\beta+\gamma} > 0, \ \frac{\partial p_i^{*B}}{\partial n_j} = -\frac{\partial p_i^{*B}}{\partial n_i} < 0. \tag{7}$$
$$\frac{\partial x_i^{*B}}{\partial m_i} = \frac{\partial x_i^{*B}}{\partial m_j} = \frac{\alpha\beta\mu}{2\beta-\gamma} > 0, \ \frac{\partial x_i^{*B}}{\partial n_i} = \frac{\alpha\beta\left(\frac{b+1}{b-1}\right)}{2\beta+\gamma} > 0, \ \frac{\partial x_i^{*B}}{\partial n_j} = -\frac{\partial x_i^{*B}}{\partial n_i} < 0. \tag{8}$$

The equilibrium profit of firm i is:

$$\Pi_i^{*B} = \beta\left(p_i^{*B} - c\right)^2.$$

We use that to replace the product market competition stage and get the reduced extensive form game.

In the first stage of the game, firm i maximizes its profit function in the reduced extensive form game:

$$\pi_i^B = \Pi_i^{*B} - c_i(m_i, n_i).$$

By Assumption 1, we ensure that the profit function of each firm is concave with respect to its own choice and then there exists a pure strategy Nash Equilibrium, which is unique and stable.[10]

[10] Fudenberg and Tirole (1991) and Nikaido (1968).

Lemma 2

1. If $\mu > \frac{2b^2+b-1}{2b^2-b-1}$, there exists a symmetric equilibrium, in which both firms invest in cooperative advertising. Cooperative advertising is a strategic complement and makes rival's profit increase.
2. If $\mu < \frac{2b^2+b-1}{2b^2-b-1}$, there exists a symmetric equilibrium, in which both firms invest in predatory advertising. Predatory advertising is a strategic substitute and makes rival's profit decrease.[11]

The rationale is the same as for Lemma 1.

When firms invest in cooperative advertising, the equilibrium value of cooperative advertising is

$$m_i^{*B} = m^{*B} = \frac{2a\left(a-c\right)b\left(b-1\right)\mu}{\left(b+1\right)\left(2b-1\right)^2 k - 4a^2 b\left(b-1\right)\mu^2}.$$

It is easy to show that:

$$\frac{\partial m^{*B}}{\partial a} > 0, \ \frac{\partial m^{*B}}{\partial c} < 0, \ \frac{\partial m^{*B}}{\partial \mu} > 0, \ \frac{\partial m^{*B}}{\partial k} < 0,$$

and

$$\frac{\partial m^{*B}}{\partial b} > 0, \ if \quad b < 1.6777, \quad \frac{\partial m^{*B}}{\partial b} < 0, \ if \quad b > 1.6777.$$

Substituting the expression for m^{*B} into the expressions for the equilibrium price p_i^{*B} and equilibrium quantity x_i^{*B}, we get their equilibrium values respectively,

$$p_i^{*B} = p^{*B}\left(m\right) = \frac{a\left(b-1\right)+bc}{2b-1} + \frac{4a^2\left(a-c\right)b\left(b-1\right)^2\mu^2}{\left(2b-1\right)\left[\left(b+1\right)\left(2b-1\right)^2 k - 4a^2 b\left(b-1\right)\mu^2\right]},$$

$$x_i^{*B} = x^{*B}\left(m\right) = \frac{\left(a-c\right)b\left(2b-1\right)k}{\left(b+1\right)\left(2b-1\right)^2 k - 4a^2 b\left(b-1\right)\mu^2}.$$

When firms invest in predatory advertising, the equilibrium value of predatory advertising is

$$n_i^{*B} = n^{*B} = \frac{2a\left(a-c\right)b}{\left(4b^2-1\right)k},$$

which is equal to the equilibrium value of predatory advertising in the Cournot case. Substituting the expression for n_i^{*B} into the expressions for the equilibrium price p_i^{*B} and equilibrium quantity x_i^{*B}, we get their equilibrium values respectively,

$$p_i^{*B} = p^{*B}\left(n\right) = \frac{a\left(b-1\right)+bc}{2b-1},$$

$$x_i^{*B} = x^{*B}\left(n\right) = \frac{\left(a-c\right)b}{\left(b+1\right)\left(2b-1\right)}.$$

[11]Of course, if $\mu = \frac{2b^2+b-1}{2b^2-b-1}$, there exists a symmetric equilibrium, in which both firms invest in both cooperative advertising and predatory advertising. We omit this case.

We summarize the main results of the analysis in the above two Subsections in the following Proposition.

Proposition 3

1. Whatever the form of product market competition, cooperative advertising will be present in equilibrium, if $\mu > \frac{2b^2+b-1}{2b^2-b-1}$, cooperative advertising will be present in equilibrium when firms play Cournot, while predatory advertising will be present in equilibrium when firms play Bertrand, if $\frac{2b^2+b-1}{2b^2-b-1} > \mu > \frac{2b+1}{2b-1}$, predatory advertising will be present in equilibrium, if $\frac{2b+1}{2b-1} > \mu$.

2. Whatever the form of product market competition, cooperative advertising is a strategic complement and makes rival's profit increase, while predatory advertising is a strategic substitute and makes rival's profit decrease.

The intuition behind the first part of Proposition 3 is fairly simple. When b is very small, i.e., the degree of product differentiation is very small,[12] only if the relative effectiveness of cooperative advertising is very large will cooperative advertising be chosen in equilibrium. Otherwise, firms will invest in predatory advertising. In other words, if the two products are very similar, a firm has a strong incentive to "steal" rival's market share. On the other hand, when b is very large, i.e., the degree of product differentiation is very large, even if the relative effectiveness of cooperative advertising is very small, cooperative advertising will be chosen in equilibrium. The reason is that the incentive to steal rival's market share diminishes and each firm wants to increase its own market scale.[13]

This completes our discussion of the basic model.

4 Policy Implications for an Open Economy

In this Section we will consider the policy implications of the basic model for an open economy in a Brander-Spencer third-country model. From the viewpoint of an export country, if there is unilateral intervention, what is the optimal policy?

We now consider a three-stage game in which we add an additional stage to the start of the basic model. In this new first stage, the government of country i sets its policy and the potential policy instruments are trade policy, a subsidy s on output, and industrial policy, a subsidy τ on advertising.[14] We assume the opportunity cost

[12] According to Singh and Vives (1984), when the market scale of one firm is equal to that of the other, $\frac{1}{b^2}$ is the measure of product differentiation in this case.

[13] See also Mantovani and Mion (2002).

[14] Note that trade policy has two effects on the subsequent game. First, it will directly change the equilibrium outcome of the product market competition. Second, it has an indirect effect on the competition as well by changing rival firm's incentive to invest in advertising. Unlike trade policy, industrial policy cannot directly influence the product market competition but has an indirect effect on that by changing rival firm's investment incentive.

of public fund is unity. The representative consumer and the market are now in a third country.[15]

Before going further, note that firms' decisions in the investment stage on whether to invest in cooperative or predatory advertising are not changed by the policy instruments, since, as we have just seen, that decision depends only on the relative effectiveness of cooperative advertising and the degree of product differentiation.

In the following Subsections we examine the first best policy combination, second best industrial policy and second best trade policy.

4.1 First best policy analysis

In this case, the government uses different instruments for different targets, in particular, trade policy s towards domestic firm's quantity or price and industrial policy τ towards domestic advertising investment. Given the equilibrium outcome in the subsequent game, the government maximizes its welfare:

$$\max_{\{s,\tau\}} W(s,\tau) = \pi_i(s,\tau) - s x_i(s,\tau) - \tau I_i(s,\tau),$$

where π_i is domestic firm's profit and $I_i \in \{m_i, n_i\}$. We shall assume that the welfare function is strictly concave, so the following two conditions characterize the unique optimal policy combination:

$$\frac{\partial W}{\partial s} = \frac{\partial \pi_i}{\partial s}(\cdot) - x_i - \frac{\partial x_i}{\partial s}(\cdot)\, s - \frac{\partial I_i}{\partial s}\tau = 0, \tag{9}$$

$$\frac{\partial W}{\partial \tau} = \frac{\partial \pi_i}{\partial \tau}(\cdot) - \frac{\partial x_i}{\partial \tau}(\cdot)\, s - I_i - \frac{\partial I_i}{\partial \tau}\tau = 0. \tag{10}$$

Denote

$$D = \frac{\partial x_i}{\partial s}(\cdot)\frac{\partial I_i}{\partial \tau} - \frac{\partial x_i}{\partial \tau}(\cdot)\frac{\partial I_i}{\partial s}, \tag{11}$$

$$D_1 = \left[\frac{\partial \pi_i}{\partial s}(\cdot) - x_i\right]\frac{\partial I_i}{\partial \tau} - \left[\frac{\partial \pi_i}{\partial \tau}(\cdot) - I_i\right]\frac{\partial I_i}{\partial s}, \tag{12}$$

$$D_2 = \left[\frac{\partial \pi_i}{\partial \tau}(\cdot) - I_i\right]\frac{\partial x_i}{\partial s}(\cdot) - \left[\frac{\partial \pi_i}{\partial s}(\cdot) - x_i\right]\frac{\partial x_i}{\partial \tau}. \tag{13}$$

We have

$$s = \frac{D_1}{D},\ \tau = \frac{D_2}{D}.\text{[16]} \tag{14}$$

[15] According to the terminology of Neary and Leahy (2000), we consider only "Government-Only-Commitment Equilibrium" in this paper.

[16] It should be noted that according to Neary and Leahy (2000) the first best policy combination should not only do the profit-shifting job but also should correct domestic firm's strategic behaviour to influence rival's decision and domestic government's decision (if possible), which is socially wasteful.

We apply these general formulae for the first best policy combination to the particular cases to obtain the following Proposition.

Proposition 4 In a Brander-Spencer third-country world the unique unilateral first-best intervention by the government of country i is as follows.

1. If $\mu > \frac{2b+1}{2b-1}$, i.e., in the subsequent game Cournot firms invest in cooperative advertising, both the optimal trade policy and the optimal industrial policy are ambiguous. If $\mu < \frac{2b+1}{2b-1}$, i.e., in the subsequent game Cournot firms invest in predatory advertising, the optimal trade policy is a trade subsidy, whereas the optimal industrial policy is ambiguous.

2. If $\mu > \frac{2b^2+b-1}{2b^2-b-1}$, i.e., in the subsequent game Bertrand firms invest in cooperative advertising, the first best policy is the combination of trade tax and advertising subsidy. If $\mu < \frac{2b^2+b-1}{2b^2-b-1}$, i.e., in the subsequent game Bertrand firms invest in predatory advertising, the optimal industrial policy is an advertising subsidy, whereas the optimal trade policy is ambiguous.

3. Irrespective of the form of market competition and type of advertising, optimal first-best policy never involves taxes on both exports and advertising.

We also have the following Corollaries.

Corollary 5 The cross derivative of the welfare function is negative whatever the form of competition and whatever the equilibrium type of advertising investment,

$$\frac{\partial^2 W}{\partial s \partial \tau} < 0.$$

That is the two policy instruments are substitutes.

Corollary 6 When firms play Cournot,

$$\left.\frac{\partial W}{\partial s}\right|_{(s,\tau)=(0,0)} > 0.$$

When firms play Bertrand, $sign \left.\frac{\partial W}{\partial s}\right|_{(s,\tau)=(0,0)}$ is indefinite.

Corollary 7 Whatever the form of competition and the equilibrium type of advertising investment,

$$\left.\frac{\partial W}{\partial \tau}\right|_{(s,\tau)=(0,0)} > 0.$$

Corollary 8

1. The marginal rate of substitution between trade policy and industrial policy (thereafter MRS), which is defined by

$$-\frac{d\tau}{ds} = \frac{\frac{\partial W}{\partial s}}{\frac{\partial W}{\partial \tau}},$$

is positive at the non-intervention point when firms play Cournot, whereas it can be positive or negative at that point when firms play Bertrand (see Figure 1 , 2).

2. The MRS is decreasing at the non-intervention point, i.e.,

$$-\left.\frac{d^2\tau}{ds^2}\right|_{(s,\tau)=(0,0)} < 0,$$

whatever the form of competition and whatever the equilibrium type of advertising investment.

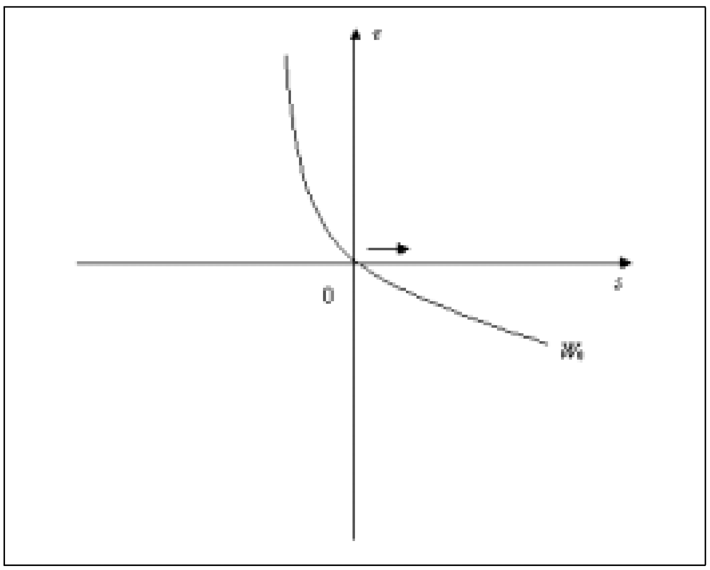

Figure 1: Positive MRS at non-intervention point

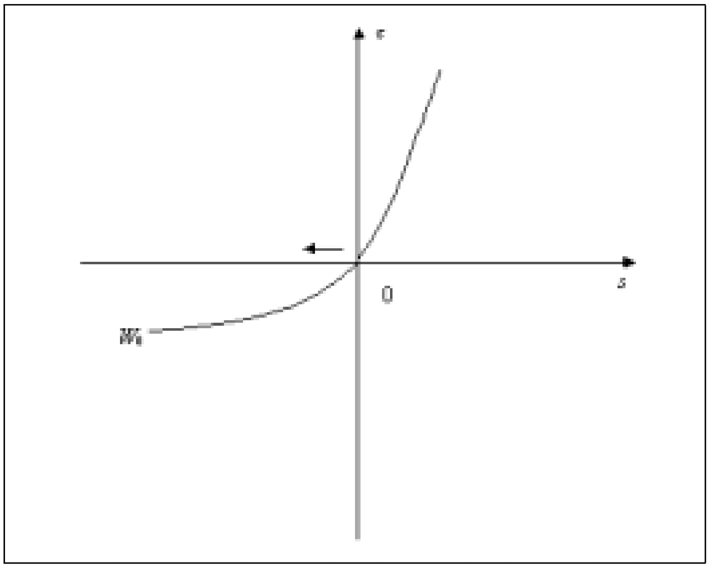

Figure 2: Negative MRS at non-intervention point

Obviously, the results on first best policy combination are not clear-cut. This is because the two policy instruments are substitutes, as stated in Corollary 5. Moreover, the substitutability between them is dependent on the fundamental characteristics of the model, i.e., the degree of product differentiation, the relative effectiveness of cooperative advertising and the coefficient of investment cost. However, note that whatever the form of competition and whatever the equilibrium type of advertising investment, the two policy instruments cannot both be zero, so the government always wants to play an active role in international competition and its optimal intervention includes at least one positive component, i.e., a subsidy.

Corollary 8 is the key to understanding the link between the first-best and the second best results, to which we now turn.

4.2 Second best policy analysis

4.2.1 Trade policy

In this case, the government can use only trade policy to intervene in international competition. Given the equilibrium outcome in the subsequent game, the government maximizes its welfare:

$$\max_{\{s\}} W\left(s\right) = \pi_i\left(s\right) - sx_i\left(s\right).$$

The following condition characterizes the optimal policy:

$$\frac{dW}{ds} = \frac{d\pi_i}{ds}\left(\cdot\right) - x_i - \frac{dx_i}{ds}s = 0. \tag{15}$$

We have

$$s = \frac{\frac{d\pi_i}{ds}(\cdot) - x_i}{\frac{dx_i}{ds}}. \tag{16}$$

Applying this general formula for second best trade policy to the particular cases, we obtain the following Proposition.

Proposition 9 In a Brander-Spencer third-country world where the government can use only trade policy, the unique unilateral intervention by the government of country i is as follows:

1. If $\mu > \frac{2b^2+b-1}{2b^2-b-1}$, i.e., in the subsequent game both Cournot and Bertrand firms invest in cooperative advertising, the optimal trade policy is to implement a trade subsidy whatever the form of product market competition, if

$$\left[\sqrt{(5b^2-1)} - (b-1)\right] k_3 > k.$$

2. If $\frac{2b^2+b-1}{2b^2-b-1} > \mu > \frac{2b+1}{2b-1}$, i.e., in the subsequent game Cournot and Bertrand firms will invest in cooperative and predatory advertising respectively, the optimal trade policy is to implement a trade subsidy whatever the form of product market competition, if

$$\left[\sqrt{(5b^2-1)} + (b+1)\right] k_4 > k.$$

3. If $\frac{2b+1}{2b-1} > \mu$, i.e., in the subsequent game both Cournot and Bertrand firms invest in predatory advertising, the optimal trade policy is to implement a trade subsidy whatever the form of product market competition, if

$$\left[\sqrt{(5b^2-1)} + (b+1)\right] k_4 > k.$$

4. In all three cases if we do not get robust trade policy, then optimal trade policy is a trade subsidy with Cournot behaviour and a trade tax with Bertrand behaviour.

Of course, the magnitude of policy instrument is not necessarily the same in each scenario.

4.2.2 Industrial policy

In this case, the government can use only industrial policy τ to intervene in international competition. Given the equilibrium outcome in the subsequent game, the government maximizes its welfare:

$$\max_{\{\tau\}} W(\tau) = \pi_i(\tau) - \tau I_i(\tau).$$

The following condition characterizes the optimal policy:

$$\frac{dW}{d\tau} = \frac{d\pi_i}{d\tau}(\cdot) - I_i - \frac{dI_i}{d\tau}\tau = 0. \tag{17}$$

We have

$$\tau = \frac{\frac{d\pi_i}{d\tau}(\cdot) - I_i}{\frac{dI_i}{d\tau}}. \tag{18}$$

We apply this general formula for second best industrial policy to the particular cases to obtain the following Proposition.

Proposition 10 In a Brander-Spencer third-country world where the government can implement only industrial policy the unique unilateral intervention is an advertising subsidy, whatever the form of product market competition and whatever the equilibrium type of advertising investment.

Of course, the magnitude of policy instrument is not necessarily the same in each scenario.

4.2.3 Discussion

Proposition 10 provides further support for the robustness of industrial policy. But the results presented in Proposition 9 about the robustness of trade policy seem to be new in the strategic trade literature. To see the rationale for these results we refer again to Figures 1 and 2. W_0 is the non-intervention welfare level. According to the second part of Corollary 8, the upper contour set $\{(s,\tau) \in R^2 : W(s,\tau) \geq W_0\}$ must be "above" the iso-welfare curve, which passes the non-intervention point in the neighbourhood of that point. Therefore, if the government is restrained to maximize its welfare along the vertical axis it is clear that the optimal policy must be an advertising subsidy whatever the form of competition and whatever the equilibrium type of advertising investment.

Things are a bit more complex in the trade policy case. From the above figures, we see that if the MRS in the neighbourhood of the non-intervention point is positive, then the robust trade policy, i.e., a trade subsidy occurs. It turns out that in the Cournot case, the MRSs are always positive, whereas those in the Bertrand case can be positive or negative. When conditions presented in Proposition 9 are satisfied, MRSs in the Bertrand case will be positive and robust trade policy in the form of a trade subsidy will be observed.[17]

[17] In summary, if the MRS is decreasing at the non-intervention point, the robust industrial policy, i.e., an industrial subsidy will be observed. If the MRS is both positive and decreasing at the non-intervention point, both an industrial subsidy and the robust trade policy, i.e., a trade subsidy will be observed.

In addition, if MRS is infinity at the non-intervention point, the second best industrial policy will be non-intervention. If MRS is zero at the non-intervention point, the second best trade policy will be a free trade one.

4.2.4 Simulation results on robust trade policy

Proposition 9 established for each of the three cases a critical value of k, k_c, such that if $k < k_c$ trade policy is robust. In this section we want to ask: how "large" is the range of k for which trade policy is robust. To answer this question we first make it more precise, and then present some numerical results.

Assumption 1 restricted k to lie in the range $\left(\underline{k}, \overline{k}\right)$, where $\underline{k}$ is a function of parameters a, b and μ. However to prove Proposition 4 we assumed that the welfare function is strictly concave. In Section 12 of the Appendix we show that if the welfare function is strictly concave, we must have $k > \underline{k}'' > \underline{k}$, where $\underline{k}''$ is a function of parameters a, b and μ. So far we have not said anything about what determines $\overline{k}$. In Section 11 of the Appendix, we derive a value for $\overline{k}$, which depends on parameters a, b, μ and the advertising sales ratio, which we define as ϕ. From Proposition 9 we know that k_c depends on parameters a, b and μ. Putting this together, for any set of parameters a, b, μ and ϕ, we can calculate $\underline{k}''$, $\overline{k}$ and k_c, and the question we ask is: for what proportion of the range of feasible values of k, $\left(\underline{k}'', \overline{k}\right)$, is trade policy robust? We denote the proportion as $l \equiv \frac{k_c - \underline{k}''}{\overline{k} - \underline{k}''}$, and so our question is how large is l?

It is shown that l is a function of parameters b, μ and ϕ in the Section 11 of the Appendix. It is not possible to derive a simple expression for l to show how it relates to these parameter values. So we have used numerical simulations. We take 100 values of b from the interval $(1, 6)$, 100 values of μ from the interval $(1, 2)$ and 100 values of ϕ from the interval $(0, 0.12)$. For each of the $1,000,000$ combinations of b, μ and ϕ, we used Proposition 3 to assign it to one of the three cases. We then calculated $\underline{k}''$, $\overline{k}$, k_c and l. Finally, for the sets of parameter values lying in each of the three cases, we calculated summary statistics of the distribution of l : the average of l, the standard deviation of l, and the maximum and minimum values of l. The results are shown in Table 1.

Table 1: Simulation results on robust trade policy

	Average of l	Standard Deviation of l	Minimum of l	Maximum of l
Case 1	0.462	0.354	0.000	1.000
Case 2	0.450	0.314	0.004	1.000
Case 3	0.671	0.359	0.005	1.000

We define the case where both Cournot and Bertrand firms invest in cooperative advertising as Case 1, the case where Cournot and Bertrand firms invest in cooperative and predatory advertising respectively as Case 2 and the case where both Cournot and Bertrand firms invest in predatory advertising as Case 3. The proportion of Case 1 is 0.529, the proportion of Case 2 is 0.033 and the proportion of Case 3 is 0.375.[18]

[18] If in an parameters combination we had $\underline{k}'' > \overline{k}$, that combination is invalid. The invalid proportion of observations is 0.063.

Two key points emerge from the simulation results. First the average fraction of the feasible range of values for k for which trade policy is robust is not trivial in every case. Second, the robustness results are more likely in Case 3 where firms invest in predatory advertising and there exists negative externality in investment.

This completes the discussions on the policy implications of the basic model. All of the results presented in this Section are summarized in Table 2.

Table 2: Summary of policy implications for an open economy

Parameter combination	Equ type of Ad	First best policy	Second best industrial policy	Second best trade policy
$\mu > \frac{2b^2+b-1}{2b^2-b-1}$	C: Coop	$sign\ s = sign\ B_1$ $sign\ \tau = sign\ B_2$	$\tau > 0$	$s > 0$
	B: Coop	$s < 0$ $\tau > 0$	$\tau > 0$	$sign\ s = sign\ B_5$
$\frac{2b^2+b-1}{2b^2-b-1} > \mu > \frac{2b+1}{2b-1}$	C: Coop	$sign\ s = sign\ B_1$ $sign\ \tau = sign\ B_2$	$\tau > 0$	$s > 0$
	B: Pred	$sign\ s = sign\ B_4$ $\tau > 0$	$\tau > 0$	$sign\ s = sign\ B_6$
$\frac{2b+1}{2b-1} > \mu$	C: Pred	$s > 0$ $sign\ \tau = sign\ B_3$	$\tau > 0$	$s > 0$
	B: Pred	$sign\ s = sign\ B_4$ $\tau > 0$	$\tau > 0$	$sign\ s = sign\ B_6$

Equ: Equilibrium, C: Cournot, B: Bertrand,

Coop Ad: Cooperative Advertising, Pred Ad: Predatory Advertising,

$B_1 = k - (2b+1)\,k_1,\ B_2 = 2b\,(2b+1)\,k_1 - k,$

$B_3 = 2b\,(2b-1)\,k_2 - k,\ B_4 = (2b+1)\,k_4 - k,$

$B_5 = \left[\sqrt{(5b^2-1)} - (b-1)\right] k_3 - k,\ B_6 = \left[\sqrt{(5b^2-1)} + (b+1)\right] k_4 - k.$

5 Conclusion and Further Extensions

In this paper, we first construct a model of advertising in a differentiated duopoly. This is modelled as a two-stage game, in which at the first stage firms decide how much to invest in cooperative advertising, or predatory advertising or both, and at the second stage they engage in product market competition. We show that firms will invest only in one type of advertising, which is determined by the relative effectiveness of the two types of advertising and the degree of product differentiation. We then use this model to explore the policy implications in the context of a Brander-Spencer third-country model of strategic trade. We first analyse optimal policies when governments use both trade and industrial policies and show that these policies are substitutes. We then study optimal policy when governments can use only one policy instrument and show that industrial policy is robust, i.e., governments will always use an advertising subsidy irrespective of the type of advertising and form of market competition. More interestingly we show that for a range of parameter values we also get robust trade policy in which governments always use a trade subsidy irrespective of the type of advertising or form of market competition.

It might be argued that this paper does not capture the increasing return effect of advertising. However, we can construct a similar model to deal with the increasing return effect of advertising.[19] Consider a three-stage game. In the first stage, firms decide how much to spend on advertising. Then follows the two-stage game described in Section 2 with two revisions: the investment cost can be an affine function but cannot exceed the budget set in the first stage. Solving this game is straightforward. Our primary task has been to examine the policy implications for an open economy with a given market structure. With the introduction of fixed cost, an obvious extension would be to endogenise market structure.[20]

This paper does not discuss bilateral intervention. However, it is a natural extension of the analysis presented in Section 4 and we will get symmetric SPNE and the main results will be unchanged. As to other possible extensions, studies of R&D and strategic trade have not explored the implications for robustness of strategic trade policy. So, an obvious question is whether the robustness result we obtained in the advertising case carries over to the R&D case? Finally, Ulph and Ulph (2001) explore the implications for trade and industrial policy of allowing for full government commitment. It would be interesting to reconsider the policy implications of our model using this approach.

We hope to report on the results of these extensions.

[19] See also Footnote 7.

[20] The classic treatment on this topic is Sutton (1991).

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A Appendix

A.1 Proof of Lemma 1

Denote $\sigma_i = (m_i, n_i)$, $i = 1, 2$. We want to find a pair $\left(\sigma_i^*, \sigma_j^*\right)$ satisfying

$$\pi_i^C \left(\sigma_i^*, \sigma_j^*\right) \geq \pi_i^C \left(\sigma_i^{'}, \sigma_j^*\right), \ i = 1, 2, \ j = 1, 2, \ i \neq j, \ \forall \sigma_i^{'} \in R_+^2.$$

Given rival's choice $\sigma_j = (m_j, n_j)$, the optimal response of firm i solves the problem

$$\max_{\{m_i, n_i\}} \pi_i^C, \ s.t. \ m_i \geq 0, \ n_i \geq 0.$$

The Lagrangian is

$$L_i \left(m_i, n_i, \omega_{m_i}, \omega_{n_i}; m_j, n_j\right) = \pi_i^C + \omega_{m_i} m_i + \omega_{n_i} n_i.$$

According to Kuhn-Tucker's Method, the solution is characterized by the following conditions

$$
\begin{gathered}
2bx_i^{*C}\left(\frac{\partial x_i^{*C}}{\partial m_i}\right) = k\left(m_i + n_i\right) - \omega_{m_i}, \\
2bx_i^{*C}\left(\frac{\partial x_i^{*C}}{\partial n_i}\right) = k\left(m_i + n_i\right) - \omega_{n_i}, \\
m_i \geq 0,\ n_i \geq 0, \\
\omega_{m_i} \geq 0,\ \omega_{n_i} \geq 0, \\
\omega_{m_i} m_i = 0,\ \omega_{n_i} n_i = 0.
\end{gathered}
$$

Note first that $m_i = 0$, $n_i = 0$ cannot be a solution because at that point marginal investment cost is zero while marginal investment revenue is strictly positive. Second, if $\mu \neq \frac{2b+1}{2b-1}$, we cannot have both $m_i > 0$ and $n_i > 0$ because marginal revenues from the two kinds of advertising would differ while marginal investment costs are the same.

In particular, when $\mu > \frac{2b+1}{2b-1}$, the marginal investment revenue from cooperative advertising is the larger one, so firm i invests only in cooperative advertising. In fact, by choosing an appropriate ω_{n_i}, all of the above conditions can be satisfied. When $\mu < \frac{2b+1}{2b-1}$, the marginal investment revenue from predatory advertising is the larger one, so firm i invests only in predatory advertising. In fact, by choosing an appropriate ω_{m_i}, all of the above conditions can be satisfied.

The above arguments are valid for both firms. Therefore, the profit function of firm i in the reduced extensive form game is given by

$$
\pi_i^C\left(m_i, n_i, m_j, n_j\right) = \begin{cases} \pi_i^C\left(m_i, m_j\right) & if \quad \mu > \frac{2b+1}{2b-1}, \\ \pi_i^C\left(n_i, n_j\right) & if \quad \mu < \frac{2b+1}{2b-1}. \end{cases}
$$

When both firms invest in cooperative advertising, since

$$
\frac{\partial^2 \pi_i^C}{\partial m_i \partial m_j} = 2b\left(\frac{\partial x_i^{*C}}{\partial m_i}\right)^2 > 0,
$$

cooperative advertising is a strategic complement. We also have

$$
\frac{\partial \pi_i^C}{\partial m_j} = 2b\left(\frac{\partial x_i^{*C}}{\partial m_i}\right) > 0.
$$

When both firms invest in predatory advertising, since

$$
\frac{\partial^2 \pi_i^C}{\partial n_i \partial n_j} = -2b\left(\frac{\partial x_i^{*C}}{\partial n_i}\right)^2 < 0,
$$

predatory advertising is a strategic substitute. We also have

$$
\frac{\partial \pi_i^C}{\partial n_j} = -2b\left(\frac{\partial x_i^{*C}}{\partial n_i}\right) < 0.
$$

Finally, when $\mu > \frac{2b+1}{2b-1}$, the two firms' first order conditions with respect to cooperative advertising simultaneously determine the equilibrium values of m_i^* and m_j^* and $\left(\sigma_i^*, \sigma_j^*\right) = \left[\left(m_i^*, 0\right), \left(m_j^*, 0\right)\right]$. It is easy to show $m_i^* = m_j^*$. When $\mu < \frac{2b+1}{2b-1}$, the two firms' first order conditions with respect to predatory advertising simultaneously determine the equilibrium values of n_i^* and n_j^* and $\left(\sigma_i^*, \sigma_j^*\right) = \left[\left(0, n_i^*\right), \left(0, n_j^*\right)\right]$. It is easy to show $n_i^* = n_j^*$.

A.2 Proof of Lemma 2

Denote $\sigma_i = (m_i, n_i)\,,\ i = 1, 2$. We want to find a pair $\left(\sigma_i^*, \sigma_j^*\right)$ satisfying

$$\pi_i^B\left(\sigma_i^*, \sigma_j^*\right) \geq \pi_i^B\left(\sigma_i^{'}, \sigma_j^*\right),\ \ i = 1, 2,\ \ j = 1, 2,\ \ i \neq j,\ \ \forall \sigma_i^{'} \in R_+^2.$$

Given rival's choice $\sigma_j = (m_j, n_j)$, the optimal response of firm i solves the problem

$$\max_{\{m_i, n_i\}} \pi_i^B,\ \ s.t.\ m_i \geq 0,\ \ n_i \geq 0.$$

The Lagrangian is

$$L_i\left(m_i, n_i, \omega_{m_i}, \omega_{n_i}; m_j, n_j\right) = \pi_i^B + \omega_{m_i} m_i + \omega_{n_i} n_i.$$

According to Kuhn-Tucker's Method, the solution is characterized by the following conditions

$$\begin{gathered}
2\beta(p_i^{*B} - c)\left(\frac{\partial p_i^{*B}}{\partial m_i}\right) = k(m_i + n_i) - \omega_{m_i}, \\
2\beta(p_i^{*B} - c)\left(\frac{\partial p_i^{*B}}{\partial n_i}\right) = k(m_i + n_i) - \omega_{n_i}, \\
m_i \geq 0,\ \ n_i \geq 0, \\
\omega_{m_i} \geq 0,\ \ \omega_{n_i} \geq 0, \\
\omega_{m_i} m_i = 0,\ \ \omega_{n_i} n_i = 0.
\end{gathered}$$

Note first that $m_i = 0$, $n_i = 0$ cannot be a solution because at that point marginal investment cost is zero while marginal investment revenue is strictly positive. Second, if $\mu \neq \frac{2b^2+b-1}{2b^2-b-1}$, we cannot have both $m_i > 0$ and $n_i > 0$ because marginal revenues from the two kinds of advertising would differ while marginal investment costs are the same.

In particular, when $\mu > \frac{2b^2+b-1}{2b^2-b-1}$, the marginal investment revenue from cooperative advertising is the larger one, so firm i invests only in cooperative advertising. In fact, by choosing an appropriate ω_{n_i}, all of the above conditions can be satisfied. When $\mu < \frac{2b^2+b-1}{2b^2-b-1}$, the marginal investment revenue from predatory advertising is the larger one, so firm i invests only in predatory advertising. In fact, by choosing an appropriate ω_{m_i}, all of the above conditions can be satisfied.

The above arguments are valid for both firms. Therefore, the profit function of firm i in the reduced extensive form game is given by

$$\pi_i^B(m_i, n_i, m_j, n_j) = \begin{cases} \pi_i^B(m_i, m_j) & if \quad \mu > \frac{2b^2+b-1}{2b^2-b-1}, \\ \pi_i^B(n_i, n_j) & if \quad \mu < \frac{2b^2+b-1}{2b^2-b-1}. \end{cases}$$

When both firms invest in cooperative advertising, since

$$\frac{\partial^2 \pi_i^B}{\partial m_i \partial m_j} = 2\beta \left(\frac{\partial p_i^{*B}}{\partial m_i} \right)^2 > 0,$$

cooperative advertising is a strategic complement. We also have

$$\frac{\partial \pi_i^B}{\partial m_j} = 2\beta \left(\frac{\partial p_i^{*B}}{\partial m_i} \right) > 0.$$

When both firms invest in predatory advertising, since

$$\frac{\partial^2 \pi_i^B}{\partial n_i \partial n_j} = -2\beta \left(\frac{\partial p_i^{*B}}{\partial n_i} \right)^2 < 0,$$

predatory advertising is a strategic substitute. We also have

$$\frac{\partial \pi_i^B}{\partial n_j} = -2\beta \left(\frac{\partial p_i^{*B}}{\partial n_i} \right) < 0.$$

Finally, When $\mu > \frac{2b^2+b-1}{2b^2-b-1}$, the two firms' first order conditions with respect to cooperative advertising simultaneously determine the equilibrium values of m_i^* and m_j^* and $\left(\sigma_i^*, \sigma_j^*\right) = \left[\left(m_i^*, 0\right), \left(m_j^*, 0\right)\right]$. It is easy to show $m_i^* = m_j^*$. When $\mu < \frac{2b^2+b-1}{2b^2-b-1}$, the two firms' first order conditions with respect to predatory advertising simultaneously determine the equilibrium values of n_i^* and n_j^* and $\left(\sigma_i^*, \sigma_j^*\right) = \left[\left(0, n_i^*\right), \left(0, n_j^*\right)\right]$. It is easy to show $n_i^* = n_j^*$.

A.3 Proof of Proposition 3

Note that

$$\frac{2b+1}{2b-1} - \frac{2b^2+b-1}{2b^2-b-1} = -\frac{2}{(b-1)\left(4b^2-1\right)} < 0.$$

Then, given Lemma 1 and Lemma 2, it is straightforward.

A.4 Proof of Proposition 4

A.4.1 Part 1

Given trade policy s and industrial policy τ, and firms' advertising levels (m_i, n_i) and (m_j, n_j), in the product market firm i maximizes

$$\Pi_i^C = (a_i - bx_i - x_j - c + s)\, x_i,$$

firm j maximizes

$$\Pi_j^C = (a_j - x_i - bx_j - c)\, x_j.$$

The Nash Equilibrium is given by

$$x_i = \frac{a\left[1 + \mu(m_i + m_j) + \left(\frac{2b+1}{2b-1}\right)(n_i - n_j)\right] - c}{2b+1} + \frac{2bs}{4b^2 - 1},$$
$$x_j = \frac{a\left[1 + \mu(m_i + m_j) + \left(\frac{2b+1}{2b-1}\right)(n_j - n_i)\right] - c}{2b+1} - \frac{s}{4b^2 - 1}.$$

Obviously, the comparative statics results on advertising investments are unchanged. However, a trade policy has a direct impact on the equilibrium outcome:

$$\frac{\partial x_i}{\partial s} = \frac{2b}{4b^2 - 1} > 0, \ \frac{\partial x_j}{\partial s} = -\frac{1}{4b^2 - 1} < 0.$$

Firm i's and j's equilibrium profits are

$$\Pi_i^C = b\,(x_i)^2\,,\ \Pi_j^C = b\,(x_j)^2\,,$$

respectively.

We use that to replace the product market competition stage and get the reduced extensive form game. Note that firms' decisions in the investment stage on whether to invest in cooperative or predatory advertising are not changed by the policy instruments, since, as we have just seen, that decision depends only on the relative effectiveness of cooperative advertising and the degree of product differentiation.

Case 1 $\mu > \frac{2b+1}{2b-1}$

If $\mu > \frac{2b+1}{2b-1}$ and both firms invest in cooperative advertising in equilibrium, the equilibrium values are characterized by the two conditions:

$$2bx_i\left(\frac{\partial x_i}{\partial m_i}\right) - km_i + \tau = 0,$$
$$2bx_j\left(\frac{\partial x_j}{\partial m_j}\right) - km_j = 0.$$

Taking derivatives with respect to s and rearranging the equations, we have

$$\begin{bmatrix} \frac{\partial^2 \pi_i^C}{\partial m_i^2} & \frac{\partial^2 \pi_i^C}{\partial m_i \partial m_j} \\ \frac{\partial^2 \pi_j^C}{\partial m_j \partial m_i} & \frac{\partial^2 \pi_j^C}{\partial m_j^2} \end{bmatrix} \begin{bmatrix} \frac{\partial m_i}{\partial s} \\ \frac{\partial m_j}{\partial s} \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2 \pi_i^C}{\partial m_i \partial s} \\ -\frac{\partial^2 \pi_j^C}{\partial m_j \partial s} \end{bmatrix}.$$

Using Cramer's Rule,

$$\frac{\partial m_i}{\partial s} = \left[\frac{k_1}{a\mu (2b-1)}\right]\left[\frac{2b(k-k_1)-k_1}{\Delta}\right],$$
$$\frac{\partial m_j}{\partial s} = \left[\frac{k_1}{a\mu (2b-1)}\right]\left[\frac{(2b+1)k_1-k}{\Delta}\right],$$

where $\Delta = k^2 - 2kk_1$.

Taking derivatives with respect to τ and rearranging the equations, we have

$$\begin{bmatrix} \frac{\partial^2 \pi_i^C}{\partial m_i^2} & \frac{\partial^2 \pi_i^C}{\partial m_i \partial m_j} \\ \frac{\partial^2 \pi_j^C}{\partial m_j \partial m_i} & \frac{\partial^2 \pi_j^C}{\partial m_j^2} \end{bmatrix} \begin{bmatrix} \frac{\partial m_i}{\partial \tau} \\ \frac{\partial m_j}{\partial \tau} \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2 \pi_i^C}{\partial m_i \partial \tau} \\ 0 \end{bmatrix}.$$

Using Cramer's Rule,

$$\frac{\partial m_i}{\partial \tau} = \frac{k-k_1}{\Delta},\ \frac{\partial m_j}{\partial \tau} = \frac{k_1}{\Delta},$$

where $\Delta = k^2 - 2kk_1$.

Given the equilibrium of the product market competition and cooperative advertising is the equilibrium type of investment, in the first stage of the game the government of country i chooses trade policy s and industrial policy τ to maximize national welfare

$$W(s,\tau) = \pi_i^C - sx_i - \tau m_i.$$

The first order conditions are as follows.

$$\frac{\partial W}{\partial s} = \left\{\left[\left(\frac{\partial a_i}{\partial m_j} - \frac{\partial x_j}{\partial m_j}\right)\frac{\partial m_j}{\partial s}\right] - \frac{\partial x_j}{\partial s}\right\} x_i - \frac{\partial x_i}{\partial s}(\cdot)\, s - \frac{\partial m_i}{\partial s}\tau = 0,$$
$$\frac{\partial W}{\partial \tau} = \left(\frac{\partial a_i}{\partial m_j} - \frac{\partial x_j}{\partial m_j}\right)\frac{\partial m_j}{\partial \tau} x_i - \frac{\partial x_i}{\partial \tau}(\cdot)\, s - \frac{\partial m_i}{\partial \tau}\tau = 0.$$

Using matrix notation, we have

$$\begin{bmatrix} \frac{\partial x_i}{\partial s}(\cdot) & \frac{\partial m_i}{\partial s} \\ \frac{\partial x_i}{\partial \tau}(\cdot) & \frac{\partial m_i}{\partial \tau} \end{bmatrix} \begin{bmatrix} s \\ \tau \end{bmatrix} = \begin{bmatrix} \left\{\left[\left(\frac{\partial a_i}{\partial m_j} - \frac{\partial x_j}{\partial m_j}\right)\frac{\partial m_j}{\partial s}\right] - \frac{\partial x_j}{\partial s}\right\} x_i \\ \left(\frac{\partial a_i}{\partial m_j} - \frac{\partial x_j}{\partial m_j}\right)\frac{\partial m_j}{\partial \tau} x_i \end{bmatrix},$$

where $\frac{\partial x_i}{\partial s}(\cdot) = \frac{\partial x_i}{\partial m_i}\frac{\partial m_i}{\partial s} + \frac{\partial x_i}{\partial m_j}\frac{\partial m_j}{\partial s} + \frac{\partial x_i}{\partial s}$, $\frac{\partial x_i}{\partial \tau}(\cdot) = \frac{\partial x_i}{\partial m_i}\frac{\partial m_i}{\partial \tau} + \frac{\partial x_i}{\partial m_j}\frac{\partial m_j}{\partial \tau}$.

We use Cramer's Rule to solve this linear equation system.

Denote the determinant of the coefficient matrix of the above linear equation system by D.

$$\begin{aligned} D &= \frac{\partial x_i}{\partial s}(\cdot)\frac{\partial m_i}{\partial \tau} - \frac{\partial x_i}{\partial \tau}(\cdot)\frac{\partial m_i}{\partial s} \\ &= \frac{\partial x_i}{\partial m_j}\left(\frac{\partial m_j}{\partial s}\frac{\partial m_i}{\partial \tau} - \frac{\partial m_i}{\partial s}\frac{\partial m_j}{\partial \tau}\right) + \frac{\partial x_i}{\partial s}\frac{\partial m_i}{\partial \tau}. \end{aligned}$$

Note that

$$\begin{aligned} & \frac{\partial m_j}{\partial s}\frac{\partial m_i}{\partial \tau} - \frac{\partial m_i}{\partial s}\frac{\partial m_j}{\partial \tau} \\ = & \left[\frac{k_1}{a\mu(2b-1)}\right]\left\{\left[\frac{(2b+1)k_1 - k}{\Delta}\right]\left(\frac{k-k_1}{\Delta}\right) - \left[\frac{2b(k-k_1)-k_1}{\Delta}\right]\left(\frac{k_1}{\Delta}\right)\right\} \\ = & -\left[\frac{k_1}{a\mu(2b-1)\Delta}\right] \\ < & \ 0. \end{aligned}$$

Then

$$\begin{aligned} D &= -\left(\frac{a\mu}{2b+1}\right)\left[\frac{k_1}{a\mu(2b-1)\Delta}\right] + \left(\frac{2b}{4b^2-1}\right)\left(\frac{k-k_1}{\Delta}\right) \\ &= \left(\frac{1}{4b^2-1}\right)\left[\frac{2b(k-k_1)-k_1}{\Delta}\right] \\ &> 0. \end{aligned}$$

Next, we have

$$\begin{aligned} D_1 &= \left\{\left[\left(\frac{\partial a_i}{\partial m_j} - \frac{\partial x_j}{\partial m_j}\right)\frac{\partial m_j}{\partial s} - \frac{\partial x_j}{\partial s}\right]\frac{\partial m_i}{\partial \tau} - \left(\frac{\partial a_i}{\partial m_j} - \frac{\partial x_j}{\partial m_j}\right)\frac{\partial m_j}{\partial \tau}\frac{\partial m_i}{\partial s}\right\} x_i \\ &= \left[\left(\frac{\partial a_i}{\partial m_j} - \frac{\partial x_j}{\partial m_j}\right)\left(\frac{\partial m_j}{\partial s}\frac{\partial m_i}{\partial \tau} - \frac{\partial m_i}{\partial s}\frac{\partial m_j}{\partial \tau}\right) - \frac{\partial x_j}{\partial s}\frac{\partial m_i}{\partial \tau}\right] x_i \\ &= \left\{-\left(\frac{2ab\mu}{2b+1}\right)\left[\frac{k_1}{a\mu(2b-1)\Delta}\right] + \left(\frac{1}{4b^2-1}\right)\left(\frac{k-k_1}{\Delta}\right)\right\} x_i \\ &= \left\{\left(\frac{1}{4b^2-1}\right)\left[\frac{k-(2b+1)k_1}{\Delta}\right]\right\} x_i, \end{aligned}$$

$$\begin{aligned}
D_2 &= \left\{\frac{\partial x_i}{\partial s}(\cdot)\left(\frac{\partial a_i}{\partial m_j}-\frac{\partial x_j}{\partial m_j}\right)\frac{\partial m_j}{\partial \tau}-\frac{\partial x_i}{\partial \tau}(\cdot)\left[\left(\frac{\partial a_i}{\partial m_j}-\frac{\partial x_j}{\partial m_j}\right)\frac{\partial m_j}{\partial s}-\frac{\partial x_j}{\partial s}\right]\right\}x_i \\
&= \left\{\begin{array}{c} \frac{\partial x_i}{\partial m_i}\left(\frac{\partial a_i}{\partial m_j}-\frac{\partial x_j}{\partial m_j}\right)\left(\frac{\partial m_i}{\partial s}\frac{\partial m_j}{\partial \tau}-\frac{\partial m_j}{\partial s}\frac{\partial m_i}{\partial \tau}\right)+\left(\frac{\partial a_i}{\partial m_j}-\frac{\partial x_j}{\partial m_j}\right)\frac{\partial x_i}{\partial s}\frac{\partial m_j}{\partial \tau} \\ +\left(\frac{\partial x_i}{\partial m_i}\frac{\partial m_i}{\partial \tau}+\frac{\partial x_i}{\partial m_j}\frac{\partial m_j}{\partial \tau}\right)\frac{\partial x_j}{\partial s} \end{array}\right\}x_i \\
&= \left\{\begin{array}{c} \left(\frac{a\mu}{2b+1}\right)\left(\frac{2ab\mu}{2b+1}\right)\left[\frac{k_1}{a\mu(2b-1)\Delta}\right]+\left(\frac{2b}{4b^2-1}\right)\left(\frac{2ab\mu}{2b+1}\right)\left(\frac{k_1}{\Delta}\right) \\ -\left(\frac{a\mu}{2b+1}\right)\left(\frac{1}{4b^2-1}\right)\left(\frac{k}{\Delta}\right) \end{array}\right\}x_i \\
&= \left\{\left[\frac{a\mu}{(2b+1)(4b^2-1)}\right]\left[\frac{2b(2b+1)k_1-k}{\Delta}\right]\right\}x_i.
\end{aligned}$$

Because

$$s=\frac{D_1}{D},\ \tau=\frac{D_2}{D},$$

we have

$$sign\ s = sign\ \left[k-(2b+1)k_1\right],\ sign\ \tau = sign\ \left[2b(2b+1)k_1-k\right]. \tag{A1}$$

Case 2 $\mu < \frac{2b+1}{2b-1}$

If $\mu < \frac{2b+1}{2b-1}$ and both firms invest in predatory advertising in equilibrium, the equilibrium values are characterized by the two conditions:

$$\begin{aligned}
2bx_i\left(\frac{\partial x_i}{\partial n_i}\right)-kn_i+\tau &= 0, \\
2bx_j\left(\frac{\partial x_j}{\partial n_j}\right)-kn_j &= 0.
\end{aligned}$$

Taking derivatives with respect to s and rearranging the equations, we have

$$\begin{bmatrix} \frac{\partial^2 \pi_i^C}{\partial n_i^2} & \frac{\partial^2 \pi_i^C}{\partial n_i \partial n_j} \\ \frac{\partial^2 \pi_j^C}{\partial n_j \partial n_i} & \frac{\partial^2 \pi_j^C}{\partial n_j^2} \end{bmatrix}\begin{bmatrix} \frac{\partial n_i}{\partial s} \\ \frac{\partial n_j}{\partial s} \end{bmatrix}=\begin{bmatrix} -\frac{\partial^2 \pi_i^C}{\partial n_i \partial s} \\ -\frac{\partial^2 \pi_j^C}{\partial n_j \partial s} \end{bmatrix}.$$

Using Cramer's Rule,

$$\begin{aligned}
\frac{\partial n_i}{\partial s} &= \left[\frac{k_2}{a(2b+1)}\right]\left[\frac{2b(k-k_2)+k_2}{\Delta}\right], \\
\frac{\partial n_j}{\partial s} &= \left[\frac{k_2}{a(2b-1)}\right]\left[\frac{-(2b-1)k_2-k}{\Delta}\right],
\end{aligned}$$

where $\Delta = k^2-2kk_2$.

Taking derivatives with respect to τ and rearranging the equations, we have

$$\begin{bmatrix} \frac{\partial^2 \pi_i^C}{\partial n_i^2} & \frac{\partial^2 \pi_i^C}{\partial n_i \partial n_j} \\ \frac{\partial^2 \pi_j^C}{\partial n_j \partial n_i} & \frac{\partial^2 \pi_j^C}{\partial n_j^2} \end{bmatrix}\begin{bmatrix} \frac{\partial n_i}{\partial \tau} \\ \frac{\partial n_j}{\partial \tau} \end{bmatrix}=\begin{bmatrix} -\frac{\partial^2 \pi_i^C}{\partial n_i \partial \tau} \\ 0 \end{bmatrix}.$$

Using Cramer's Rule,

$$\frac{\partial n_i}{\partial \tau} = \frac{k - k_2}{\Delta}, \ \frac{\partial n_j}{\partial \tau} = -\frac{k_2}{\Delta},$$

where $\Delta = k^2 - 2kk_2$.

Given the equilibrium of the product market competition and predatory advertising is the equilibrium type of investment, in the first stage of the game the government of country i chooses trade policy s and industrial policy τ to maximize national welfare

$$W(s, \tau) = \pi_i^C - sx_i - \tau n_i.$$

The first order conditions are as follows.

$$\frac{\partial W}{\partial s} = \left\{ \left[\left(\frac{\partial a_i}{\partial n_j} - \frac{\partial x_j}{\partial n_j} \right) \frac{\partial n_j}{\partial s} \right] - \frac{\partial x_j}{\partial s} \right\} x_i - \frac{\partial x_i}{\partial s} (\cdot) \, s - \frac{\partial n_i}{\partial s} \tau = 0,$$
$$\frac{\partial W}{\partial \tau} = \left(\frac{\partial a_i}{\partial n_j} - \frac{\partial x_j}{\partial n_j} \right) \frac{\partial n_j}{\partial \tau} x_i - \frac{\partial x_i}{\partial \tau} (\cdot) \, s - \frac{\partial n_i}{\partial \tau} \tau = 0.$$

Using matrix notation, we have

$$\begin{bmatrix} \frac{\partial x_i}{\partial s}(\cdot) & \frac{\partial n_i}{\partial s} \\ \frac{\partial x_i}{\partial \tau}(\cdot) & \frac{\partial n_i}{\partial \tau} \end{bmatrix} \begin{bmatrix} s \\ \tau \end{bmatrix} = \begin{bmatrix} \left\{ \left[\left(\frac{\partial a_i}{\partial n_j} - \frac{\partial x_j}{\partial n_j} \right) \frac{\partial n_j}{\partial s} \right] - \frac{\partial x_j}{\partial s} \right\} x_i \\ \left(\frac{\partial a_i}{\partial n_j} - \frac{\partial x_j}{\partial n_j} \right) \frac{\partial n_j}{\partial \tau} x_i \end{bmatrix},$$

where $\frac{\partial x_i}{\partial s}(\cdot) = \frac{\partial x_i}{\partial n_i}\frac{\partial n_i}{\partial s} + \frac{\partial x_i}{\partial n_j}\frac{\partial n_j}{\partial s} + \frac{\partial x_i}{\partial s}$, $\frac{\partial x_i}{\partial \tau}(\cdot) = \frac{\partial x_i}{\partial n_i}\frac{\partial n_i}{\partial \tau} + \frac{\partial x_i}{\partial n_j}\frac{\partial n_j}{\partial \tau}$.

we use Cramer's Rule to solve this linear equation system.

Denote the determinant of the coefficient matrix of the above linear equation system by D.

$$\begin{aligned} D &= \frac{\partial x_i}{\partial s}(\cdot)\frac{\partial n_i}{\partial \tau} - \frac{\partial x_i}{\partial \tau}(\cdot)\frac{\partial n_i}{\partial s} \\ &= \frac{\partial x_i}{\partial n_j}\left(\frac{\partial n_j}{\partial s}\frac{\partial n_i}{\partial \tau} - \frac{\partial n_i}{\partial s}\frac{\partial n_j}{\partial \tau} \right) + \frac{\partial x_i}{\partial s}\frac{\partial n_i}{\partial \tau}. \end{aligned}$$

Note that

$$\begin{aligned} & \frac{\partial n_j}{\partial s}\frac{\partial n_i}{\partial \tau} - \frac{\partial n_i}{\partial s}\frac{\partial n_j}{\partial \tau} \\ = & \left[\frac{k_2}{a(2b+1)} \right] \left\{ \left[\frac{-(2b-1)k_2 - k}{\Delta} \right] \left(\frac{k-k_2}{\Delta} \right) - \left[\frac{2b(k-k_2)+k_2}{\Delta} \right] \left(-\frac{k_2}{\Delta} \right) \right\} \\ = & -\left[\frac{k_2}{a(2b+1)\Delta} \right] \\ < & \ 0. \end{aligned}$$

Then

$$
\begin{aligned}
D &= \left(\frac{a}{2b-1}\right)\left[\frac{k_2}{a\,(2b+1)\,\Delta}\right]+\left(\frac{2b}{4b^2-1}\right)\left(\frac{k-k_2}{\Delta}\right)\\
&= \left(\frac{1}{4b^2-1}\right)\left[\frac{2b\,(k-k_2)+k_2}{\Delta}\right]\\
&> 0.
\end{aligned}
$$

Next, we have

$$
\begin{aligned}
D_1 &= \left\{\left[\left(\frac{\partial a_i}{\partial n_j}-\frac{\partial x_j}{\partial n_j}\right)\frac{\partial n_j}{\partial s}-\frac{\partial x_j}{\partial s}\right]\frac{\partial n_i}{\partial \tau}-\left(\frac{\partial a_i}{\partial n_j}-\frac{\partial x_j}{\partial n_j}\right)\frac{\partial n_j}{\partial \tau}\frac{\partial n_i}{\partial s}\right\}x_i\\
&= \left[\left(\frac{\partial a_i}{\partial n_j}-\frac{\partial x_j}{\partial n_j}\right)\left(\frac{\partial n_j}{\partial s}\frac{\partial n_i}{\partial \tau}-\frac{\partial n_i}{\partial s}\frac{\partial n_j}{\partial \tau}\right)-\frac{\partial x_j}{\partial s}\frac{\partial n_i}{\partial \tau}\right]x_i\\
&> 0.
\end{aligned}
$$

This is because

$$
\frac{\partial a_i}{\partial n_j}-\frac{\partial x_j}{\partial n_j}<0,\ \frac{\partial n_j}{\partial s}\frac{\partial n_i}{\partial \tau}-\frac{\partial n_i}{\partial s}\frac{\partial n_j}{\partial \tau}<0,\ \frac{\partial x_j}{\partial s}<0,\ \frac{\partial n_i}{\partial \tau}>0.
$$

$$
\begin{aligned}
D_2 &= \left\{\frac{\partial x_i}{\partial s}(\cdot)\left(\frac{\partial a_i}{\partial m_j}-\frac{\partial x_j}{\partial m_j}\right)\frac{\partial m_j}{\partial \tau}-\frac{\partial x_i}{\partial \tau}(\cdot)\left[\left(\frac{\partial a_i}{\partial m_j}-\frac{\partial x_j}{\partial m_j}\right)\frac{\partial m_j}{\partial s}-\frac{\partial x_j}{\partial s}\right]\right\}x_i\\
&= \left\{\begin{array}{c}\frac{\partial x_i}{\partial n_i}\left(\frac{\partial a_i}{\partial n_j}-\frac{\partial x_j}{\partial n_j}\right)\left(\frac{\partial n_i}{\partial s}\frac{\partial n_j}{\partial \tau}-\frac{\partial n_j}{\partial s}\frac{\partial n_i}{\partial \tau}\right)+\left(\frac{\partial a_i}{\partial n_j}-\frac{\partial x_j}{\partial n_j}\right)\frac{\partial x_i}{\partial s}\frac{\partial n_j}{\partial \tau}\\ +\left(\frac{\partial x_i}{\partial n_i}\frac{\partial n_i}{\partial \tau}+\frac{\partial x_i}{\partial n_j}\frac{\partial n_j}{\partial \tau}\right)\frac{\partial x_j}{\partial s}\end{array}\right\}x_i\\
&= \left\{\begin{array}{c}\left(\frac{a}{2b-1}\right)\left(-\frac{2ab}{2b-1}\right)\left[\frac{k_2}{a(2b+1)\Delta}\right]+\left(\frac{2b}{4b^2-1}\right)\left(-\frac{2ab}{2b-1}\right)\left(-\frac{k_2}{\Delta}\right)\\ -\left(\frac{a}{2b-1}\right)\left(\frac{1}{4b^2-1}\right)\left(\frac{k}{\Delta}\right)\end{array}\right\}x_i\\
&= \left\{\left[\frac{a}{(2b-1)\,(4b^2-1)}\right]\left[\frac{2b\,(2b-1)\,k_2-k}{\Delta}\right]\right\}x_i.
\end{aligned}
$$

Because

$$
s=\frac{D_1}{D},\ \tau=\frac{D_2}{D},
$$

we have

$$
s>0,\ sign\ \tau=sign\ \left[2b\,(2b-1)\,k_2-k\right]. \tag{A2}
$$

A.4.2 Part 2

Given trade policy s and industrial policy τ, and firms' advertising levels (m_i, n_i) and (m_j, n_j), in the product market firm i maximizes

$$
\Pi_i^B=(p_i-c+s)\,(\alpha_i-\beta p_i+\gamma p_j)\,,
$$

firm j maximizes

$$\Pi_j^B = (p_j - c)(\alpha_j - \beta p_j + \gamma p_i).$$

The Nash Equilibrium is given by

$$p_i = \frac{\alpha\left[1+\mu(m_i+m_j)+\left(\frac{b+1}{b-1}\right)\left(\frac{2\beta-\gamma}{2\beta+\gamma}\right)(n_i-n_j)\right]+\beta c}{2\beta-\gamma} - \frac{2\beta^2 s}{4\beta^2-\gamma^2},$$
$$p_j = \frac{\alpha\left[1+\mu(m_i+m_j)+\left(\frac{b+1}{b-1}\right)\left(\frac{2\beta-\gamma}{2\beta+\gamma}\right)(n_j-n_i)\right]+\beta c}{2\beta-\gamma} - \frac{\beta\gamma s}{4\beta^2-\gamma^2}.$$

Obviously, the comparative statics results on advertising investments are unchanged. However, a trade policy has a direct impact on the equilibrium outcome:

$$\frac{\partial p_i}{\partial s} = -\frac{2\beta^2}{4\beta^2-\gamma^2} = -\frac{2b^2}{4b^2-1} < 0,$$
$$\frac{\partial p_j}{\partial s} = -\frac{\beta\gamma}{4\beta^2-\gamma^2} = -\frac{b}{4b^2-1} < 0.$$

Firm i's and j's equilibrium profits are

$$\Pi_i^B = \beta(p_i - c + s)^2, \ \Pi_j^B = \beta(p_j - c)^2,$$

respectively.

We use that to replace the product market competition stage and get the reduced extensive form game. Note that firms' decisions in the investment stage on whether to invest in cooperative or predatory advertising are not changed by the policy instruments, since, as we have just seen, that decision depends only on the relative effectiveness of cooperative advertising and the degree of product differentiation.

Case 1 $\mu > \frac{2b^2+b-1}{2b^2-b-1}$

If $\mu > \frac{2b^2+b-1}{2b^2-b-1}$ and both firms invest in cooperative advertising in equilibrium, the equilibrium values are characterized by the two conditions:

$$2\beta(p_i - c + s)\left(\frac{\partial p_i}{\partial m_i}\right) - km_i + \tau = 0,$$
$$2\beta(p_j - c)\left(\frac{\partial p_j}{\partial m_j}\right) - km_j = 0.$$

Taking derivatives with respect to s and rearranging the equations, we have

$$\begin{bmatrix} \frac{\partial^2 \pi_i^B}{\partial m_i^2} & \frac{\partial^2 \pi_i^B}{\partial m_i \partial m_j} \\ \frac{\partial^2 \pi_j^B}{\partial m_j \partial m_i} & \frac{\partial^2 \pi_j^B}{\partial m_j^2} \end{bmatrix} \begin{bmatrix} \frac{\partial m_i}{\partial s} \\ \frac{\partial m_j}{\partial s} \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2 \pi_i^B}{\partial m_i \partial s} \\ -\frac{\partial^2 \pi_j^B}{\partial m_j \partial s} \end{bmatrix}.$$

Using Cramer's Rule,

$$\frac{\partial m_i}{\partial s} = \left[\frac{k_3}{a\mu\left(b-1\right)\left(2b+1\right)}\right]\left[\frac{\left(2b^2-1\right)\left(k-k_3\right)-bk_3}{\Delta}\right],$$
$$\frac{\partial m_j}{\partial s} = \left[\frac{k_3}{a\mu\left(b-1\right)\left(2b+1\right)}\right]\left[\frac{\left(b+1\right)\left(2b-1\right)k_3-bk}{\Delta}\right],$$

where $\Delta = k^2 - 2kk_3$.

Taking derivatives with respect to τ and rearranging the equations, we have

$$\begin{bmatrix} \frac{\partial^2 \pi_i^B}{\partial m_i^2} & \frac{\partial^2 \pi_i^B}{\partial m_i \partial m_j} \\ \frac{\partial^2 \pi_j^B}{\partial m_j \partial m_i} & \frac{\partial^2 \pi_j^B}{\partial m_j^2} \end{bmatrix} \begin{bmatrix} \frac{\partial m_i}{\partial \tau} \\ \frac{\partial m_j}{\partial \tau} \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2 \pi_i^B}{\partial m_i \partial \tau} \\ 0 \end{bmatrix}.$$

Using Cramer's Rule,

$$\frac{\partial m_i}{\partial \tau} = \frac{k-k_3}{\Delta}, \ \frac{\partial m_j}{\partial \tau} = \frac{k_3}{\Delta},$$

where $\Delta = k^2 - 2kk_3$.

Given the equilibrium of the product market competition and cooperative advertising is the equilibrium type of investment, in the first stage of the game the government of country i chooses trade policy s and industrial policy τ to maximize national welfare

$$W\left(s,\tau\right) = \pi_i^B - sx_i - \tau m_i.$$

The first order conditions are as follows.

$$\frac{\partial W}{\partial s} = \left\{\left[\left(\frac{\partial \alpha_i}{\partial m_j} + \gamma\frac{\partial p_j}{\partial m_j}\right)\frac{\partial m_j}{\partial s}\right] + \gamma\frac{\partial p_j}{\partial s}\right\}\left(p_i - c + s\right) - \frac{\partial x_i}{\partial s}\left(\cdot\right)s - \frac{\partial m_i}{\partial s}\tau = 0,$$
$$\frac{\partial W}{\partial \tau} = \left(\frac{\partial \alpha_i}{\partial m_j} + \gamma\frac{\partial p_j}{\partial m_j}\right)\frac{\partial m_j}{\partial \tau}\left(p_i - c + s\right) - \frac{\partial x_i}{\partial \tau}\left(\cdot\right)s - \frac{\partial m_i}{\partial \tau}\tau = 0.$$

Using matrix notation, we have

$$\begin{bmatrix} \frac{\partial x_i}{\partial s}\left(\cdot\right) & \frac{\partial m_i}{\partial s} \\ \frac{\partial x_i}{\partial \tau}\left(\cdot\right) & \frac{\partial m_i}{\partial \tau} \end{bmatrix}\begin{bmatrix} s \\ \tau \end{bmatrix} = \begin{bmatrix} \left\{\left[\left(\frac{\partial \alpha_i}{\partial m_j} + \gamma\frac{\partial p_j}{\partial m_j}\right)\frac{\partial m_j}{\partial s}\right] + \gamma\frac{\partial p_j}{\partial s}\right\}\left(p_i - c + s\right) \\ \left(\frac{\partial \alpha_i}{\partial m_j} + \gamma\frac{\partial p_j}{\partial m_j}\right)\frac{\partial m_j}{\partial \tau}\left(p_i - c + s\right) \end{bmatrix},$$

where $\frac{\partial x_i}{\partial s}\left(\cdot\right) = \beta\frac{\partial(p_i-c+s)}{\partial s} = \beta\left(\frac{\partial p_i}{\partial m_i}\frac{\partial m_i}{\partial s} + \frac{\partial p_i}{\partial m_j}\frac{\partial m_j}{\partial s} + \frac{\partial p_i}{\partial s} + 1\right)$, $\frac{\partial x_i}{\partial \tau}\left(\cdot\right) = \beta\frac{\partial(p_i-c+s)}{\partial \tau} = \beta\left(\frac{\partial p_i}{\partial m_i}\frac{\partial m_i}{\partial \tau} + \frac{\partial p_i}{\partial m_j}\frac{\partial m_j}{\partial \tau}\right)$.

We use Cramer's Rule to solve this linear equation system.

Denote the determinant of the coefficient matrix of the above linear equation system by D.

$$\begin{aligned} D &= \frac{\partial x_i}{\partial s}\left(\cdot\right)\frac{\partial m_i}{\partial \tau} - \frac{\partial x_i}{\partial \tau}\left(\cdot\right)\frac{\partial m_i}{\partial s} \\ &= \beta\left[\frac{\partial p_i}{\partial m_j}\left(\frac{\partial m_j}{\partial s}\frac{\partial m_i}{\partial \tau} - \frac{\partial m_i}{\partial s}\frac{\partial m_j}{\partial \tau}\right) + \left(\frac{\partial p_i}{\partial s} + 1\right)\frac{\partial m_i}{\partial \tau}\right]. \end{aligned}$$

Note that

$$\begin{aligned}
& \frac{\partial m_j}{\partial s}\frac{\partial m_i}{\partial \tau}-\frac{\partial m_i}{\partial s}\frac{\partial m_j}{\partial \tau} \\
= & \left[\frac{k_3}{a\mu (b-1)(2b+1)}\right]\left[\frac{(b+1)(2b-1)k_3-bk}{\Delta}\right]\left(\frac{k-k_3}{\Delta}\right) \\
& -\left[\frac{k_3}{a\mu (b-1)(2b+1)}\right]\left[\frac{(2b^2-1)(k-k_3)-bk_3}{\Delta}\right]\left(\frac{k_3}{\Delta}\right) \\
= & -\left[\frac{bk_3}{a\mu (b-1)(2b+1)\Delta}\right] \\
< & \ 0.
\end{aligned}$$

Then

$$\begin{aligned}
D &= \left(\frac{b}{b^2-1}\right)\left\{\left[\frac{a\mu (b-1)}{2b-1}\right]\left[-\frac{bk_3}{a\mu (b-1)(2b+1)\Delta}\right]+\left(\frac{2b^2-1}{4b^2-1}\right)\left(\frac{k-k_3}{\Delta}\right)\right\} \\
&= \left(\frac{b}{b^2-1}\right)\left(\frac{1}{4b^2-1}\right)\left[\frac{(2b^2-1)(k-k_3)-bk_3}{\Delta}\right] \\
&> 0.
\end{aligned}$$

Next, we have

$$\begin{aligned}
D_1 &= \left\{\left[\left(\frac{\partial \alpha_i}{\partial m_j}+\gamma\frac{\partial p_j}{\partial m_j}\right)\frac{\partial m_j}{\partial s}+\gamma\frac{\partial p_j}{\partial s}\right]\frac{\partial m_i}{\partial \tau}-\left(\frac{\partial \alpha_i}{\partial m_j}+\gamma\frac{\partial p_j}{\partial m_j}\right)\frac{\partial m_j}{\partial \tau}\frac{\partial m_i}{\partial s}\right\}(p_i-c+s) \\
&= \left[\left(\frac{\partial \alpha_i}{\partial m_j}+\gamma\frac{\partial p_j}{\partial m_j}\right)\left(\frac{\partial m_j}{\partial s}\frac{\partial m_i}{\partial \tau}-\frac{\partial m_i}{\partial s}\frac{\partial m_j}{\partial \tau}\right)+\gamma\frac{\partial p_j}{\partial s}\frac{\partial m_i}{\partial \tau}\right](p_i-c+s) \\
&< 0.
\end{aligned}$$

This is because

$$\frac{\partial \alpha_i}{\partial m_j}+\gamma\frac{\partial p_j}{\partial m_j}>0,\ \frac{\partial m_j}{\partial s}\frac{\partial m_i}{\partial \tau}-\frac{\partial m_i}{\partial s}\frac{\partial m_j}{\partial \tau}<0,\ \frac{\partial p_j}{\partial s}<0,\ \frac{\partial m_i}{\partial \tau}>0.$$

$$\begin{aligned}
D_2 &= \left\{\frac{\partial x_i}{\partial s}(\cdot)\left(\frac{\partial \alpha_i}{\partial m_j}+\gamma\frac{\partial p_j}{\partial m_j}\right)\frac{\partial m_j}{\partial \tau}-\frac{\partial x_i}{\partial \tau}(\cdot)\left[\left(\frac{\partial \alpha_i}{\partial m_j}+\gamma\frac{\partial p_j}{\partial m_j}\right)\frac{\partial m_j}{\partial s}+\gamma\frac{\partial p_j}{\partial s}\right]\right\}(p_i-c+s) \\
&= \beta\left\{\begin{array}{c} \frac{\partial p_i}{\partial m_i}\left(\frac{\partial \alpha_i}{\partial m_j}+\gamma\frac{\partial p_j}{\partial m_j}\right)\left(\frac{\partial m_i}{\partial s}\frac{\partial m_j}{\partial \tau}-\frac{\partial m_j}{\partial s}\frac{\partial m_i}{\partial \tau}\right)+\left(\frac{\partial \alpha_i}{\partial m_j}+\gamma\frac{\partial p_j}{\partial m_j}\right)\left(\frac{\partial p_i}{\partial s}+1\right)\frac{\partial m_j}{\partial \tau} \\ -\left(\frac{\partial p_i}{\partial m_i}\frac{\partial m_i}{\partial \tau}+\frac{\partial p_i}{\partial m_j}\frac{\partial m_j}{\partial \tau}\right)\gamma\frac{\partial p_j}{\partial s} \end{array}\right\}(p_i-c+s) \\
&> 0.
\end{aligned}$$

This is because

$$\frac{\partial p_i}{\partial m_i}>0,\ \frac{\partial p_i}{\partial m_j}>0,\ \frac{\partial \alpha_i}{\partial m_j}+\gamma\frac{\partial p_j}{\partial m_j}>0,\ \frac{\partial m_j}{\partial s}\frac{\partial m_i}{\partial \tau}-\frac{\partial m_i}{\partial s}\frac{\partial m_j}{\partial \tau}<0,$$
$$\frac{\partial p_i}{\partial s}+1>0,\ \frac{\partial p_j}{\partial s}<0,\ \frac{\partial m_i}{\partial \tau}>0,\ \frac{\partial m_j}{\partial \tau}>0.$$

Because

$$s = \frac{D_1}{D},\ \tau = \frac{D_2}{D},$$

we have

$$s < 0,\ \tau > 0. \tag{A3}$$

Case 2 $\mu < \frac{2b^2+b-1}{2b^2-b-1}$

If $\mu < \frac{2b^2+b-1}{2b^2-b-1}$ and both firms invest in predatory advertising in equilibrium, the equilibrium values are characterized by the two conditions:

$$2\beta\left(p_i - c + s\right)\left(\frac{\partial p_i}{\partial n_i}\right) - kn_i + \tau = 0,$$
$$2\beta\left(p_j - c\right)\left(\frac{\partial p_j}{\partial n_j}\right) - kn_j = 0.$$

Taking derivatives with respect to s and rearranging the equations, we have

$$\begin{bmatrix} \frac{\partial^2 \pi_i^B}{\partial n_i^2} & \frac{\partial^2 \pi_i^B}{\partial n_i \partial n_j} \\ \frac{\partial^2 \pi_j^B}{\partial n_j \partial n_i} & \frac{\partial^2 \pi_j^B}{\partial n_j^2} \end{bmatrix} \begin{bmatrix} \frac{\partial n_i}{\partial s} \\ \frac{\partial n_j}{\partial s} \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2 \pi_i^B}{\partial n_i \partial s} \\ -\frac{\partial^2 \pi_j^B}{\partial n_j \partial s} \end{bmatrix}.$$

Using Cramer's Rule,

$$\frac{\partial n_i}{\partial s} = \left[\frac{k_4}{a\left(b+1\right)\left(2b-1\right)}\right]\left[\frac{\left(2b^2-1\right)\left(k-k_4\right)+bk_4}{\Delta}\right],$$
$$\frac{\partial n_j}{\partial s} = \left[\frac{k_4}{a\left(b+1\right)\left(2b-1\right)}\right]\left[\frac{-\left(b-1\right)\left(2b+1\right)k_4 - bk}{\Delta}\right].$$

where $\Delta = k^2 - 2kk_4$.

Taking derivatives with respect to τ and rearranging the equations, we have

$$\begin{bmatrix} \frac{\partial^2 \pi_i^B}{\partial n_i^2} & \frac{\partial^2 \pi_i^B}{\partial n_i \partial n_j} \\ \frac{\partial^2 \pi_j^B}{\partial n_j \partial n_i} & \frac{\partial^2 \pi_j^B}{\partial n_j^2} \end{bmatrix} \begin{bmatrix} \frac{\partial n_i}{\partial \tau} \\ \frac{\partial n_j}{\partial \tau} \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2 \pi_i^B}{\partial n_i \partial \tau} \\ 0 \end{bmatrix}.$$

Using Cramer's Rule,

$$\frac{\partial n_i}{\partial \tau} = \frac{k-k_4}{\Delta},\ \frac{\partial n_j}{\partial \tau} = -\frac{k_4}{\Delta},$$

where $\Delta = k^2 - 2kk_4$.

Given the equilibrium of the product market competition and predatory advertising is the equilibrium type of investment, in the first stage of the game the government of country i chooses trade policy s and industrial policy τ to maximize national welfare

$$W\left(s,\tau\right) = \pi_i^B - sx_i - \tau n_i.$$

The first order conditions are as follows.

$$\frac{\partial W}{\partial s}=\left\{\left[\left(\frac{\partial \alpha_i}{\partial n_j}+\gamma\frac{\partial p_j}{\partial n_j}\right)\frac{\partial n_j}{\partial s}\right]+\gamma\frac{\partial p_j}{\partial s}\right\}(p_i-c+s)-\frac{\partial x_i}{\partial s}(\cdot)\,s-\frac{\partial n_i}{\partial s}\tau=0,$$

$$\frac{\partial W}{\partial \tau}=\left(\frac{\partial \alpha_i}{\partial n_j}+\gamma\frac{\partial p_j}{\partial n_j}\right)\frac{\partial n_j}{\partial \tau}(p_i-c+s)-\frac{\partial x_i}{\partial \tau}(\cdot)\,s-\frac{\partial n_i}{\partial \tau}\tau=0.$$

Using matrix notation, we have

$$\begin{bmatrix}\frac{\partial x_i}{\partial s}(\cdot) & \frac{\partial n_i}{\partial s}\\ \frac{\partial x_i}{\partial \tau}(\cdot) & \frac{\partial n_i}{\partial \tau}\end{bmatrix}\begin{bmatrix} s\\ \tau\end{bmatrix}=\begin{bmatrix}\left\{\left[\left(\frac{\partial \alpha_i}{\partial n_j}+\gamma\frac{\partial p_j}{\partial n_j}\right)\frac{\partial n_j}{\partial s}\right]+\gamma\frac{\partial p_j}{\partial s}\right\}(p_i-c+s)\\ \left(\frac{\partial \alpha_i}{\partial n_j}+\gamma\frac{\partial p_j}{\partial n_j}\right)\frac{\partial n_j}{\partial \tau}(p_i-c+s)\end{bmatrix},$$

where $\frac{\partial x_i}{\partial s}(\cdot)=\beta\frac{\partial(p_i-c+s)}{\partial s}=\beta\left(\frac{\partial p_i}{\partial n_i}\frac{\partial n_i}{\partial s}+\frac{\partial p_i}{\partial n_j}\frac{\partial n_j}{\partial s}+\frac{\partial p_i}{\partial s}+1\right)$, $\frac{\partial x_i}{\partial \tau}(\cdot)=\beta\frac{\partial(p_i-c+s)}{\partial \tau}=\beta\left(\frac{\partial p_i}{\partial n_i}\frac{\partial n_i}{\partial \tau}+\frac{\partial p_i}{\partial n_j}\frac{\partial n_j}{\partial \tau}\right)$.

We use Cramer's Rule to solve this linear equation system.

Denote the determinant of the coefficient matrix of the above linear equation system by D.

$$\begin{aligned}D &= \frac{\partial x_i}{\partial s}(\cdot)\frac{\partial n_i}{\partial \tau}-\frac{\partial x_i}{\partial \tau}(\cdot)\frac{\partial n_i}{\partial s}\\ &= \beta\left[\frac{\partial p_i}{\partial n_j}\left(\frac{\partial n_j}{\partial s}\frac{\partial n_i}{\partial \tau}-\frac{\partial n_i}{\partial s}\frac{\partial n_j}{\partial \tau}\right)+\left(\frac{\partial p_i}{\partial s}+1\right)\frac{\partial n_i}{\partial \tau}\right].\end{aligned}$$

Note that

$$\begin{aligned}&\frac{\partial n_j}{\partial s}\frac{\partial n_i}{\partial \tau}-\frac{\partial n_i}{\partial s}\frac{\partial n_j}{\partial \tau}\\ =&\left[\frac{k_4}{a(b+1)(2b-1)}\right]\left[\frac{-(b-1)(2b+1)k_4-bk}{\Delta}\right]\left(\frac{k-k_4}{\Delta}\right)\\ &-\left[\frac{k_4}{a(b+1)(2b-1)}\right]\left[\frac{(2b^2-1)(k-k_4)+bk_4}{\Delta}\right]\left(-\frac{k_4}{\Delta}\right)\\ =&-\left[\frac{bk_4}{a(b+1)(2b-1)\Delta}\right]\\ <&\ 0.\end{aligned}$$

Then

$$\begin{aligned}D &= \left(\frac{b}{b^2-1}\right)\left\{\left[-\frac{a(b+1)}{2b+1}\right]\left[-\frac{bk_4}{a(b+1)(2b-1)\Delta}\right]+\left(\frac{2b^2-1}{4b^2-1}\right)\left(\frac{k-k_4}{\Delta}\right)\right\}\\ &= \left(\frac{b}{b^2-1}\right)\left(\frac{1}{4b^2-1}\right)\left[\frac{(2b^2-1)(k-k_4)+bk_4}{\Delta}\right]\\ &> 0.\end{aligned}$$

Next, we have

$$
\begin{aligned}
D_1 &= \left\{ \left[\left(\frac{\partial \alpha_i}{\partial n_j} + \gamma \frac{\partial p_j}{\partial n_j} \right) \frac{\partial n_j}{\partial s} + \gamma \frac{\partial p_j}{\partial s} \right] \frac{\partial n_i}{\partial \tau} - \left(\frac{\partial \alpha_i}{\partial n_j} + \gamma \frac{\partial p_j}{\partial n_j} \right) \frac{\partial n_j}{\partial \tau} \frac{\partial n_i}{\partial s} \right\} (p_i - c + s) \\
&= \left[\left(\frac{\partial \alpha_i}{\partial n_j} + \gamma \frac{\partial p_j}{\partial n_j} \right) \left(\frac{\partial n_j}{\partial s} \frac{\partial n_i}{\partial \tau} - \frac{\partial n_i}{\partial s} \frac{\partial n_j}{\partial \tau} \right) + \gamma \frac{\partial p_j}{\partial s} \frac{\partial n_i}{\partial \tau} \right] (p_i - c + s) \\
&= \left(-\frac{2ab}{(b-1)(2b+1)} \right) \left(-\frac{bk_4}{a(b+1)(2b-1)\Delta} \right) (p_i - c + s) \\
&\quad + \left(\frac{1}{b^2-1} \right) \left(-\frac{b}{4b^2-1} \right) \left(\frac{k-k_4}{\Delta} \right) (p_i - c + s) \\
&= \left[\frac{b}{(b^2-1)(4b^2-1)} \right] \left[\frac{(2b+1)k_4 - k}{\Delta} \right],
\end{aligned}
$$

$$
\begin{aligned}
D_2 &= \left\{ \frac{\partial x_i}{\partial s}(\cdot) \left(\frac{\partial \alpha_i}{\partial n_j} + \gamma \frac{\partial p_j}{\partial n_j} \right) \frac{\partial n_j}{\partial \tau} - \frac{\partial x_i}{\partial \tau}(\cdot) \left[\left(\frac{\partial \alpha_i}{\partial n_j} + \gamma \frac{\partial p_j}{\partial n_j} \right) \frac{\partial n_j}{\partial s} + \gamma \frac{\partial p_j}{\partial s} \right] \right\} (p_i - c + s) \\
&= \beta \left\{ \begin{array}{c} \frac{\partial p_i}{\partial n_i} \left(\frac{\partial \alpha_i}{\partial n_j} + \gamma \frac{\partial p_j}{\partial n_j} \right) \left(\frac{\partial n_i}{\partial s} \frac{\partial n_j}{\partial \tau} - \frac{\partial n_j}{\partial s} \frac{\partial n_i}{\partial \tau} \right) + \left(\frac{\partial \alpha_i}{\partial n_j} + \gamma \frac{\partial p_j}{\partial n_j} \right) \left(\frac{\partial p_i}{\partial s} + 1 \right) \frac{\partial n_j}{\partial \tau} \\ - \left(\frac{\partial p_i}{\partial n_i} \frac{\partial n_i}{\partial \tau} + \frac{\partial p_i}{\partial n_j} \frac{\partial n_j}{\partial \tau} \right) \gamma \frac{\partial p_j}{\partial s} \end{array} \right\} (p_i - c + s) \\
&= \beta \left\{ \begin{array}{c} \left[\frac{a(b+1)}{2b+1} \right] \left[-\frac{2ab}{(b-1)(2b+1)} \right] \left[\frac{bk_4}{a(b+1)(2b-1)\Delta} \right] + \\ \left[-\frac{2ab}{(b-1)(2b+1)} \right] \left(\frac{2b^2-1}{4b^2-1} \right) \left(-\frac{k_4}{\Delta} \right) - \left[\frac{a(b+1)}{2b+1} \right] \left(\frac{k}{\Delta} \right) \left(\frac{1}{b^2-1} \right) \left(-\frac{b}{4b^2-1} \right) \end{array} \right\} (p_i - c + s) \\
&= \beta \left\{ \left[\frac{ab}{(b-1)(2b+1)(4b^2-1)} \right] \left[\frac{2(b-1)(2b+1)k_4 + k}{\Delta} \right] \right\} (p_i - c + s) \\
&> 0.
\end{aligned}
$$

Because

$$
s = \frac{D_1}{D}, \ \tau = \frac{D_2}{D},
$$

we have

$$
sign \ s = sign \ \left[(2b+1) k_4 - k \right], \ \tau > 0. \tag{A4}
$$

A.4.3 Part 3

This is implied by the above two Parts.

A.5 Proof of Corollary 5

By the assumption that the welfare function is strictly concave we must have

$$
\underline{k}' = \max \left\{ \left(\frac{3}{2} + \frac{1}{2}\sqrt{5} \right) k_1, \left(\frac{3}{2} + \frac{1}{2}\sqrt{13} \right) k_4 \right\} .^{21} \tag{A5}
$$

[21] See the last Section of this Appendix.

A.5.1 Cournot competition

First consider Cournot competition.

Case 1 $\mu > \frac{2b+1}{2b-1}$

Step 1. Note that in this case,

$$\begin{aligned}\frac{\partial^2 W}{\partial s \partial \tau} &= \frac{\partial x_i}{\partial \tau}(\cdot)\left[\left(\frac{\partial a_i}{\partial m_j}-\frac{\partial x_j}{\partial m_j}\right)\frac{\partial m_j}{\partial s}-\frac{\partial x_j}{\partial s}\right]-\frac{\partial m_i}{\partial s} \\ &= \frac{\partial x_i}{\partial s}(\cdot)\left[\left(\frac{\partial a_i}{\partial m_j}-\frac{\partial x_j}{\partial m_j}\right)\frac{\partial m_j}{\partial \tau}\right]-\frac{\partial x_i}{\partial \tau}(\cdot),\end{aligned}$$

and

$$sign\ \frac{\partial^2 W}{\partial s \partial \tau} = sign\ \left[\left(1-4b^2\right)k^2+\left(12b^2-2\right)k_1 k-2b\left(2b+1\right)\left(k_1\right)^2\right].$$

Let

$$\left(1-4b^2\right)k^2+\left(12b^2-2\right)k_1 k-2b\left(2b+1\right)\left(k_1\right)^2=0.$$

The solution is as follows.

$$k \in \left\{\begin{array}{l}\frac{1}{2(-1+4b^2)}\left(12b^2-2-2\sqrt{(20b^4-8b^2+1+2b-8b^3)}\right)k_1, \\ \frac{1}{2(-1+4b^2)}\left(12b^2-2+2\sqrt{(20b^4-8b^2+1+2b-8b^3)}\right)k_1\end{array}\right\}.$$

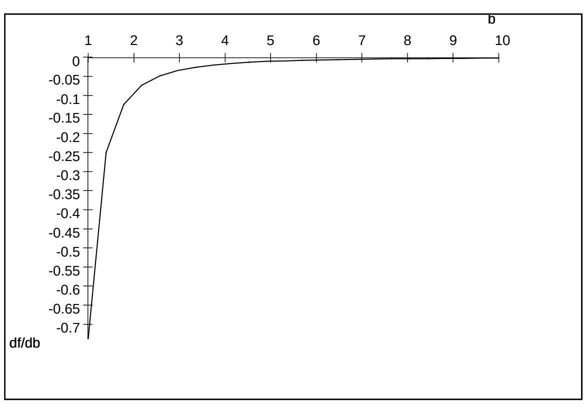
Step 2. Consider the first root. Let

$$f=\frac{1}{2\left(-1+4b^2\right)}\left(12b^2-2-2\sqrt{(20b^4-8b^2+1+2b-8b^3)}\right).$$

Then,

$$\begin{aligned}\frac{df}{db} &= \frac{b}{64b^4-32b^2+4}\left(32\sqrt{2b-8b^2-8b^3+20b^4+1}-192b^2+32\right) \\ &\quad +\frac{1}{8b^2-2}\left(24b-\frac{80b^3-24b^2-16b+2}{\sqrt{2b-8b^2-8b^3+20b^4+1}}\right).\end{aligned}$$

From the graph of $\frac{df}{db}$,



we know that f is a decreasing function. Take the following limit,

$$\lim_{b\to 1^+}\frac{1}{2\left(-1+4b^2\right)}\left(12b^2-2-2\sqrt{\left(20b^4-8b^2+1+2b-8b^3\right)}\right)=\frac{5}{3}-\frac{1}{3}\sqrt{7},$$

which is smaller than $\underline{k}'$.

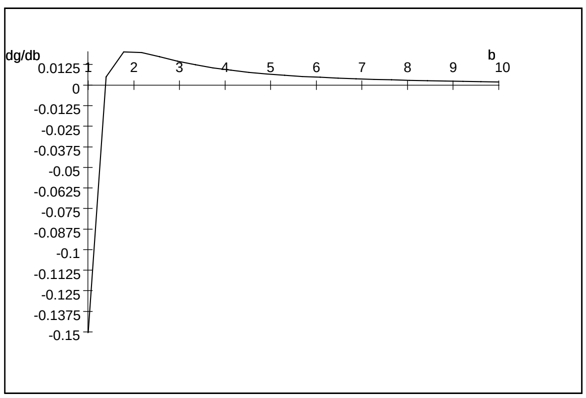
Step 3. Consider the second root. Let

$$g=\frac{1}{2\left(-1+4b^2\right)}\left(12b^2-2+2\sqrt{\left(20b^4-8b^2+1+2b-8b^3\right)}\right).$$

Then,

$$\begin{aligned}\frac{dg}{db} &= \frac{b}{64b^4-32b^2+4}\left(32-32\sqrt{2b-8b^2-8b^3+20b^4+1}-192b^2\right)\\ &\quad+\frac{1}{8b^2-2}\left(24b+\frac{80b^3-24b^2-16b+2}{\sqrt{2b-8b^2-8b^3+20b^4+1}}\right).\end{aligned}$$

From the graph of $\frac{dg}{db}$,



we know that g has a critical point and it is a minimum. Take the following limits,

$$\begin{aligned}&\lim_{b\to 1^+}\frac{1}{2\left(-1+4b^2\right)}\left(12b^2-2+2\sqrt{\left(20b^4-8b^2+1+2b-8b^3\right)}\right)\\ =\ &\frac{5}{3}+\frac{1}{3}\sqrt{7},\\ &\lim_{b\to +\infty}\frac{1}{2\left(-1+4b^2\right)}\left(12b^2-2+2\sqrt{\left(20b^4-8b^2+1+2b-8b^3\right)}\right)\\ =\ &\frac{3}{2}+\frac{1}{2}\sqrt{5}.\end{aligned}$$

It is easy to show that $\frac{3}{2}+\frac{1}{2}\sqrt{5}>\frac{5}{3}+\frac{1}{3}\sqrt{7}$.

Step 4. Therefore, we have

$$\frac{\partial^2 W}{\partial s\partial\tau}<0, \tag{A6}$$

if $k > \underline{k}'$.

Case 2 $\mu < \frac{2b+1}{2b-1}$

Step 1. Note that in this case,

$$\begin{aligned}\frac{\partial^2 W}{\partial s \partial \tau} &= \frac{\partial x_i}{\partial \tau}(\cdot)\left[\left(\frac{\partial a_i}{\partial n_j} - \frac{\partial x_j}{\partial n_j}\right)\frac{\partial n_j}{\partial s} - \frac{\partial x_j}{\partial s}\right] - \frac{\partial n_i}{\partial s} \\ &= \frac{\partial x_i}{\partial s}(\cdot)\left[\left(\frac{\partial a_i}{\partial n_j} - \frac{\partial x_j}{\partial n_j}\right)\frac{\partial n_j}{\partial \tau}\right] - \frac{\partial x_i}{\partial \tau}(\cdot),\end{aligned}$$

and

$$sign\ \frac{\partial^2 W}{\partial s \partial \tau} = sign\ \left[\left(1-4b^2\right)k^2 + \left(12b^2-2\right)k_2 k - 2b\left(2b-1\right)\left(k_2\right)^2\right].$$

Let

$$\left(1-4b^2\right)k^2 + \left(12b^2-2\right)k_2 k - 2b\left(2b-1\right)\left(k_2\right)^2 = 0.$$

The solution is as follows.

$$k \in \left\{ \begin{array}{l} \frac{1}{2(-1+4b^2)}\left(12b^2 - 2 - 2\sqrt{(20b^4 - 8b^2 + 1 - 2b + 8b^3)}\right)k_2, \\ \frac{1}{2(-1+4b^2)}\left(12b^2 - 2 + 2\sqrt{(20b^4 - 8b^2 + 1 - 2b + 8b^3)}\right)k_2 \end{array} \right\}.$$

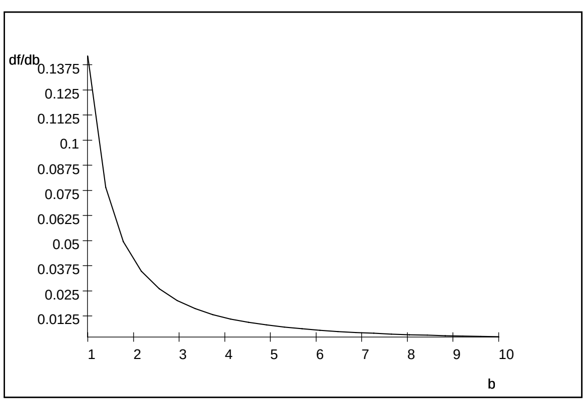
Step 2. consider the first root. Let

$$f = \frac{1}{2\left(-1+4b^2\right)}\left(12b^2 - 2 - 2\sqrt{(20b^4 - 8b^2 + 1 - 2b + 8b^3)}\right).$$

Then

$$\begin{aligned}\frac{df}{db} &= \frac{b}{64b^4 - 32b^2 + 4}\left(32\sqrt{8b^3 - 8b^2 - 2b + 20b^4 + 1} - 192b^2 + 32\right) \\ &\quad + \frac{1}{8b^2 - 2}\left(24b - \frac{24b^2 - 16b + 80b^3 - 2}{\sqrt{8b^3 - 8b^2 - 2b + 20b^4 + 1}}\right).\end{aligned}$$

From the graph of $\frac{df}{db}$,



we know that f is an increasing function. Take the following limit,

$$\lim_{b\to+\infty}\frac{1}{2\left(-1+4b^2\right)}\left(12b^2-2-2\sqrt{(20b^4-8b^2+1-2b+8b^3)}\right)=\frac{3}{2}-\frac{1}{2}\sqrt{5},$$

which is smaller than $\underline{k}'$.

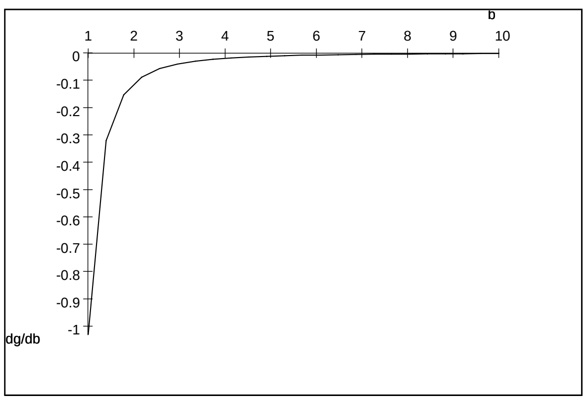
Step 3. Consider the second root. Let

$$g=\frac{1}{2\left(-1+4b^2\right)}\left(12b^2-2+2\sqrt{(20b^4-8b^2+1-2b+8b^3)}\right).$$

Then

$$\begin{aligned}\frac{dg}{db} &= \frac{b}{64b^4-32b^2+4}\left(32-32\sqrt{8b^3-8b^2-2b+20b^4+1}-192b^2\right)\\ &\quad+\frac{1}{8b^2-2}\left(24b+\frac{24b^2-16b+80b^3-2}{\sqrt{8b^3-8b^2-2b+20b^4+1}}\right).\end{aligned}$$

From the graph of $\frac{dg}{db}$,



we know that g is a decreasing function. Take the following limit,

$$\lim_{b\to1^+}\frac{1}{2\left(-1+4b^2\right)}\left(12b^2-2+2\sqrt{(20b^4-8b^2+1-2b+8b^3)}\right)=\frac{5}{3}+\frac{1}{3}\sqrt{19}.$$

Step 4. Since

$$\left(\frac{3}{2}+\frac{1}{2}\sqrt{13}\right)k_2>\left(\frac{5}{3}+\frac{1}{3}\sqrt{19}\right)k_2,$$

we have

$$\frac{\partial^2 W}{\partial s\partial\tau}<0, \tag{A7}$$

if $k>\underline{k}'$.

A.5.2 Bertrand competition

Second consider Bertrand competition.

Case 1 $\mu > \frac{2b^2+b-1}{2b^2-b-1}$

Step 1. Note that in this case,

$$\frac{\partial^2 W}{\partial s \partial \tau} = \frac{\partial p_i}{\partial \tau}(\cdot)\left\{\left[\left(\frac{\partial \alpha_i}{\partial m_j}+\gamma\frac{\partial p_j}{\partial m_j}\right)\frac{\partial m_j}{\partial s}\right]+\gamma\frac{\partial p_j}{\partial s}\right\}-\frac{\partial m_i}{\partial s}$$
$$\left[\frac{\partial p_i}{\partial s}(\cdot)+1\right]\left(\frac{\partial \alpha_i}{\partial m_j}+\gamma\frac{\partial p_j}{\partial m_j}\right)\frac{\partial m_j}{\partial \tau}-\frac{\partial x_i}{\partial \tau}(\cdot),$$

and

$$sign\ \frac{\partial^2 W}{\partial s \partial \tau} = sign\ \left[\left(1-4b^2\right)k^2+\left(12b^2-4\right)k_3 k-2\left(b+1\right)\left(2b-1\right)\left(k_3\right)^2\right].$$

Let

$$\left(1-4b^2\right)k^2+\left(12b^2-4\right)k_3 k-2\left(b+1\right)\left(2b-1\right)\left(k_3\right)^2=0.$$

The solution is as follows.

$$k \in \left\{\begin{array}{l} \frac{1}{2(-1+4b^2)}\left(-4+12b^2-2\sqrt{(2-12b^2+20b^4+2b-8b^3)}\right)k_3, \\ \frac{1}{2(-1+4b^2)}\left(-4+12b^2+2\sqrt{(2-12b^2+20b^4+2b-8b^3)}\right)k_3 \end{array}\right\}.$$

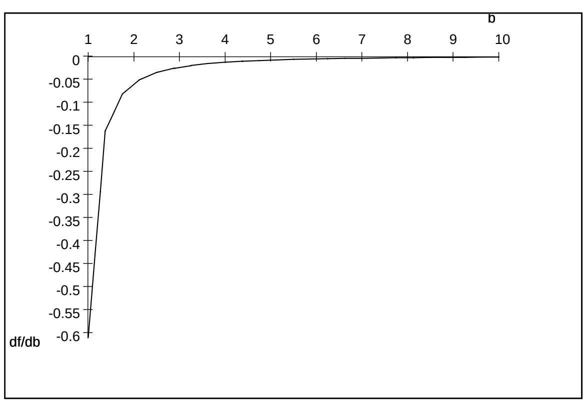
Step 2. Consider the first root. Let

$$f=\frac{1}{2\left(-1+4b^2\right)}\left(-4+12b^2-2\sqrt{(2-12b^2+20b^4+2b-8b^3)}\right).$$

Then

$$\frac{df}{db} = \frac{b}{64b^4-32b^2+4}\left(32\sqrt{2}\sqrt{b-6b^2-4b^3+10b^4+1}-192b^2+64\right)$$
$$+\frac{1}{8b^2-2}\left(24b-\sqrt{2}\frac{40b^3-12b^2-12b+1}{\sqrt{b-6b^2-4b^3+10b^4+1}}\right).$$

From the graph of $\frac{df}{db}$,



we know that f is a decreasing function. Take the following limit,

$$\lim_{b\to 1^+} \frac{1}{2\left(-1+4b^2\right)}\left(-4+12b^2-2\sqrt{(2-12b^2+20b^4+2b-8b^3)}\right)=\frac{2}{3},$$

which is smaller than $\underline{k}'$.

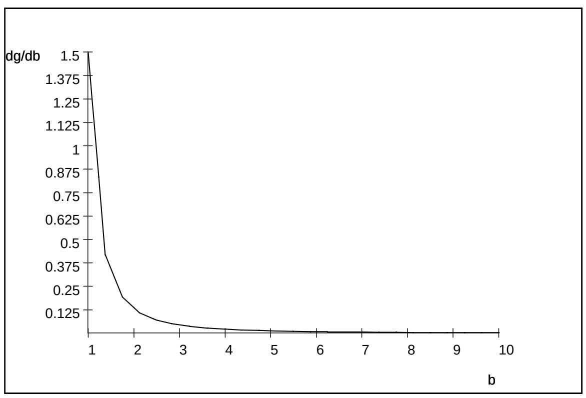
Step 3. Consider the second root. Let

$$g=\frac{1}{2\left(-1+4b^2\right)}\left(-4+12b^2+2\sqrt{(2-12b^2+20b^4+2b-8b^3)}\right).$$

Then

$$\begin{aligned}\frac{dg}{db} &= \frac{b}{64b^4-32b^2+4}\left(64-32\sqrt{2}\sqrt{b-6b^2-4b^3+10b^4+1}-192b^2\right)\\ &\quad +\frac{1}{8b^2-2}\left(24b+\sqrt{2}\frac{40b^3-12b^2-12b+1}{\sqrt{b-6b^2-4b^3+10b^4+1}}\right).\end{aligned}$$

From the graph of $\frac{dg}{db}$,



we know that g is an increasing function. Take the following limit,

$$\lim_{b\to +\infty} \frac{1}{2\left(-1+4b^2\right)}\left(-4+12b^2+2\sqrt{(2-12b^2+20b^4+2b-8b^3)}\right)=\frac{3}{2}+\frac{1}{2}\sqrt{5}.$$

Step 4. Hence,

$$\frac{\partial^2 W}{\partial s\partial\tau}<0, \tag{A8}$$

if $k>\underline{k}'$.

Case 2 $\mu<\frac{2b^2+b-1}{2b^2-b-1}$

Step 1. Note that in this case,

$$\begin{aligned}\frac{\partial^2 W}{\partial s\partial\tau} &= \frac{\partial p_i}{\partial\tau}(\cdot)\left\{\left[\left(\frac{\partial\alpha_i}{\partial n_j}+\gamma\frac{\partial p_j}{\partial n_j}\right)\frac{\partial n_j}{\partial s}\right]+\gamma\frac{\partial p_j}{\partial s}\right\}-\frac{\partial n_i}{\partial s}\\ &\quad \left[\frac{\partial p_i}{\partial s}(\cdot)+1\right]\left(\frac{\partial\alpha_i}{\partial n_j}+\gamma\frac{\partial p_j}{\partial n_j}\right)\frac{\partial n_j}{\partial\tau}-\frac{\partial x_i}{\partial\tau}(\cdot),\end{aligned}$$

and

$$sign\ \frac{\partial^2 W}{\partial s \partial \tau} = sign\ \left[\left(1-4b^2\right)k^2 + \left(12b^2-4\right)k_4 k - 2\left(b-1\right)\left(2b+1\right)\left(k_4\right)^2\right].$$

Let

$$\left(1-4b^2\right)k^2 + \left(12b^2-4\right)k_4 k - 2\left(b-1\right)\left(2b+1\right)\left(k_4\right)^2 = 0.$$

The solution is as follows.

$$k \in \left\{ \begin{array}{l} \frac{1}{2(-1+4b^2)}\left(-4+12b^2-2\sqrt{(2-12b^2+20b^4-2b+8b^3)}\right)k_4, \\ \frac{1}{2(-1+4b^2)}\left(-4+12b^2+2\sqrt{(2-12b^2+20b^4-2b+8b^3)}\right)k_4 \end{array} \right\}.$$

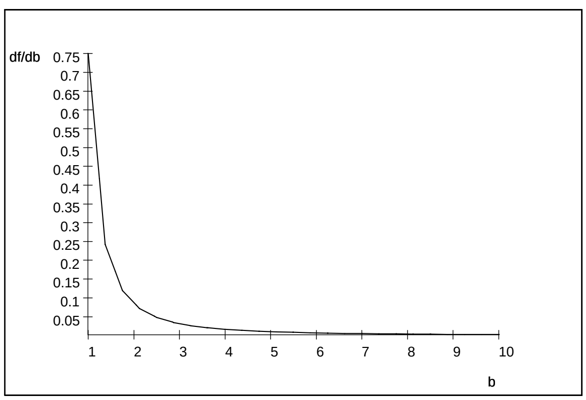
Step 2. Consider the first root. Let

$$f = \frac{1}{2\left(-1+4b^2\right)}\left(-4+12b^2-2\sqrt{(2-12b^2+20b^4-2b+8b^3)}\right).$$

Then

$$\begin{aligned} \frac{df}{db} &= \frac{b}{64b^4-32b^2+4}\left(32\sqrt{2}\sqrt{4b^3-6b^2-b+10b^4+1}-192b^2+64\right) \\ &\quad + \frac{1}{8b^2-2}\left(24b-\sqrt{2}\frac{12b^2-12b+40b^3-1}{\sqrt{4b^3-6b^2-b+10b^4+1}}\right). \end{aligned}$$

From the graph of $\frac{df}{db}$,



we know that f is an increasing function. Take the following limit,

$$\lim_{b\to+\infty} \frac{1}{2\left(-1+4b^2\right)}\left(-4+12b^2-2\sqrt{(2-12b^2+20b^4-2b+8b^3)}\right) = \frac{3}{2}-\frac{1}{2}\sqrt{5},$$

which is smaller than $\underline{k}'$.

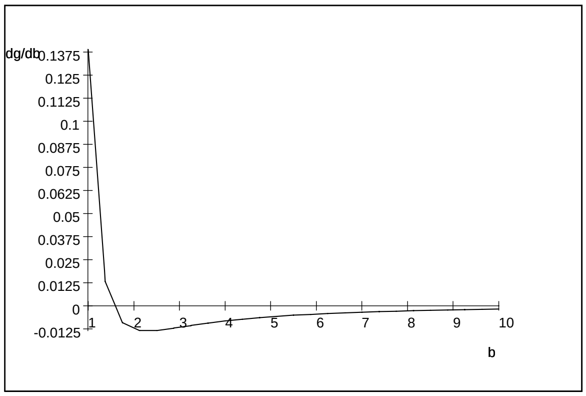
Step 3. Consider the second root. Let

$$g = \frac{1}{2\left(-1+4b^2\right)}\left(-4+12b^2+2\sqrt{(2-12b^2+20b^4-2b+8b^3)}\right).$$

Then

$$\frac{dg}{db} = \frac{b}{64b^4 - 32b^2 + 4}\left(64 - 32\sqrt{2}\sqrt{4b^3 - 6b^2 - b + 10b^4 + 1} - 192b^2\right)$$
$$+\frac{1}{8b^2 - 2}\left(24b + \sqrt{2}\frac{12b^2 - 12b + 40b^3 - 1}{\sqrt{4b^3 - 6b^2 - b + 10b^4 + 1}}\right).$$

From the graph of $\frac{dg}{db}$,



we know that g has a critical point and it is a maximum. In addition, the critical point is $b = 1.5253$, and

$$g|_{b=1.5253} = 2.6891.$$

Step 4. Since

$$\left(\frac{3}{2} + \frac{1}{2}\sqrt{13}\right) k_4 > (2.6891)\, k_4,$$

we have

$$\frac{\partial^2 W}{\partial s \partial \tau} < 0, \tag{A9}$$

if $k > \underline{k}'$.

A.6 Proof of Corollary 6

First consider Cournot case.

1. When both firms invest in cooperative advertising in equilibrium,

$$\left.\frac{\partial W}{\partial s}\right|_{(s,\tau)=(0,0)} = \left\{\left[\left(\frac{\partial a_i}{\partial m_j} - \frac{\partial x_j}{\partial m_j}\right)\frac{\partial m_j}{\partial s}\right] - \frac{\partial x_j}{\partial s}\right\} x_i$$
$$= \left(\frac{1}{4b^2 - 1}\right)\left\{\frac{2bk_1\left[(2b+1)\,k_1 - k\right] + \Delta}{\Delta}\right\} x_i$$
$$> 0,$$

where $\Delta = k^2 - 2kk_1$, and note that

$$2bk_1 \left[(2b+1)\, k_1 - k\right] + \Delta = \left[k - (b+1)\, k_1\right]^2 + \left(3b^2 - 1\right)(k_1)^2 > 0.$$

2. When both firms invest in predatory advertising in equilibrium,

$$\left.\frac{\partial W}{\partial s}\right|_{(s,\tau)=(0,0)} = \left\{\left[\left(\frac{\partial a_i}{\partial n_j} - \frac{\partial x_j}{\partial n_j}\right)\frac{\partial n_j}{\partial s}\right] - \frac{\partial x_j}{\partial s}\right\} x_i > 0,$$

because $\left(\frac{\partial a_i}{\partial n_j} - \frac{\partial x_j}{\partial n_j}\right) < 0$, $\frac{\partial n_j}{\partial s} < 0$, $\frac{\partial x_j}{\partial s} < 0$.

Next, consider Bertrand case.

1. When both firms invest in cooperative advertising in equilibrium,

$$\left.\frac{\partial W}{\partial s}\right|_{(s,\tau)=(0,0)} = \left\{\left[\left(\frac{\partial \alpha_i}{\partial m_j} + \gamma\frac{\partial p_j}{\partial m_j}\right)\frac{\partial m_j}{\partial s}\right] + \gamma\frac{\partial p_j}{\partial s}\right\}(p_i - c + s)$$

$$= \left[\frac{b}{(b^2-1)(4b^2-1)}\right]\left\{\frac{2k_3\left[(b+1)(2b-1)\,k_3 - bk\right] - \Delta}{\Delta}\right\}(p_i - c + s),$$

where $\Delta = k^2 - 2kk_3$. Because

$$2k_3\left[(b+1)(2b-1)\,k_3 - bk\right] - \Delta = \left(5b^2-1\right)(k_3)^2 - \left[k + (b-1)k_3\right]^2,$$

we have

$$\begin{cases} sign\ \left.\frac{\partial W}{\partial s}\right|_{(s,\tau)=(0,0)} > 0 & if\ \left[\sqrt{(5b^2-1)} - (b-1)\right]k_3 > k, \\ sign\ \left.\frac{\partial W}{\partial s}\right|_{(s,\tau)=(0,0)} < 0 & if\ \left[\sqrt{(5b^2-1)} - (b-1)\right]k_3 < k. \end{cases}$$

2. When both firms invest in predatory advertising in equilibrium,

$$\left.\frac{\partial W}{\partial s}\right|_{(s,\tau)=(0,0)} = \left\{\left[\left(\frac{\partial \alpha_i}{\partial n_j} + \gamma\frac{\partial p_j}{\partial n_j}\right)\frac{\partial n_j}{\partial s}\right] + \gamma\frac{\partial p_j}{\partial s}\right\}(p_i - c + s)$$

$$= \left[\frac{b}{(b^2-1)(4b^2-1)}\right]\left\{\frac{2k_4\left[(b-1)(2b+1)\,k_4 + bk\right] - \Delta}{\Delta}\right\}(p_i - c + s),$$

where $\Delta = k^2 - 2kk_4$. Because

$$2k_4\left[(b-1)(2b+1)\,k_4 + bk\right] - \Delta = \left(5b^2-1\right)(k_4)^2 - \left[k - (b+1)k_4\right]^2,$$

we have

$$\begin{cases} sign\ \left.\frac{\partial W}{\partial s}\right|_{(s,\tau)=(0,0)} > 0 & if\ \left[\sqrt{(5b^2-1)} + (b+1)\right]k_3 > k, \\ sign\ \left.\frac{\partial W}{\partial s}\right|_{(s,\tau)=(0,0)} < 0 & if\ \left[\sqrt{(5b^2-1)} + (b+1)\right]k_3 < k. \end{cases}$$

A.7 Proof of Corollary 7

First consider Cournot case.

1. When both firms invest in cooperative advertising in equilibrium,

$$\left.\frac{\partial W}{\partial \tau}\right|_{(s,\tau)=(0,0)} = \left(\frac{\partial a_i}{\partial m_j} - \frac{\partial x_j}{\partial m_j}\right)\frac{\partial m_j}{\partial \tau} x_i > 0,$$

   because $\left(\frac{\partial a_i}{\partial m_j} - \frac{\partial x_j}{\partial m_j}\right) > 0$, $\frac{\partial m_j}{\partial \tau} > 0$.

2. When both firms invest in predatory advertising in equilibrium,

$$\left.\frac{\partial W}{\partial \tau}\right|_{(s,\tau)=(0,0)} = \left(\frac{\partial a_i}{\partial n_j} - \frac{\partial x_j}{\partial n_j}\right)\frac{\partial n_j}{\partial \tau} x_i > 0,$$

   because $\left(\frac{\partial a_i}{\partial n_j} - \frac{\partial x_j}{\partial n_j}\right) < 0$, $\frac{\partial n_j}{\partial \tau} < 0$.

Next consider Bertrand case.

1. When both firms invest in cooperative advertising in equilibrium,

$$\left.\frac{\partial W}{\partial \tau}\right|_{(s,\tau)=(0,0)} = \left(\frac{\partial \alpha_i}{\partial m_j} + \gamma\frac{\partial p_j}{\partial m_j}\right)\frac{\partial m_j}{\partial \tau}(p_i - c + s) > 0,$$

   because $\left(\frac{\partial \alpha_i}{\partial m_j} + \gamma\frac{\partial p_j}{\partial m_j}\right) > 0$, $\frac{\partial m_j}{\partial \tau} > 0$.

2. When both firms invest in predatory advertising in equilibrium,

$$\left.\frac{\partial W}{\partial \tau}\right|_{(s,\tau)=(0,0)} = \left(\frac{\partial \alpha_i}{\partial n_j} + \gamma\frac{\partial p_j}{\partial n_j}\right)\frac{\partial n_j}{\partial \tau}(p_i - c + s) > 0,$$

   because $\left(\frac{\partial \alpha_i}{\partial n_j} + \gamma\frac{\partial p_j}{\partial n_j}\right) < 0$, $\frac{\partial n_j}{\partial \tau} < 0$.

A.8 Proof of Corollary 8

1. It is implied by Corollary 6 and 7.

2. Note that

$$-\frac{d^2\tau}{ds^2} = \frac{\frac{\partial^2 W}{\partial s^2}\left(\frac{\partial W}{\partial \tau}\right)^2 - 2\frac{\partial^2 W}{\partial s \partial \tau}\frac{\partial W}{\partial s}\frac{\partial W}{\partial \tau} + \frac{\partial^2 W}{\partial \tau^2}\left(\frac{\partial W}{\partial s}\right)^2}{\left(\frac{\partial W}{\partial \tau}\right)^3}$$

   By the assumption that the welfare function is strictly concave it is strictly quasiconcave. Hence, $\frac{\partial^2 W}{\partial s^2}\left(\frac{\partial W}{\partial \tau}\right)^2 - 2\frac{\partial^2 W}{\partial s \partial \tau}\frac{\partial W}{\partial s}\frac{\partial W}{\partial \tau} + \frac{\partial^2 W}{\partial \tau^2}\left(\frac{\partial W}{\partial s}\right)^2 < 0$. This fact and Corollary 7 imply decreasing marginal rate of substitution at non-intervention point.

A.9 Proof of Proposition 9

Note that the second best trade policy analysis is equivalent to the constraint-augmented first best policy analysis where the constraint is $\tau = 0$. Therefore, this Proposition is implied by Corollary 6.

A.10 Proof of Proposition 10

Note that the second best industrial policy analysis is equivalent to the constraint-augmented first best policy analysis where the constraint is $s = 0$. Therefore, this Proposition is implied by Corollary 7.

A.11 The Simulation Design

A.11.1 The general form of "robust" fraction

Denote l the fraction of the feasible range of values for k for which trade policy is robust. The general form of l can be written as follows.

$$l = \frac{k_c - \underline{k}''}{\overline{k} - \underline{k}''} = \begin{cases} 0 & if \quad k_c < \underline{k}'', \\ l \in (0,1) & if \quad \underline{k}'' < k_c < \overline{k}, \\ 1 & if \quad k_c > \overline{k}. \end{cases}$$

We have

$$k_c \in \{k_5, k_6\},$$

where

$$k_5 = \left[\sqrt{(5b^2 - 1)} - (b - 1)\right] k_3, \; k_6 = \left[\sqrt{(5b^2 - 1)} + (b + 1)\right] k_4,$$

and

$$\underline{k}'' = \max\left\{\left(1 + \sqrt{2}\right) k_1, \left(\frac{3}{2} + \frac{1}{2}\sqrt{13}\right) k_4\right\}.[22]$$

Note that when $l = 0$, we cannot get robust trade policy in equilibrium. In particular, when firms play Cournot, optimal trade policy is a subsidy while firms play Bertrand, it is a tax. When $l = 1$, we can definitely get robust trade policy in equilibrium and it is a trade subsidy whatever the form of product market competition.

A.11.2 Calibrating $\overline{k}$

We use the advertising to sales ratios to calibrate $\overline{k}$.

First, define a firm's advertising cost to profit ratio κ_i as the proportion of total advertising investment cost in its product market profit.

[22] See the last Subsection of this Appendix.

1. When firms play Cournot and in equilibrium invest in cooperative advertising, the advertising cost to profit ratio is

$$\kappa_i^C(m) = \kappa^C(m) = \frac{k_1}{k}.$$

2. When firms play Cournot and in equilibrium invest in predatory advertising, the advertising cost to profit ratio is

$$\kappa_i^C(n) = \kappa^C(n) = \frac{k_2}{k}.$$

3. When firms play Bertrand and in equilibrium invest in cooperative advertising, the advertising cost to profit ratio is

$$\kappa_i^B(m) = \kappa^B(m) = \frac{k_3}{k}.$$

4. When firms play Bertrand and in equilibrium invest in predatory advertising, the advertising cost to profit ratio is

$$\kappa_i^B(n) = \kappa^B(n) = \frac{k_4}{k}.$$

It can be easily shown that

$$\kappa^C(m) > \kappa^B(m),\ \kappa^C(n) < \kappa^B(n).$$

Next, how do we use the above results to impose an "appropriate" upper bound on k? First, in empirical work, industrial organization economists often care about advertising to sales ratios, which is smaller than advertising cost to profit ratio. Given this fact, it is possible to calibrate four upper bounds for k using the data collected from the real world or the estimation results of empirical researches.

In particular, if in a given industry, the advertising to sales ratio is ϕ, then

1. if firms play Cournot and in equilibrium invest in cooperative advertising, we must have $\frac{k_1}{k} > \phi$. So, the upper bound calibrated should be $\frac{k_1}{\phi}$,

2. if firms play Cournot and in equilibrium invest in predatory advertising, we must have $\frac{k_2}{k} > \phi$. So, the upper bound calibrated should be $\frac{k_2}{\phi}$,

3. if firms play Bertrand and in equilibrium invest in cooperative advertising, we must have $\frac{k_3}{k} > \phi$. So, the upper bound calibrated should be $\frac{k_3}{\phi}$,

4. if firms play Bertrand and in equilibrium invest in predatory advertising, we must have $\frac{k_4}{k} > \phi$. So, the upper bound calibrated should be $\frac{k_4}{\phi}$.

In the simulation, we treat

$$\overline{k} = \frac{\min\{k_2, k_3\}}{\phi},$$

as the upper bound on k.

Why is that? In general, we do not know the market conduct and the equilibrium investment behaviour of the firms and what is available is the data or the estimation result on advertising to sales ratios. Hence, we should follow a prudential strategy that given an observed ϕ, whatever the form of competition and whatever the equilibrium type of advertising, the advertising cost to profit ratio should be greater than ϕ.

A.11.3 Using k_1 to represent k_i, k_c, $\underline{k}''$, and $\overline{k}$

According to Assumption 1, it is easy to show that

$$k_2 = \left[\frac{(2b+1)^2}{(2b-1)^2\mu^2}\right]k_1,$$

$$k_3 = \left[\frac{(b-1)(2b+1)^2}{(b+1)(2b-1)^2}\right]k_1,$$

$$k_4 = \left[\frac{(b+1)}{(b-1)\mu^2}\right]k_1,$$

and

$$k_5 = \left[\sqrt{(5b^2-1)} - (b-1)\right]\left[\frac{(b-1)(2b+1)^2}{(b+1)(2b-1)^2}\right]k_1,$$

$$k_6 = \left[\sqrt{(5b^2-1)} + (b+1)\right]\left[\frac{(b+1)}{(b-1)\mu^2}\right]k_1.$$

Note that we can write k_c as follows.

$$k_c = k_1\delta,$$

where

$$\delta \in \left\{ \begin{array}{c} \left[\sqrt{(5b^2-1)} - (b-1)\right]\left[\frac{(b-1)(2b+1)^2}{(b+1)(2b-1)^2}\right], \\ \left[\sqrt{(5b^2-1)} + (b+1)\right]\left[\frac{(b+1)}{(b-1)\mu^2}\right] \end{array} \right\}.$$

In addition,

$$\underline{k}'' = k_1 \max\left\{\left(1+\sqrt{2}\right), \left(\frac{3}{2}+\frac{1}{2}\sqrt{13}\right)\left[\frac{(b+1)}{(b-1)\mu^2}\right]\right\},$$

$$\overline{k} = k_1\left[\frac{\min\left\{\frac{(2b+1)^2}{(2b-1)^2\mu^2}, \frac{(b-1)(2b+1)^2}{(b+1)(2b-1)^2}\right\}}{\phi}\right].$$

A.11.4 Simulation on the robust proportion

Given the above results, we have

$$\begin{aligned} l &= \frac{k_c - \underline{k}''}{\overline{k} - \underline{k}''} \\ &= \frac{\delta - \max\left\{\left(1+\sqrt{2}\right), \left(\frac{3}{2}+\frac{1}{2}\sqrt{13}\right)\left[\frac{(b+1)}{(b-1)\mu^2}\right]\right\}}{\frac{\min\left\{\frac{(2b+1)^2}{(2b-1)^2\mu^2}, \frac{(b-1)(2b+1)^2}{(b+1)(2b-1)^2}\right\}}{\phi} - \max\left\{\left(1+\sqrt{2}\right), \left(\frac{3}{2}+\frac{1}{2}\sqrt{13}\right)\left[\frac{(b+1)}{(b-1)\mu^2}\right]\right\}}. \end{aligned}$$

Furthermore, given the values of parameters b and μ, and the data or estimation results on ϕ, we can calculate the two potential values of δ and the other three numbers presented in the above formula. In addition, according to Proposition 3, we can infer from the values of b and μ that in equilibrium, whether firms make cooperative or predatory advertising investments. Hence, we can decide in that case which value of δ we should use to calculate l.

According to the relationship between b and μ, we can calculate l in three cases, i.e.,

1. $\mu > \frac{2b^2+b-1}{2b^2-b-1}$, and whatever the form of product market competition, cooperative advertising will be present in equilibrium,

2. $\frac{2b^2+b-1}{2b^2-b-1} > \mu > \frac{2b+1}{2b-1}$, and cooperative advertising will be present in equilibrium when firms play Cournot, while predatory advertising will be present in equilibrium when firms play Bertrand.

3. $\frac{2b+1}{2b-1} > \mu$, and whatever the form of product market competition, predatory advertising will be present in equilibrium.

Note that, given b and μ, if we find a ϕ such that the calibrated $\overline{k}$ is smaller than $\underline{k}''$, then that case should be ignored.

A.12 Further Discussion on the Second Order Condition of Welfare Maximization

In the text we directly assume that the welfare function is strictly concave because to identify the condition that guarantees strict concavity is not easy. However, to explore the implications of this assumption will prove to be very helpful. In particular, strict concavity implies that

$$\frac{\partial^2 W}{\partial s^2} < 0,\ \frac{\partial^2 W}{\partial \tau^2} < 0.$$

This condition could enable us identify a "reasonable" lower bound on k in each case of policy analysis. As A.5 shows, in the case of first best policy analysis, such a lower bound help us prove Corollary 5; As A.11 shows, in the case of second best trade policy analysis, such a lower bound help us do simulation. Of course, like before, we also have $k < \overline{k}$.

A.12.1 Cournot competition

First consider Cournot competition.

Case 1 $\mu > \frac{2b+1}{2b-1}$

Step 1. Note that in this case,

$$\frac{\partial^2 W}{\partial s^2} = \frac{\partial x_i}{\partial s}(\cdot)\left[\left(\frac{\partial a_i}{\partial m_j} - \frac{\partial x_j}{\partial m_j}\right)\frac{\partial m_j}{\partial s} - \frac{\partial x_j}{\partial s} - 1\right],$$

and

$$sign\ \frac{\partial^2 W}{\partial s^2} = sign\ \left[\left(1-2b^2\right)k^2 + \left(4b^2-b-2\right)k_1 k + \left(2b^2+b\right)(k_1)^2\right].$$

Let

$$\left(1-2b^2\right)k^2 + \left(4b^2-b-2\right)k_1 k + \left(2b^2+b\right)(k_1)^2 = 0.$$

The solution is as follows.

$$k \in \left\{ \begin{array}{c} \frac{1}{2(-1+2b^2)}\left(-b+4b^2-2-\sqrt{(-23b^2+32b^4+4)}\right)k_1, \\ \frac{1}{2(-1+2b^2)}\left(-b+4b^2-2+\sqrt{(-23b^2+32b^4+4)}\right)k_1 \end{array} \right\}.$$

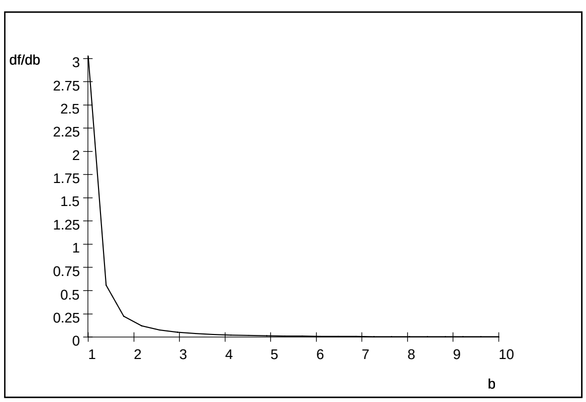
Step 2. Consider the first root. Let

$$f = \frac{1}{2\left(-1+2b^2\right)}\left(-b+4b^2-2-\sqrt{(-23b^2+32b^4+4)}\right).$$

Then

$$\begin{aligned}\frac{df}{db} &= \frac{b}{16b^4-16b^2+4}\left(8b-32b^2+8\sqrt{32b^4-23b^2+4}+16\right) \\ &\quad + \frac{1}{4b^2-2}\left(8b-\frac{1}{2}\frac{128b^3-46b}{\sqrt{32b^4-23b^2+4}}-1\right).\end{aligned}$$

From the graph of $\frac{df}{db}$,



we know that it is an increasing function. Take the following limit,

$$\lim_{b\to+\infty}\frac{1}{2\left(-1+2b^2\right)}\left(-b+4b^2-2-\sqrt{\left(-23b^2+32b^4+4\right)}\right)=1-\sqrt{2}<0.$$

This implies that we always have

$$k=\frac{1}{2\left(-1+2b^2\right)}\left(-b+4b^2-2-\sqrt{\left(-23b^2+32b^4+4\right)}\right)k_1<0,$$

which violates Assumption 1.

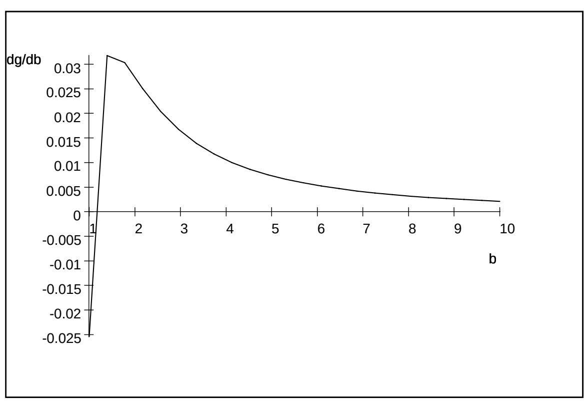
Step 3. Consider the second root. Let

$$g=\frac{1}{2\left(-1+2b^2\right)}\left(-b+4b^2-2+\sqrt{\left(-23b^2+32b^4+4\right)}\right).$$

Then

$$\begin{aligned}\frac{dg}{db} &= \frac{b}{16b^4-16b^2+4}\left(8b-32b^2-8\sqrt{32b^4-23b^2+4}+16\right)\\ &\quad+\frac{1}{4b^2-2}\left(8b+\frac{1}{2}\frac{128b^3-46b}{\sqrt{32b^4-23b^2+4}}-1\right).\end{aligned}$$

From the graph of $\frac{dg}{db}$,



we know that g has a critical point and it is a minimum. Take the following limits,

$$\lim_{b\to1^+}\frac{1}{2\left(-1+2b^2\right)}\left(-b+4b^2-2+\sqrt{\left(-23b^2+32b^4+4\right)}\right)=\frac{1}{2}+\frac{1}{2}\sqrt{13},$$
$$\lim_{b\to+\infty}\frac{1}{2\left(-1+2b^2\right)}\left(-b+4b^2-2+\sqrt{\left(-23b^2+32b^4+4\right)}\right)=1+\sqrt{2},$$

It could be easily shown that $1+\sqrt{2}>\frac{1}{2}+\frac{1}{2}\sqrt{13}>2$. The last inequality implies that Assumption 1 is satisfied.

Hence, we have

$$\frac{\partial^2 W}{\partial s^2} < 0 \Rightarrow k > \left(1+\sqrt{2}\right) k_1. \tag{A10}$$

Step 4. Note that in this case,

$$\frac{\partial^2 W}{\partial \tau^2} = \frac{\partial x_i}{\partial \tau}(\cdot)\left[\left(\frac{\partial a_i}{\partial m_j} - \frac{\partial x_j}{\partial m_j}\right)\frac{\partial m_j}{\partial \tau}\right] - \frac{\partial m_i}{\partial \tau},$$

and

$$sign\ \frac{\partial^2 W}{\partial \tau^2} = sign\ \left[-k^2 + 3k_1 k - (k_1)^2\right].$$

Let

$$k^2 - 3k_1 k + (k_1)^2 = 0.$$

The solution is as follows.

$$k \in \left\{\left(\frac{3}{2} - \frac{1}{2}\sqrt{5}\right) k_1,\ \left(\frac{3}{2} + \frac{1}{2}\sqrt{5}\right) k_1\right\}$$

Obviously, the first root violates Assumption 1. Hence, we have

$$\frac{\partial^2 W}{\partial \tau^2} < 0 \Rightarrow k > \left(\frac{3}{2} + \frac{1}{2}\sqrt{5}\right) k_1. \tag{A11}$$

Step 5. Note that

$$\left(\frac{3}{2} + \frac{1}{2}\sqrt{5}\right) k_1 > \left(1+\sqrt{2}\right) k_1.$$

Therefore,

$$\begin{cases} \frac{\partial^2 W}{\partial s^2} < 0 \\ \frac{\partial^2 W}{\partial \tau^2} < 0 \end{cases} \Rightarrow k > \left(\frac{3}{2} + \frac{1}{2}\sqrt{5}\right) k_1. \tag{A12}$$

Case 2 $\mu < \frac{2b+1}{2b-1}$

Step 1. Note that in this case,

$$\frac{\partial^2 W}{\partial s^2} = \frac{\partial x_i}{\partial s}(\cdot)\left[\left(\frac{\partial a_i}{\partial n_j} - \frac{\partial x_j}{\partial n_j}\right)\frac{\partial n_j}{\partial s} - \frac{\partial x_j}{\partial s} - 1\right],$$

and

$$sign\ \frac{\partial^2 W}{\partial s^2} = sign\ \left[\left(1-2b^2\right) k^2 + \left(4b^2 + b - 2\right) k_2 k + \left(2b^2 - b\right)(k_2)^2\right].$$

Let

$$\left(1-2b^2\right) k^2 + \left(4b^2 + b - 2\right) k_2 k + \left(2b^2 - b\right)(k_2)^2 = 0.$$

The solution is as follows.

$$k \in \left\{ \begin{array}{c} \frac{1}{2(-1+2b^2)}\left(4b^2-2+b-\sqrt{(32b^4-23b^2+4)}\right)k_2, \\ \frac{1}{2(-1+2b^2)}\left(4b^2-2+b+\sqrt{(32b^4-23b^2+4)}\right)k_2 \end{array} \right\}.$$

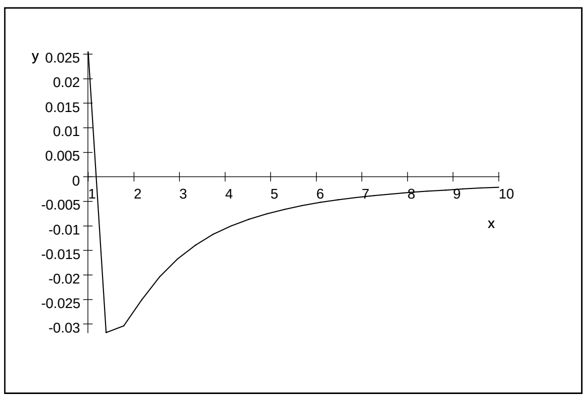
Step 2. Consider the first root. Let

$$f = \frac{1}{2\left(-1+2b^2\right)}\left(4b^2-2+b-\sqrt{(32b^4-23b^2+4)}\right).$$

Then

$$\begin{aligned} \frac{df}{db} &= \frac{b}{16b^4-16b^2+4}\left(8\sqrt{32b^4-23b^2+4}-32b^2-8b+16\right) \\ &\quad +\frac{1}{4b^2-2}\left(8b-\frac{1}{2}\frac{128b^3-46b}{\sqrt{32b^4-23b^2+4}}+1\right). \end{aligned}$$

From the graph of $\frac{df}{db}$,



we know that f has a critical point and it is a maximum. In addition, the critical point is $b = 1.0679$ and

$$f|_{b=1.0679} = -0.302.$$

Hence, Assumption 1 is violated.

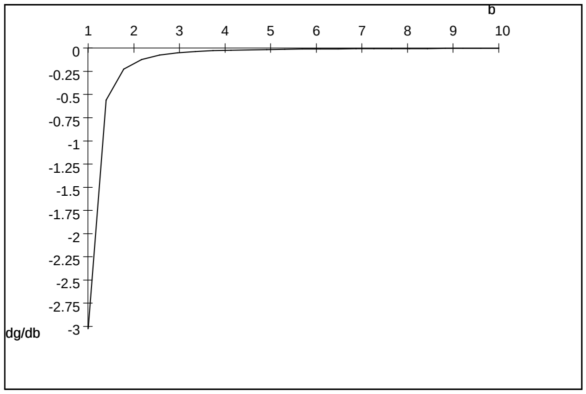
Step 3. Consider the second root. Let

$$g = \frac{1}{2\left(-1+2b^2\right)}\left(4b^2-2+b+\sqrt{(32b^4-23b^2+4)}\right).$$

Then

$$\begin{aligned} \frac{dg}{db} &= \frac{b}{16b^4-16b^2+4}\left(16-32b^2-8\sqrt{32b^4-23b^2+4}-8b\right) \\ &\quad +\frac{1}{4b^2-2}\left(8b+\frac{1}{2}\frac{128b^3-46b}{\sqrt{32b^4-23b^2+4}}+1\right) \end{aligned}$$

From the graph of $\frac{dg}{db}$,



we know that g is a decreasing function. Take the following limit,

$$\lim_{b \to 1^+} \frac{1}{2(-1+2b^2)} \left(4b^2 - 2 + b + \sqrt{(32b^4 - 23b^2 + 4)}\right) = \frac{3}{2} + \frac{1}{2}\sqrt{13},$$

which satisfies Assumption 1.

Hence, we have

$$\frac{\partial^2 W}{\partial s^2} < 0 \Rightarrow k > \left(\frac{3}{2} + \frac{1}{2}\sqrt{13}\right) k_2. \tag{A13}$$

Step 4. Note that in this case,

$$\frac{\partial^2 W}{\partial \tau^2} = \frac{\partial x_i}{\partial \tau}(\cdot)\left[\left(\frac{\partial a_i}{\partial n_j} - \frac{\partial x_j}{\partial n_j}\right)\frac{\partial n_j}{\partial \tau}\right] - \frac{\partial n_i}{\partial \tau},$$

and

$$sign\ \frac{\partial^2 W}{\partial \tau^2} = sign\ \left[-k^2 + 3k_2 k - (k_2)^2\right].$$

Let

$$k^2 - 3k_2 k + (k_2)^2 = 0.$$

The solution is as follows.

$$k \in \left\{\left(\frac{3}{2} - \frac{1}{2}\sqrt{5}\right) k_2,\ \left(\frac{3}{2} + \frac{1}{2}\sqrt{5}\right) k_2\right\}.$$

Obviously, the first root violates Assumption 1. Hence, we have

$$\frac{\partial^2 W}{\partial \tau^2} < 0 \Rightarrow \left(\frac{3}{2} + \frac{1}{2}\sqrt{5}\right) k_2. \tag{A14}$$

Step 5. Note that

$$\left(\frac{1}{2}\sqrt{13} + \frac{3}{2}\right) k_2 > \left(\frac{3}{2} + \frac{1}{2}\sqrt{5}\right) k_2.$$

Hence,

$$\begin{cases} \frac{\partial^2 W}{\partial s^2} < 0 \\ \frac{\partial^2 W}{\partial \tau^2} < 0 \end{cases} \Rightarrow k > \left(\frac{1}{2}\sqrt{13} + \frac{3}{2}\right) k_2. \tag{A15}$$

A.12.2 Bertrand competition

Second consider Bertrand competition.

Case 1 $\mu > \frac{2b^2+b-1}{2b^2-b-1}$

Step 1. Note that in this case,

$$\frac{\partial^2 W}{\partial s^2} = \left[\frac{\partial p_i}{\partial s}(\cdot)+1\right]\left\{\left[\left(\frac{\partial \alpha_i}{\partial m_j}+\gamma\frac{\partial p_j}{\partial m_j}\right)\frac{\partial m_j}{\partial s}\right]+\gamma\frac{\partial p_j}{\partial s}\right\}-\frac{\partial x_i}{\partial s}(\cdot),$$

and

$$sign\,\frac{\partial^2 W}{\partial s^2} = sign\left[-2b^2k^2+\left(4b^2-b\right)k_3k+(b+1)(2b-1)(k_3)^2\right].$$

Let

$$-2b^2k^2+\left(4b^2-b\right)k_3k+(b+1)(2b-1)(k_3)^2=0.$$

The solution is as follows.

$$k \in \left\{\left(b-\frac{1}{4}-\frac{1}{4}\sqrt{(32b^2-7)}\right)\frac{k_3}{b},\left(b-\frac{1}{4}+\frac{1}{4}\sqrt{(32b^2-7)}\right)\frac{k_3}{b}\right\}.$$

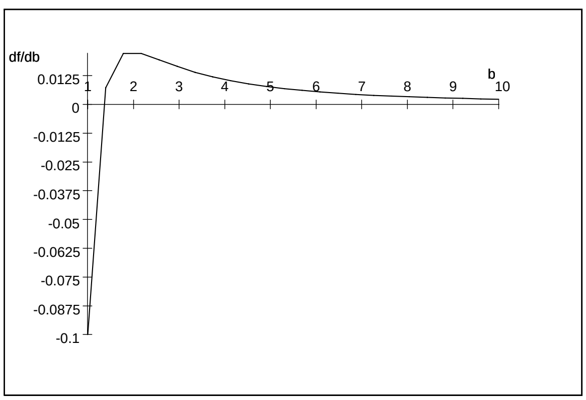
Step 2. Consider the first root. Let

$$f=\left(b-\frac{1}{4}-\frac{1}{4}\sqrt{(32b^2-7)}\right)\frac{1}{b}.$$

Then,

$$\frac{df}{db}=\frac{1}{b}\left(1-8\frac{b}{\sqrt{32b^2-7}}\right)+\frac{1}{b^2}\left(\frac{1}{4}\sqrt{32b^2-7}-b+\frac{1}{4}\right).$$

From the graph of $\frac{df}{db}$,



we know that f has a critical point and it is a minimum. Take the following limits,

$$\lim_{b\to 1^+}\left(b-\frac{1}{4}-\frac{1}{4}\sqrt{(32b^2-7)}\right)\frac{1}{b}=-\frac{1}{2},$$

$$\lim_{b\to +\infty}\left(b-\frac{1}{4}-\frac{1}{4}\sqrt{(32b^2-7)}\right)\frac{1}{b}=1-\sqrt{2}.$$

These results implies that we always have

$$k = \left(b - \frac{1}{4} - \frac{1}{4}\sqrt{(32b^2 - 7)}\right)\frac{k_3}{b} < 0,$$

which violates Assumption 1.

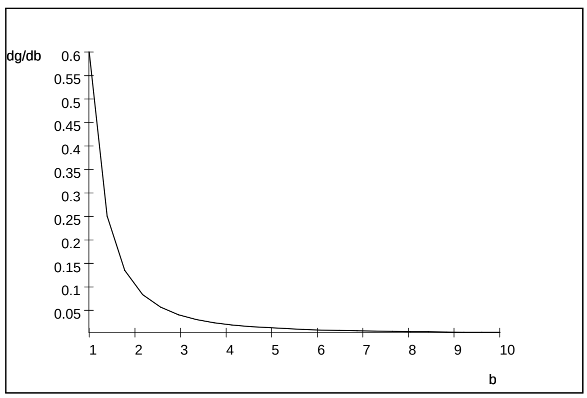
Step 3. Consider the second root. Let

$$g = \left(b - \frac{1}{4} + \frac{1}{4}\sqrt{(32b^2 - 7)}\right)\frac{1}{b}.$$

Then

$$\frac{dg}{db} = \frac{1}{b}\left(8\frac{b}{\sqrt{32b^2 - 7}} + 1\right) + \frac{1}{b^2}\left(\frac{1}{4} - \frac{1}{4}\sqrt{32b^2 - 7} - b\right).$$

From the graph of $\frac{dg}{db}$,



we know that g is an increasing function. Take the following limit,

$$\lim_{b \to +\infty}\left(b - \frac{1}{4} + \frac{1}{4}\sqrt{(32b^2 - 7)}\right)\frac{1}{b} = 1 + \sqrt{2},$$

which satisfies Assumption 1.

Hence, we have

$$\frac{\partial^2 W}{\partial s^2} < 0 \Rightarrow k > \left(1 + \sqrt{2}\right)k_3. \tag{A16}$$

Step 4. Note that in this case,

$$\frac{\partial^2 W}{\partial \tau^2} = \frac{\partial p_i}{\partial \tau}(\cdot)\left(\frac{\partial \alpha_i}{\partial m_j} + \gamma\frac{\partial p_j}{\partial m_j}\right)\frac{\partial m_j}{\partial \tau} - \frac{\partial m_i}{\partial \tau},$$

and

$$sign\,\frac{\partial^2 W}{\partial \tau^2} = sign\left[-k^2 + 3k_3 k - (k_3)^2\right].$$

Let

$$k^2 - 3k_3k + (k_3)^2 = 0.$$

The solution is as follows.

$$k \in \left\{ \left(\frac{3}{2} - \frac{1}{2}\sqrt{5}\right) k_3, \ \left(\frac{3}{2} + \frac{1}{2}\sqrt{5}\right) k_3 \right\}.$$

Obviously, the first root violates Assumption 1. Hence, we have

$$\frac{\partial^2 W}{\partial \tau^2} < 0 \Rightarrow \left(\frac{3}{2} + \frac{1}{2}\sqrt{5}\right) k_3. \tag{A17}$$

Step 5. Note that

$$\left(1 + \sqrt{2}\right) k_3 < \left(\frac{3}{2} + \frac{1}{2}\sqrt{5}\right) k_3.$$

Hence,

$$\left\{ \begin{array}{l} \frac{\partial^2 W}{\partial s^2} < 0 \\ \frac{\partial^2 W}{\partial \tau^2} < 0 \end{array} \right. \Rightarrow k > \left(\frac{3}{2} + \frac{1}{2}\sqrt{5}\right) k_3. \tag{A18}$$

Case 2 $\mu < \frac{2b^2+b-1}{2b^2-b-1}$

Step 1. Note that in this case,

$$\frac{\partial^2 W}{\partial s^2} = \left[\frac{\partial p_i}{\partial s}(\cdot) + 1\right] \left\{ \left[\left(\frac{\partial \alpha_i}{\partial n_j} + \gamma \frac{\partial p_j}{\partial n_j} \right) \frac{\partial n_j}{\partial s} \right] + \gamma \frac{\partial p_j}{\partial s} \right\} - \frac{\partial x_i}{\partial s}(\cdot),$$

and

$$sign\ \frac{\partial^2 W}{\partial s^2} = sign\left[-2b^2k^2 + \left(4b^2 + b\right) k_4 k + (b-1)(2b+1)(k_4)^2\right].$$

Let

$$-2b^2k^2 + \left(4b^2 + b\right) k_4 k + (b-1)(2b+1)(k_4)^2 = 0.$$

The solutions is as follows.

$$k \in \left\{ \begin{array}{c} \left(b + \frac{1}{4} - \frac{1}{4}\sqrt{(32b^2 - 7)}\right) \frac{k_4}{b}, \\ \left(b + \frac{1}{4} + \frac{1}{4}\sqrt{(32b^2 - 7)}\right) \frac{k_4}{b} \end{array} \right\}.$$

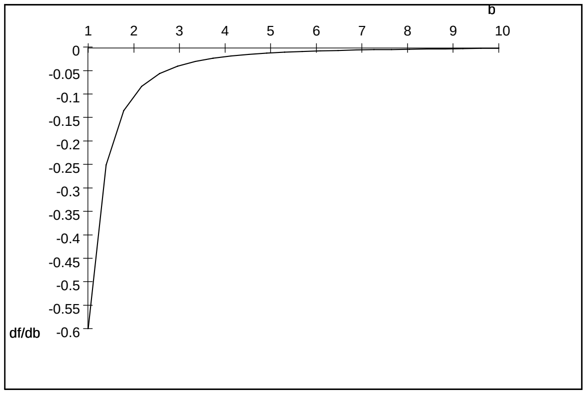
Step 2. Consider the first root. Let

$$f = \left(b + \frac{1}{4} - \frac{1}{4}\sqrt{(32b^2 - 7)}\right) \frac{1}{b}.$$

Then

$$\frac{df}{db} = \frac{1}{b}\left(1 - 8\frac{b}{\sqrt{32b^2 - 7}}\right) + \frac{1}{b^2}\left(\frac{1}{4}\sqrt{32b^2 - 7} - b - \frac{1}{4}\right).$$

From the graph of $\frac{df}{db}$,



we know that f is a decreasing function. Take the following limit,

$$\lim_{b\to 1^+}\left(b+\frac{1}{4}-\frac{1}{4}\sqrt{(32b^2-7)}\right)\frac{1}{b}=0,$$

which violates Assumption 1.

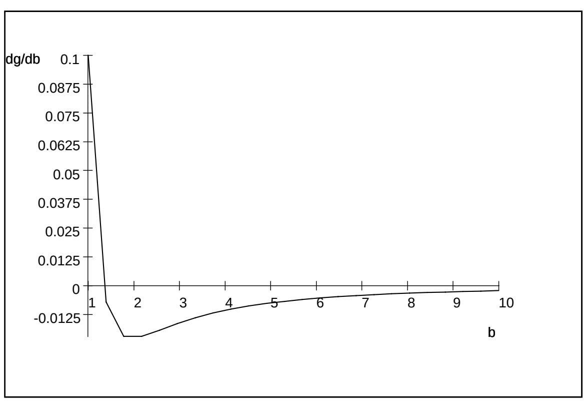
Step 3. Consider the second root. Let

$$g=\left(b+\frac{1}{4}+\frac{1}{4}\sqrt{(32b^2-7)}\right)\frac{1}{b}.$$

Then

$$\frac{dg}{db}=\frac{1}{b}\left(8\frac{b}{\sqrt{32b^2-7}}+1\right)+\frac{1}{b^2}\left(-b-\frac{1}{4}\sqrt{32b^2-7}-\frac{1}{4}\right).$$

From the graph of $\frac{dg}{db}$,



we know that g has a critical point and it is a maximum. In addition, the critical point is $b = 1.3229$ and

$$g|_{b=1.3229}=2.5119.$$

which satisfies Assumption 1.

Hence, we have

$$\frac{\partial^2 W}{\partial s^2} < 0 \Rightarrow k > (2.5119)\, k_4. \tag{A19}$$

Step 4. Note that in this case,

$$\frac{\partial^2 W}{\partial \tau^2} = \frac{\partial p_i}{\partial \tau} (\cdot) \left(\frac{\partial \alpha_i}{\partial n_j} + \gamma \frac{\partial p_j}{\partial n_j} \right) \frac{\partial n_j}{\partial \tau} - \frac{\partial n_i}{\partial \tau},$$

and

$$sign\, \frac{\partial^2 W}{\partial \tau^2} = sign \left[-k^2 + 3k_4 k - (k_4)^2 \right].$$

Let

$$k^2 - 3k_4 k + (k_4)^2 = 0.$$

The solution is as follows.

$$k \in \left\{ \left(\frac{3}{2} - \frac{1}{2}\sqrt{5} \right) k_4,\ \left(\frac{3}{2} + \frac{1}{2}\sqrt{5} \right) k_4 \right\}.$$

Obviously, the first root violates Assumption 1. Hence, we have

$$\frac{\partial^2 W}{\partial \tau^2} < 0 \Rightarrow \left(\frac{3}{2} + \frac{1}{2}\sqrt{5} \right) k_4. \tag{A20}$$

Step 5. Note that

$$(2.5119)\, k_4 < \left(\frac{3}{2} + \frac{1}{2}\sqrt{5} \right) k_4.$$

Hence,

$$\left\{ \begin{array}{l} \frac{\partial^2 W}{\partial s^2} < 0 \\ \frac{\partial^2 W}{\partial \tau^2} < 0 \end{array} \right. \Rightarrow k > \left(\frac{3}{2} + \frac{1}{2}\sqrt{5} \right) k_4. \tag{A21}$$

A.12.3 Summary

Based on the above discussions, we may draw a conclusion.

1. In the first best policy analysis, the "reasonable" lower bound on k is

$$\underline{k}' = \max \left\{ \left(\frac{3}{2} + \frac{1}{2}\sqrt{5} \right) k_1, \left(\frac{3}{2} + \frac{1}{2}\sqrt{13} \right) k_4 \right\}. \tag{A22}$$

   So, in fact, we require in this case $\underline{k}' < k < \overline{k}$.

2. In the second best trade policy analysis, the "reasonable" lower bound on k is

$$\underline{k}'' = \max \left\{ \left(1 + \sqrt{2} \right) k_1, \left(\frac{3}{2} + \frac{1}{2}\sqrt{13} \right) k_4 \right\}. \tag{A23}$$

   So, in fact, we require in this case $\underline{k}'' < k < \overline{k}$.

3. In the second best industrial policy analysis, the "reasonable" lower bound on k is

$$\underline{k}''' = \max\left\{\left(\frac{3}{2}+\frac{1}{2}\sqrt{5}\right)k_1, \left(\frac{3}{2}+\frac{1}{2}\sqrt{5}\right)k_4\right\}. \tag{A24}$$

So, in fact, we require in this case $\underline{k}''' < k < \overline{k}$.