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Computation of LQ Approximations to Optimal Policy Problems in Different Information Settings under Zero Lower Bound Constraints

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Abstract

This paper describes a series of algorithms that are used to compute optimal policy under full and imperfect information. Firstly we describe how to obtain linear quadratic (LQ) approximations to a nonlinear optimal policy problem. We develop novel algorithms that are required as a result of having agents with forward-looking expectations, that go beyond the scope of those that are used when all equations are backward-looking; these are utilised to generate impulse response functions and second moments for the case of imperfect information. We describe algorithms for reducing a system to minimal form that are based on conventional approaches, and that are necessary to ensure that a solution for fully optimal policy can be computed. Finally we outline a computational algorithm that is used to generate solutions when there is a zero lower bound constraint for the nominal interest rate.

Contents

1 Introduction

A Linear-Quadratic (LQ) approach to nonlinear dynamic optimization problems in macroeconomics is widely used for a number of reasons. First, for LQ problems the characterization of time-consistent and commitment equilibria for a single policy maker, and even more so for many interacting policymakers, are well understood. Second, the certainty equivalence property results in optimal rules that are robust in the sense that they are independent of the variance-covariance matrix of additive disturbances. Third, policy can be decomposed into deterministic and stochastic components. This is a very convenient property since it enables the stochastic stabilization component to be pursued using simple Taylor-type feedback rules rather than the exceedingly complex optimal counterpart. Fourth, in an imperfect information context the conditional welfare loss (in deviation form about the deterministic steady state) conveniently decomposes into a deterministic component and two stochastic components one of which describes the effect of imperfect information. Finally for sufficiently simple models, LQ approximation allows analytical rather than numerical solution.

The solution to linear rational expectations models goes back to Blanchard and Kahn (1980) and has since been generalized in various dimensions by Pearlman *et al.* (1986), Klein (2000) and Sims (2003). The early literature on optimal policy with commitment developed LQ infinite time horizon control theory for engineering, non-forward-looking models into a rational expectations (RE) forward-looking context (Driffill (1982), Calvo (1978)), Miller and Salmon (1985), Levine and Currie (1987)).

In a stochastic environment the feedback representation of policy is crucial. For the standard infinite time horizon LQ engineering problem, optimal policy can be represented as a linear time-invariant feedback rule on the state variables; but this is no longer the case when RE are introduced. Then as is shown in Levine and Currie (1987) the optimal policy can only be implemented as a form of integral control. The added complexity of such a rule adds force to the case for designing policy in the form of simple optimized, but sub-optimal rules. The *normative case* for such rules was first put forward by Vines *et al.* (1983), Levine and Currie (1985), Currie and Levine (1985) and Currie and Levine (1993). This early literature considered both monetary and fiscal policy and in the case of Vines *et al.* (1983) incomes and exchange rate targeting policies. The *positive case* for a particular form of monetary policy interest rate rule feeding back on current inflation the output gap was advocated by Taylor (1999), so simple 'Maciejowski-Meade-Vines-Currie-Levine Rules' eventually became known as 'Taylor Rules'. More recently, in the context of DSGE models, we have seen a renewed interest in simple rules in general (referred to by Woodford (2003) as 'explicit instrument rules') and interest rate rules in particular.

Following the pioneering contributions of Kydland and Prescott (1977) and Barro and Gordon (1983), the credibility problem associated with monetary policy has stimulated a huge academic literature that has been influential with policymakers. The central message underlying these contributions is the existence of significant macroeconomic gains, in some sense, from 'enhancing credibility' through formal commitment to a policy rule or through institutional arrangements for central banks such as independence, transparency, and forward-looking inflation targets, that achieve the same outcome. The technical reason for this result is that optimal policy formulated by Pontryagin's maximum principle is time-inconsistent - the simple passage of time, even in a deterministic environment, leads to an incentive to re-optimize and renege on the initial optimal plan. Appreciation of this problem has motivated the examination of policies that are optimal within the constraint of being time consistent (Levine and Currie (1985), Miller and Salmon (1985), Currie and Levine (1987), Cohen and Michel (1988) and Söderlind (1999))

Comparing optimal policy with and without commitment enables us then to quantify the stabilization gains from commitment. A number of papers have addressed this question (see, for example, Vestin (2001), Ehrmann and Smets (2003), McCallum and Nelson (2004), and Dennis and Söderström (2006)), but only in the context of econometric models without micro-foundations and using an ad hoc loss function, or both, or for rudimentary New Keynesian models. The credibility issue only arises because the decisions of consumers and firms are forward looking and depends on expectations of future policy. In the earlier generation of econometric models lacking micro-foundations, many aspects of such forward-looking behaviour were lacking and therefore important sources of time-inconsistency were missing. Although for simple New Keynesian models a quadratic approximation of the representative consumer's utility coincides with the standard ad hoc loss that penalizes variances of the output gap and inflation, in more developed DSGE models this is far from the case. By utilizing an influential empirical micro-founded DSGE model, the euro area model of Smets and Wouters (2003), Levine *et al.* (2008b) use a quadratic approximation of the representative household's utility as the welfare criterion, toe remedy these deficiencies of earlier estimates of commitment gains.

An further important consideration when addressing the gains from commitment, and missing from these earlier studies, is the existence of a nominal interest rate zero lower bound. A number of papers have studied optimal commitment policy with this constraint (for example, Coenen and Wieland (2003), Eggertsson and Woodford (2003), Woodford (2003), chapter 6). In an important contribution to the credibility literature, Adam and Billi (2007) show that ignoring the zero lower bound constraint for the setting of the nominal interest rate can result in considerably underestimating the stabilization gain from commitment. The reason for this is that under discretion the monetary authority cannot make credible promises about future policy. For a given setting of future interest rates the volatility of inflation is driven up by the expectations of the private sector that the monetary authority will re-optimize in the future. This means that to achieve a given low volatility of inflation the lower bound is reached more often under discretion than under commitment. All these authors study a simple New Keynesaian and are able to employ non-linear techniques. In a more developed model such as Smets and Wouters (2003), Levine *et al.* (2008b) adopt the more tractable linear-quadratic (LQ) framework reviewed above.

Further work on policy design takes LQ approximation as given and addresses issues of robustness or the zero lower bound constraint for interest rates, and this is a further Dynare theme that will be addressed by the authors in the future.

Section 2 focuses on the quadratic approximation to the welfare function via the quadratic expansion of the Lagrangian about the long-run optimum, while Section 3 provides a detailed account of the linear approximation to the constraints. Sections 4 and 5 describe how this is used to generate impulse response functions and second moments when agents have imperfect information. Section 6 describes how the likelihood function is computed under symmetric imperfect information between agents and econometrician. Section 7 shows how to obtain a reduced form of the state space which is controllable and observable - both essential for computational purposes. Section 8 discusses issues of optimal policy when the nominal interest rate is constrained not to be below zero with a given (small) probability; we outline an algorithm which has good convergence properties in practice. Section 9 concludes.

2 The LQ Approximation

But what is the correct procedure for replacing a stochastic nonlinear optimization problem with a LQ approximation? As pointed out by Judd (1998), some common methods employed by economists have produced poor approximations which fail to consistently incorporate all relevant second-order terms and thus open up the possibility of spurious results. These pitfalls are also very neatly exposed in Kim and Kim (2003) and Kim and Kim (2006).

Judd (1998), pages 507-509, draws attention to a general Hamiltonian framework for approximating a nonlinear problem by an LQ one due to Magill (1977a), who appears to be the first to have applied it in the economics literature, albeit in a continuous-time framework.[1] This paper is the precursor to a recent literature led by Michael Woodford that considers an LQ approximation to the Ramsey problem in the context of DSGE models.[2] Levine *et al.* (2008a) also apply the Hamiltonian approach to nonlinear prob-

[1] See also, Magill (1977b).

[2] See Woodford (2003), Benigno and Woodford (2003, 2005), Altissimo *et al.* (2005), Benigno and Woodford (2008) for one-country models and Benigno and Benigno (2006) for a two-country generalization. The large distortions' case of Benigno and Woodford (2003) and Benigno and Benigno (2006) uses the method of undetermined coefficients, but their more recent work uses what amounts to the Hamiltonian approach of Magill which involves less algebraic manipulation and provides a more convenient algorithm suitable for numerical computation.

lems in a two-country context to obtain an LQ approximation. It should be noted that the Judd second-order perturbation and Hamiltonian approaches generate the same LQ approximation.

Both Benigno and Woodford (2008) and Levine *et al.* (2008a) develop the Magill framework in presenting a discrete-time version of his results generalized to rational expectations models with forward-looking variables. These results include second-order necessary conditions for non-concave intertemporal problems which are rarely discussed in the literature and have not been published anywhere for forward-looking systems. Levine *et al.* (2008a) explain how these conditions relate to the non-optimality of zero inflation for certain parameter combinations in a New Keynesian setting.

The underlying idea behind LQ approximation is that it is an approximation that is valid in the vicinity of the steady state of the optimal solution to the policy problem. This poses no problems for a purely backward-looking system, but is potentially controversial in economics, given that some behaviour is forward-looking. It would seem therefore that there is potentially one steady state that is a solution to the policy problem when the policymaker can commit, and another when the policymaker cannot. In the first case, the steady state may be solved from the steady state of the first-order conditions for an optimum; this is identical to the case when all forward-looking expectations are treated as though they were dependent on the other variables in the equations in which they appear, so that in effect they are backward-looking[3]. In the second case, the time-consistent solution must be Markov perfect, which requires that forward expectations be expressed in terms of variables which are backward looking; the optimal solution must then be consistent with this assumed behaviour. However, apart from LQ problems there is no known way to calculate the solutions to these time-consistent problems in which the policymaker cannot commit. In addition there is the issue of whether there are further possible steady states which take account of the policymaker merely applying an optimal simple (e.g. Taylor-type monetary) rule.

The literature however appears to have converged to a view that most policymakers have the ability to commit to a long-run value of the policy variable, but there is no guarantee that they have the ability or institutional power to pre-commit to responses to shocks. It follows that one can use an LQ approximation derived from perturbations about the deterministic long run of the fully optimal (pre-commitment) solution, which yields a linear approximation to the dynamics and the quadratic approximation to the welfare. This in turn can be applied to solve the response to shocks under any further behavioural assumption - pre-commitment, time consistency or commitment to simple rules. A variation on the optimal rule is the timeless approach due to Woodford (1999), which has been shown by Blake and Kirsanova (2004) to be sometimes inferior to time-

[3] Of course the *dynamic behaviour* is different from the forward-looking case, as the initial conditions in the latter case can jump.

consistent policy and by Ellison *et al.* (2009) to suffer from lack of transparency.

A useful property of the LQ approximation is that when it is extended to include shocks as well, then the quadratic approximation of the welfare can be expressed in terms of targets or 'bliss points' for linear combinations of macroeconomic variables. Such a property fits in with the notion of targeting rules proposed by Svensson (2003, 2005).

For two decades or more many macroeconomists 'forgot' the work of Magill (1977a), and proceeded by linearizing the dynamics and taking quadratic approximations of the welfare function, which leads to wrong results. For a number of years from about 2000-2006, the LQ approximation to the Ramsey problem was analysed for the 'efficient case' (where subsidies eliminate distortions in the steady state due typically to price or wage frictions) and the 'small distortions case' where such subsidies are not available, but for which the steady state was similar to that of efficiency. However with the resurrection of the Hamiltonian approach the so-called 'large distortions' or LQ approach is becoming the norm.

The problem is to maximize $E_0 \sum \beta^t u(Y_t, W_t)$ such that

$$E_t g(Y_t, Y_{t+1}, W_t, \varepsilon_t) = 0 \qquad h(Y_t, Y_{t-1}, W_t, \varepsilon_t) = 0 \tag{1}$$

We write the problem in this way so that, for convenience, there are no 2nd order derivatives in Y_{t+1} and Y_{t-1}; thus the main constraint is that there are no nonlinear interactions between Y_{t+1} and Y_{t-1}. If there are, then just define a new set of required variables $YL_{it} = Y_{i,t-1}$, and append the latter equations to $h(\, , \,)$ and the new variables to Y_t.

Write the Lagrangian as

$$\sum \beta^t [u(Y_t, W_t) + \lambda^T f(Y_t, Y_{t+1}, Y_{t-1}, W_t, \varepsilon_t)] \tag{2}$$

where $f^T = [g^T \;\; h^T]$. First-order conditions are given by

$$\begin{aligned} \frac{\partial L}{\partial W_t} &= u_2 + \lambda^T f_4(Y_t, Y_{t+1}, Y_{t-1}, W_t, \varepsilon_t) & (3) \\ \frac{\partial L}{\partial Y_t} &= u_1 + \lambda^T f_1(Y_t, Y_{t+1}, Y_{t-1}, W_t, \varepsilon_t) + \frac{1}{\beta}\lambda^T f_2(Y_{t-1}, Y_t, Y_{t-2}, W_{t-1}, \varepsilon_{t-1}) \\ &\quad + \beta \lambda^T f_3(Y_{t+1}, Y_{t+2}, Y_t, W_{t+1}, \varepsilon_{t+1}) & (4) \end{aligned}$$

Second-order terms are given by

$$\begin{gathered} L_{WW} = u_{22} + \lambda^T f_{44} \qquad L_{WY} = u_{21} + \lambda^T f_{41} \\ L_{YY} = u_{11} + \lambda^T f_{11} + \frac{1}{\beta}\lambda^T f_{22} + \beta \lambda^T f_{33} \end{gathered} \tag{5}$$

$$L_{\varepsilon\varepsilon} = \lambda^T f_{55} \qquad L_{W\varepsilon} = \lambda^T f_{45} \qquad L_{Y\varepsilon} = \lambda^T f_{15} \tag{6}$$

Additional terms across time periods are as follows:

$$L_{W_{-1}Y} = \frac{1}{\beta}\lambda^T f_{42} \qquad L_{Y_{-1}Y} = \lambda^T f_{31} + \frac{1}{\beta}\lambda^T f_{12} \qquad L_{WY_{-1}} = \lambda^T h_{43} \qquad L_{\varepsilon_{-1}Y} = \frac{1}{\beta}\lambda^T f_{52} \tag{7}$$

Note that because of our assumption about no interaction between Y_{t+1} and Y_{t-1}, it follows that $L_{Y_{-1}Y_{+1}} = 0$. Since it is expectation of the utility function that is to be maximized, we can ignore $L_{Y_{-1}\varepsilon} = 0$ because $E_0 Y_{t-1}\varepsilon_t = 0$. We shall also assume for convenience that $f_{52} = 0$, so that the last term of (7) is zero; the state space setup derived below requires a new variable which is equal to the shocks, so this is not an unreasonable requirement.

As we shall see below, the linearized state space setup of the dynamics at time t will contain a linear combination of $\Delta Y_t \equiv Y_t - Y_{t-1}$, as well as a ΔY_{t-1}, but in order to accommodate the lags in ΔW_t, as in (7), we need these in the state space as well.

From now on we shall express all linearized variables apart from the shocks as proportional deviations from the steady state of the optimum e.g. $y_{it} = \frac{Y_{it}-\bar{Y}_i}{\bar{Y}_i}$, and all second-derivatives of the Lagrangian are transformed to correspond to this e.g. $L_{wy} = diag(\bar{W}_1, ..., \bar{W}_k) L_{WY} diag(\bar{Y}_1, ..., \bar{Y}_n)$. An exception to this is when $\bar{Y}_i = 0$, in which case we use deviations and not proportional deviations.

Suppose we write the linearized proportional deviation approximation of (1) as

$$A_0 y_{t+1,t} + A_1 y_t = A_2 y_{t-1} + B_1 w_t + B_2 \varepsilon_t \tag{8}$$

In general A_0 will not be of full rank, and its rank could either be (a) less than the number of forward-looking variables or (b) less than the number of equations in which a forward-looking variable appears.

As we shall see in the next section, one can rewrite (8) in Blanchard-Kahn format as

$$\begin{aligned}\begin{bmatrix} \varepsilon_{t+1} \\ s_t \\ x_t \\ x_{t+1,t} \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ G_1 & G_2 & G_3 & G_4 \\ 0 & 0 & 0 & I \\ H_1 & H_2 & H_3 & H_4 \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ s_{t-1} \\ x_{t-1} \\ x_t \end{bmatrix} \\ &+ \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} \varepsilon_{t+1} + \begin{bmatrix} 0 \\ N_1 \\ 0 \\ N_3 \end{bmatrix} w_t \end{aligned} \tag{9}$$

where $y_t = V_1 x_t + V_2 s_t$. However because of the requirement for lags in the instruments to enter the state space, we need to expand (34) to

$$\begin{bmatrix} \varepsilon_{t+1} \\ w_t \\ s_t \\ x_t \\ x_{t+1,t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ G_1 & 0 & G_2 & G_3 & G_4 \\ 0 & 0 & 0 & 0 & I \\ H_1 & 0 & H_2 & H_3 & H_4 \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ w_{t-1} \\ s_{t-1} \\ x_{t-1} \\ x_t \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \varepsilon_{t+1} + \begin{bmatrix} 0 \\ I \\ N_1 \\ 0 \\ N_3 \end{bmatrix} w_t \tag{10}$$

so that $y_t = V_1 x_t + V_2 s_t = [V_2 G_1 \quad 0 \quad V_2 G_2 \quad V_2 G_3 \quad V_2 G_4 + V_1] z_t \equiv \Gamma_y z_t$, where $z_t^T = [\varepsilon_t^T \ w_{t-1}^T \ s_{t-1}^T \ x_{t-1}^T \ x_t^T]$. Also note that $y_{t-1} = [0 \ 0 \ V_2 \ V_1 \ 0] z_t \equiv \Gamma_{y_{-1}} z_t$ and $w_{t-1} = [0 \ I \ 0 \ 0 \ 0] z_t \equiv \Gamma_w z_t$ and that $\varepsilon_t = [I \ 0 \ 0 \ 0 \ 0] z_t \equiv \Gamma_\varepsilon z_t$.

Thus the welfare approximation in each period may be expressed as

$$WelfApprox = \frac{1}{2}\left[z_t^T W_{zz} z_t + w_t^T W_{ww} w_t + z_t^T W_{zw} w_t + w_t^T W_{wz} z_t \right] \tag{11}$$

where

$$\begin{aligned} W_{zz} &= \Gamma_y^T L_{yy} \Gamma_y + \Gamma_\varepsilon^T L_{\varepsilon\varepsilon} \Gamma_\varepsilon + \Gamma_y^T L_{y\varepsilon} \Gamma_\varepsilon + \Gamma_\varepsilon^T L_{\varepsilon y} \Gamma_y \\ &\quad + \Gamma_w^T L_{w_{-1} y} \Gamma_y + \Gamma_y^T L_{y w_{-1}} \Gamma_w + \Gamma_{y_{-1}}^T L_{y_{-1} y} \Gamma_y + \Gamma_y^T L_{y y_{-1}} \Gamma_{y_{-1}} \end{aligned} \tag{12}$$

$$W_{ww} = L_{ww} \qquad W_{wz} = L_{wy} \Gamma_y + L_{w\varepsilon} \Gamma_\varepsilon + L_{w y_{-1}} \Gamma_{y_{-1}} \tag{13}$$

3 From the Sims to the Blanchard-Kahn State Space Form

The aim of this section is to describe an algorithm for turning the state space setup (8) of Dynare, into one that is suitable for obtaining the partial information setup that conforms to that of Pearlman *et al.* (1986):

$$\begin{bmatrix} z_{t+1} \\ x_{t+1,t} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + \begin{bmatrix} C \\ 0 \end{bmatrix} \varepsilon_{t+1} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} w_t \tag{14}$$

We assume that the information set is expressed in linearized form as

$$m_t = L y_t + v_t \tag{15}$$

where typically there is no observation error ($v_t = 0$) and L picks out most of the economic variables, typically excluding capital stock, Tobin's q and shocks. However more generally there is observation error, most notably when using historical data revisions.

Agents' measurements in the Pearlman *et al.* (1986) setup are then given by

$$m_t = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} z_t \\ x_t \end{bmatrix} + v_t \tag{16}$$

(14) can then be used in conjunction with the welfare loss of the previous section to generate fully optimal, time consistent and optimized simple rules for both the full information case using Currie and Levine (1993), and the case when agents have only partial information of the form , using the results of Pearlman (1992). In addition, when estimating a system with given rules, one can generate the likelihood function under partial information (see below).

The algorithm proceeds as follows. For the moment define $u_t^T = [w_t^T \;\; \varepsilon_t^T]$, and $B = [B_1 \;\; B_2]$, so that we may write (8) as

$$A_0 y_{t+1,t} + A_1 y_t = A_2 y_{t-1} + B u_t \tag{17}$$

To repeat, all shocks $\bar{m}_t$ to the system at time t are dated as though they were m_{t-1}. The procedure for conversion to a form suitable for filtering is then as follows:

1. Obtain the singular value decomposition for matrix A_0: $A_0 = UDV^T$, where U, V are unitary matrices. Assuming that only the first m values of the diagonal matrix D are non-zero, we can rewrite this as $A_0 = U_1 D_1 V_1^T$, where U_1 are the first m columns of U, D_1 is the first $m \times m$ block of D and V_1^T are the first m rows of V^T.

2. Multiply (17) by $D_1^{-1} U_1^T$, which yields

$$V_1^T y_{t+1,t} + D_1^{-1} U_1^T A_1 y_t = D_1^{-1} U_1^T A_2 y_{t-1} + D_1^{-1} U_1^T B u_t \tag{18}$$

   Now define $x_t = V_1^T y_t$, $s_t = V_2^T y_t$, and use the fact that $I = VV^T = V_1 V_1^T + V_2 V_2^T$ to rewrite this as:

$$x_{t+1,t} + D_1^{-1} U_1^T A_1 (V_1 x_t + V_2 s_t) = D_1^{-1} U_1^T A_2 (V_1 x_{t-1} + V_2 s_{t-1}) + D_1^{-1} U_1^T B u_t \tag{19}$$

3. Multiply (17) by U_2^T which yields

$$U_2^T A_1 y_t = U_2^T A_2 y_{t-1} + U_2^T B u_t \tag{20}$$

   which can be rewritten as

$$U_2^T A_1 (V_1 x_t + V_2 s_t) = U_2^T A_2 (V_1 x_{t-1} + V_2 s_{t-1}) + U_2^T B u_t \tag{21}$$

4. Suppose that $U_2^T A_1 V_2$ is not invertible. We then need a more sophisticated approach, which reduces the dimension of the forward-looking variables (and increases the dimension of the backward-looking variables), and which may require a loop:

(a) For convenience, rewrite (19) and (21) as

$$x_{t+1,t} + F_1 x_t + F_2 s_t = F_3 x_{t-1} + F_4 s_{t-1} + F_5 u_t \tag{22}$$

$$C_1 x_t + C_2 s_t = C_3 x_{t-1} + C_4 s_{t-1} + C_5 u_t \tag{23}$$

and obtain the SVD $C_2 = J_1 K_1 L_1^T$.

(b) Multiply (23) through by J_2^T where J_2 is orthogonal to J_1 to yield

$$J_2^T C_1 x_t = J_2^T C_3 x_{t-1} + J_2^T C_4 s_{t-1} + J_2^T C_5 u_t \tag{24}$$

Note that the vector s_t will be augmented by $J_2^T C_1 x_t$

(c) Find a matrix M that has the same number of columns as $J_2^T C_1$ and is made up of rows that are orthogonal to $J_2^T C_1$, and define

$$\begin{bmatrix} \bar{x}_t \\ \hat{x}_t \end{bmatrix} = \begin{bmatrix} M \\ J_2^T C_1 \end{bmatrix} x_t \qquad x_t = M_1 \bar{x}_t + M_2 \hat{x}_t \qquad \text{where } [M_1 \ \ M_2] = \begin{bmatrix} M \\ J_2^T C_1 \end{bmatrix}^{-1} \tag{25}$$

Now shift (24) one period forward and take expectations; the expectation $E_t \varepsilon_{t+1} = 0$ automatically, but if any of the coefficients in $J_2^T C_5$ corresponding to $E_t w_{t+1}$ are non-zero, record an error - this will be sorted out at a much later stage. Then equate this to the product of (22) by $J_2^T C_1$ which yields

$$J_2^T C_3 x_t + J_2^T C_4 s_t = J_2^T C_1(-F_1 x_t - F_2 s_t + F_3 x_{t-1} + F_4 s_{t-1} + F_5 u_t) \tag{26}$$

or equivalently

$$(J_2^T C_3 + J_2^T C_1 F_1) x_t + (J_2^T C_4 + J_2^T C_1 F_2) s_t = J_2^T C_1 (F_3 x_{t-1} + F_4 s_{t-1} + F_5 u_t) \tag{27}$$

(d) Thus we can rewrite the system (22), (23) in terms of forward-looking variables $\bar{x}_t$ and backward-looking variables s_t, $\hat{x}_t$:

$$\bar{x}_{t+1,t} + MF_1 M_1 \bar{x}_t + [MF_2 \ \ MF_1 M_2] \begin{bmatrix} s_t \\ \hat{x}_t \end{bmatrix} \tag{28}$$

$$= MF_3 M_1 \bar{x}_{t-1} + [MF_4 \ \ MF_3 M_2] \begin{bmatrix} s_{t-1} \\ \hat{x}_{t-1} \end{bmatrix} + MF_5 u_t$$

$$\begin{bmatrix} C_1 M_1 \\ J_2^T (C_3 + C_1 F_1) M_1 \end{bmatrix} \bar{x}_t + \begin{bmatrix} C_2 & C_1 M_2 \\ J_2^T (C_4 + C_1 F_2) & J_2^T (C_3 + C_1 F_1) M_2 \end{bmatrix} \begin{bmatrix} s_t \\ \hat{x}_t \end{bmatrix} \tag{29}$$

$$= \begin{bmatrix} C_3 M_1 \\ J_2^T C_1 F_3 M_1 \end{bmatrix} \bar{x}_{t-1} + \begin{bmatrix} C_4 & C_3 M_2 \\ J_2^T C_1 F_4 & J_2^T C_1 F_3 M_2 \end{bmatrix} \begin{bmatrix} s_{t-1} \\ \hat{x}_{t-1} \end{bmatrix} + \begin{bmatrix} C_5 \\ J_2^T C_1 F_5 \end{bmatrix} u_t$$

(e) Thus the number of forward-looking states has decreased because $\bar{x}_t = M_1 x_t$, and the number of backward-looking states $\bar{s}_t = \begin{bmatrix} s_t \\ \hat{x}_t \end{bmatrix}$ has increased by the same amount. In addition the relationship $y_t = V_1 x_t + V_2 s_t$ has changed to

$$y_t = V_1 M_1 \bar{x}_t + \begin{bmatrix} V_2 \\ V_1 M_2 \end{bmatrix} \bar{s}_t \tag{30}$$

(f) We then re-define the matrices $C_1, C_2, C_3, C_4, C_5, F_1, F_2, F_3, F_4, F_5$ accordingly, and check whether C_2 is invertible. If not, then go back to (a); otherwise continue.

5. Now that C_2 is invertible, we can rewrite (19) and (21) as

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ F_2 & 0 & I \end{bmatrix} \begin{bmatrix} s_t \\ x_t \\ x_{t+1,t} \end{bmatrix} = \begin{bmatrix} C_2^{-1}C_4 & C_2^{-1}C_3 & -C_2^{-1}C_1 \\ 0 & 0 & I \\ F_4 & F_3 & -F_1 \end{bmatrix} \begin{bmatrix} s_{t-1} \\ x_{t-1} \\ x_t \end{bmatrix} + \begin{bmatrix} C_2^{-1}C_5 \\ 0 \\ F_5 \end{bmatrix} u_t \tag{31}$$

which can be further rewritten as

$$\begin{bmatrix} s_t \\ x_t \\ x_{t+1,t} \end{bmatrix} = \begin{bmatrix} C_2^{-1}C_4 & C_2^{-1}C_3 & -C_2^{-1}C_1 \\ 0 & 0 & I \\ F_4 - F_2C_2^{-1}C_4 & F_3 - F_2C_2^{-1}C_3 & -F_1 + F_2C_2^{-1}C_1 \end{bmatrix} \begin{bmatrix} s_{t-1} \\ x_{t-1} \\ x_t \end{bmatrix} + \begin{bmatrix} H_1 \\ 0 \\ H_3 - FH_1 \end{bmatrix} u_t \tag{32}$$

6. The measurements $m_t = Ly_t + v_t$ can be written in terms of the states as $m_t = L(V_1 x_t + V_2 s_t) + v_t$. To write the system in a form which corresponds to that of Pearlman *et al.* (1986) we need to write the measurements in terms of the forward-looking variables x_t and in terms of the backward-looking variables s_{t-1}, x_{t-1}. We do this by substituting for s_t from (32); but this introduces a term in u_t into the expression, and Pearlman *et al.* (1986) assume that shock terms in the dynamics and in the measurements are uncorrelated with one another. To remedy this, we incorporate ε_t into the predetermined variables, but we can retain w_t as it stands.

Defining

$$\begin{bmatrix} H_1 \\ 0 \\ H_3 - FH_1 \end{bmatrix} u_t = \begin{bmatrix} P_1 \\ 0 \\ P_3 \end{bmatrix} \varepsilon_t + \begin{bmatrix} N_1 \\ 0 \\ N_3 \end{bmatrix} w_t \tag{33}$$

we may rewrite the dynamics and measurement equations in the form:

$$\begin{bmatrix} \varepsilon_{t+1} \\ s_t \\ x_t \\ x_{t+1,t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ P_1 & C_2^{-1}C_4 & C_2^{-1}C_3 & -C_2^{-1}C_1 \\ 0 & 0 & 0 & I \\ P_3 & F_4 - F_2C_2^{-1}C_4 & F_3 - F_2C_2^{-1}C_3 & -F_1 + F_2C_2^{-1}C_1 \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ s_{t-1} \\ x_{t-1} \\ x_t \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} \varepsilon_{t+1} + \begin{bmatrix} 0 \\ N_1 \\ 0 \\ N_3 \end{bmatrix} w_t \tag{34}$$

$$m_t = \begin{bmatrix} LV_2P_1 & LV_2C_2^{-1}C_4 & LV_2C_2^{-1}C_3 & LV_1 - LV_2C_2^{-1}C_1 \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ s_{t-1} \\ x_{t-1} \\ x_t \end{bmatrix} + LV_2N_1w_t + v_t \tag{35}$$

Thus the setup is as required, with the vector of predetermined variables given by $[\varepsilon_t' \; s_{t-1}' \; x_{t-1}']'$, and the vector of jump variables given by x_t. Note that there is an issue not covered by Pearlman (1992), namely that the instrument w_t is part of the measurement equation; if we assume that the instruments are observed, then there is no problem to modify the theory.

Note that this means that the relationship between the underlying variables y_t and the state space variables and instruments is given by

$$\begin{bmatrix} y_t \\ w_t \end{bmatrix} = \Gamma \begin{bmatrix} \varepsilon_t \\ s_{t-1} \\ x_{t-1} \\ x_t \\ w_t \end{bmatrix} \quad \text{where } \Gamma = \begin{bmatrix} V_2P_1 & V_2G_{11} & V_2G_{12} & V_1 - V_2G_{13} & V_2N_1 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \tag{36}$$

4 Impulse Response Functions

We distinguish between two cases: agents having full information, and agents having partial information m_t at time t. Assume that the system (14) contains no instruments and is already saddlepath stable, so that $D_1 = 0, D_2 = 0$, and that the relationship (36) can be written as $y_t = \Gamma_1 z_t + \Gamma_2 x_t$.

4.1 Full Information Case:

It is well-known that the impulse response functions can be generated from

$$z_{t+1} = (A_{11} - A_{12}N)z_t + C\varepsilon_{t+1} \qquad x_t = -Nz_t \qquad y_t = \Gamma_1 z_t + \Gamma_2 x_t \tag{37}$$

where

$$\begin{bmatrix} N & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \Lambda^U \begin{bmatrix} N & I \end{bmatrix} \tag{38}$$

and Λ^U is a square matrix with unstable eigenvalues equal to those of the system.

4.2 Partial Information Case:

The reduced-form solution, that can be used to generate the impulse response functions is then given by:

$$\begin{aligned} System: \quad z_{t+1} &= Fz_t + (A-F)\tilde{z}_t \\ &\quad +(F-A)PH^T(HPH^T+V)^{-1}(H\tilde{z}_t+v_t) + C\varepsilon_{t+1} && (39) \\ x_t &= -Nz_t + (N - A_{22}^{-1}A_{21})\tilde{z}_t \\ &\quad -(N-A_{22}^{-1}A_{21})PH^T(HPH^T+V)^{-1}(H\tilde{z}_t+v_t) && (40) \\ Innovations: \quad \tilde{z}_{t+1} &= A\tilde{z}_t - APH^T(HPH^T+V)^{-1}(H\tilde{z}_t+v_t) + C\varepsilon_{t+1} && (41) \\ Measurement: \quad m_t &= Ez_t + (H-E)\tilde{z}_t + v_t \\ &\quad -(H-E)PH^T(HPH^T+V)^{-1}(H\tilde{z}_t+v_t) \\ &= Ez_{t,t-1} + (EPH^T+V)(HPH^T+V)^{-1}(H\tilde{z}_t+v_t) && (42) \\ Variables: \quad y_t &= \Gamma_1 z_t + \Gamma_2 x_t && (43) \end{aligned}$$

$where \quad F = A_{11} - A_{12}N \quad A = A_{11} - A_{12}A_{22}^{-1}A_{21} \quad E = K_1 - K_2N \quad H = K_1 - K_2A_{22}^{-1}A_{21}$

V is the covariance matrix of the measurement errors, and P is the solution of the Riccati equation given by

$$P = APA^T - APH^T(HPH^T+V)^{-1}HPA^T + CUC^T \tag{44}$$

and U is the covariance matrix of the shocks to the system.

5 Covariances and Autocovariances

Pearlman *et al.* (1986) show that

$$cov \begin{bmatrix} \tilde{z}_t \\ z_t \end{bmatrix} = \begin{bmatrix} P & P \\ P & P+M \end{bmatrix} \equiv P_0 \tag{45}$$

where M satisfies

$$M = FMF^T + FPH^T(HPH^T+V)^{-1}HPF^T \tag{46}$$

To calculate the covariances and autocovariances of y_t, we note from the previous section that y_t can also be written as $y_t = V_1x_t + V_2s_t$, and that the last bottom part of the vector

z_t is given by $[s_{t-1}^T \; x_{t-1}^T]^T$, of dimension n, say. Then defining Ω_0 as the bottom right $n \times n$ matrix of $(P + M)$, it follows that

$$cov(y_t) = [\; V_2 \quad V_1 \;]\Omega_0 \begin{bmatrix} V_2^T \\ V_1^T \end{bmatrix} \equiv R_0 \tag{47}$$

To calculate the autocovariances, define

$$\Phi = \begin{bmatrix} A(I - PH^T(HPH^T + V)^{-1}H) & 0 \\ (A - F)(I - PH^T(HPH^T + V)^{-1}H) & F \end{bmatrix} \tag{48}$$

Then the sequence of auto-covariance matrices of y_t are defined as follows:

$$E\left(\begin{bmatrix} \tilde{z}_{t+k} \\ z_{t+k} \end{bmatrix}, \begin{bmatrix} \tilde{z}_t \\ z_t \end{bmatrix}\right) \equiv P_k = \Phi^k P_0 = \Phi P_{k-1} \tag{49}$$

Defining Ω_k as the bottom right $n \times n$ matrix of P_k, it follows that

$$cov(y_{t+k}, y_t) = E(y_{t+k} y_t^T) = [\; V_2 \quad V_1 \;]\Omega_k \begin{bmatrix} V_2^T \\ V_1^T \end{bmatrix} \equiv R_k \tag{50}$$

It follows that

1. the autocorrelation function of the ith element of Y is given by the sequence $\frac{(R_1)_{ii}}{(R_0)_{ii}}, \frac{(R_2)_{ii}}{(R_0)_{ii}}, \frac{(R_3)_{ii}}{(R_0)_{ii}}, \ldots$.

2. the correlation matrix of the y_t variables is defined as

$$Corr = \hat{\Delta} R_0 \hat{\Delta}^T \;\text{ where }\; \hat{\Delta} = diag(\surd(R_0)_{11}, \surd(R_0)_{22}, \surd(R_0)_{33}, \ldots) \tag{51}$$

6 Calculation of the Likelihood Function

Once again we assume that there are no policy instruments w_t and that the system is saddlepath stable. In addition we assume that agents have the same information set as the econometrician.

From the perspective of the econometrician, who starts out with no information other than the structure of the system, the reduced form is given by (39) and (42), with covariance matrices as calculated above. In order to reduce the amount of notation, we assume that the measurement errors are incorporated into the shocks so that the vector ε_{t+1} is augmented by v_{t+1}. After some algebraic manipulation it can be shown that the optimal estimate of $\tilde{z}_t$ using information up to $t - 1$ is equal to 0, from which it follows that the Kalman filtering equation for z_t is given by

$$z_{t+1,t} = F z_{t,t-1} + F Z_t E^T (E Z_t E^T)^{-1} e_t \tag{52}$$

where $e_t = m_t - Ez_{t,t-1}$ and

$$Z_{t+1} = FZ_tF^T + PH^T(HPH^T)^{-1}HP - FZ_tE^T(EZ_tE^T)^{-1}EZ_tF^T \tag{53}$$

the latter being a time-dependent Ricatti equation.

The likelihood function is standard:

$$2lnL = -\sum lndet(cov(e_t) - \sum e_t^T(cov(e_t))^{-1}e_t \tag{54}$$

where

$$cov(e_t) = EZ_tE^T \tag{55}$$

The system is initialised at $z_{1,0} = 0$, and Z_1 is initialised by the solution of the Lyapunov equation

$$Z_1 = FZ_1F^T + PH^T(HPH^T)^{-1}HP \tag{56}$$

where P is the steady state of the Riccati equation above.

7 Controllable and Observable Forms

For the calculation of optimal policy based on the linearized version of the nonlinear dynamics, there are two numerical problems. Firstly, if for example one is studying a small open economy then there will be a part of the model (without an exogenous instrument) describing the large economy that typically involves forward looking variables. This will therefore have unstable eigenvalues that the period-doubling solution for the Riccati equation cannot handle - this is the controllability problem. Secondly, the linearization of the New Keynesian Phillips curve at an inflation level of 0 generates a set of at least two dynamic equations that can be collapsed into just one (see Appendix for example); the number of variables needs to be collapsed down as well, otherwise this also generates problems with the period-doubling solution - this is the observability problem.

Write the system in the form

$$\begin{bmatrix} z_{t+1} \\ x_{t+1,t} \end{bmatrix} = A\begin{bmatrix} z_t \\ x_t \end{bmatrix} + B\varepsilon_{t+1} + Dw_t \qquad y_t = \Gamma\begin{bmatrix} z_t \\ x_t \end{bmatrix} \tag{57}$$

.

7.1 Controllable Form

Suppose that the system is not controllable, so that there exist eigenvalues λ and eigenvectors m^T such that

$$m^TA = \lambda m^T \qquad m^TB = 0 \qquad m^TD = 0 \tag{58}$$

This implies that $m^T x_{t+1} = \lambda m^T x_t$ i.e. $m^T x_t$ evolves independently of any instruments or shocks.

If this were a backward-looking system, then the following algorithm is appropriate:

Define matrix T and its inverse, so that it is made up of m^T and a set of row vectors F orthogonal to m^T:

$$T = \begin{bmatrix} F \\ m^T \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} \hat{F} & n \end{bmatrix} \qquad TT^{-1} = \begin{bmatrix} F \\ m^T \end{bmatrix} \begin{bmatrix} \hat{F} & n \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix} \tag{59}$$

We can then rewrite the dynamic equations as

$$\begin{bmatrix} F \\ m^T \end{bmatrix} x_{t+1} = \begin{bmatrix} F \\ m^T \end{bmatrix} A \begin{bmatrix} \hat{F} & n \end{bmatrix} \begin{bmatrix} F \\ m^T \end{bmatrix} x_t + \begin{bmatrix} F \\ m^T \end{bmatrix} B w_t \tag{60}$$

which can be rewritten as

$$\begin{bmatrix} F x_{t+1} \\ m^T x_{t+1} \end{bmatrix} = \begin{bmatrix} FA\hat{F} & Fn \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} F x_t \\ m^T x_t \end{bmatrix} + \begin{bmatrix} FBu_t \\ 0 \end{bmatrix} \tag{61}$$

so the reduced form is

$$F x_{t+1} = FA\hat{F}(F x_t) + FBw_t \qquad y_t = \Gamma \hat{F}(F x_t) \tag{62}$$

Thus the system is now written in terms of $F x_t$.

However for RE systems one has to proceed with more care. Firstly, if m^T contains non-zero elements corresponding to both forward and backward-looking variables, then this will imply a potential saddlepath relationship **whether or not** $m^T B = 0$, provided that the eigenvalue λ has modulus greater than 1. Secondly, if $m^T B = 0$, then one one only reduces the system if all the the non-zero elements of m^T correspond only to backward or only to forward-looking variables.

Thus the program proceeds as follows:

1. Find λ, m^T. If $m^T D \neq 0$ do nothing, but if $m^T D = 0$ then
   - if m^T contains non-zero elements corresponding to **both** forward and backward variables **and** $|\lambda| > 1$, proceed to 2; reduce the number of FL variables by 1. Otherwise do nothing
   - if m^T contains non-zero elements corresponding to **only** forward or **only** backward variables **and** $m^T B = 0$, proceed to 2; reduce either the number of FL or the number of BL variables by 1 accordingly. Otherwise do nothing.
2. Choose F to create T as in (60); find T^{-1} and hence $\hat{F}$
3. Calculate $FA\hat{F}, FB, \Gamma\hat{F}$, and record whether the dimension of forward or backward looking variables is reduced.

Remark: Typically all the eigenvalues and eigenvectors of the non-controllable states will be grouped together in one sweep, rather than dealt with one by one.

7.2 Observable Form

Suppose there are eigenvalues μ and eigenvectors s such that

$$\Gamma s = 0 \qquad As = \mu s \tag{63}$$

Now define matrix T and its inverse, so that it is made up of s and a set of column vectors G each having a single 1 with remaining values 0:

$$T = \begin{bmatrix} G & s \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} H \\ v^T \end{bmatrix} \qquad T^{-1}T = \begin{bmatrix} H \\ v^T \end{bmatrix} \begin{bmatrix} G & s \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix} \tag{64}$$

We can then rewrite the dynamic equations as

$$\begin{bmatrix} H \\ v^T \end{bmatrix} x_{t+1} = \begin{bmatrix} H \\ v^T \end{bmatrix} A \begin{bmatrix} G & s \end{bmatrix} \begin{bmatrix} H \\ v^T \end{bmatrix} x_t + \begin{bmatrix} H \\ v^T \end{bmatrix} Bw_t \qquad y_t = \Gamma \begin{bmatrix} G & s \end{bmatrix} \begin{bmatrix} H \\ v^T \end{bmatrix} x_t \tag{65}$$

which can be rewritten as

$$\begin{bmatrix} Hx_{t+1} \\ v^T x_{t+1} \end{bmatrix} = \begin{bmatrix} HAG & 0 \\ v^T AG & \lambda \end{bmatrix} \begin{bmatrix} Hx_t \\ v^T x_t \end{bmatrix} + \begin{bmatrix} HBu_t \\ v^T Bu_t \end{bmatrix} \qquad y_t = \Gamma G(Hx_t) \tag{66}$$

The dynamics of $v^T x_t$ then play no role in the measurement y_t, so we can rewrite this in reduced form as

$$Hx_{t+1} = HAG(Hx_t) + HBw_t \qquad y_t = \Gamma G(Hx_t) \tag{67}$$

Thus the system is now written in terms of Hx_t.

To program this, we proceed as follows:

- Find μ, s. Select an s that contains only non-zero entries corresponding to backward-looking variables **or** only forward-looking variables.
- Choose G to create T as in (65); find T^{-1} and hence H.
- Calculate $HAG, HB, \Gamma G$, and record whether the dimension of forward or backward looking variables is reduced.
- If the only s that are left contain non-zero elements in the positions of both FL and BL variables, record as error and exit.

8 Optimal Policy and the Zero Lower Bound

Details of optimal policy for the full information case can be viewed in Currie and Levine (1993) and for the partial information case are in Pearlman (1992), so are not detailed here. Three types of optimal policy are studied - fully optimal, time-consistent and optimized simple rules. If the variances of shocks are sufficiently large, this will lead to a large nominal interest rate variability and the possibility of the nominal interest rate becoming negative.

To rule out this possibility but remain within the tractable LQ framework, we follow Woodford (2003), chapter 6, and modify our interest-rate rules to approximately impose an interest rate ZLB so that this event hardly ever occurs. Write the quadratic approximation to the single-period loss function can be written as L_t. As in Woodford (2003), chapter 6, the ZLB constraint is implemented by modifying the single period welfare loss to $L_t + w_r r_t^2$.

Then following Levine *et al.* (2008b), the policymaker's optimization problem is to choose w_r and the unconditional distribution for R_t (characterized by the steady state variance) shifted to the right about a new non-zero steady state inflation rate and a higher nominal interest rate, such that the probability, p, of the interest rate hitting the lower bound is very low. This is implemented by calibrating the weight w_r *different for each policy rule - fully optimal (OPT), time-consistent (TCT) or optimized simple (SIM)* so that $z_0(p)\sigma_r < R_n$ where $z_0(p)$ is the critical value of a standard normally distributed variable Z such that prob $(Z \leq z_0) = p$, $R_n^* = (1+\pi^*)R + \pi^*$ is the steady state nominal interest rate, R is the steady state real interest rate, $\sigma_r^2 = \text{var}(R_n)$ is the unconditional variance and π^* is the new steady state inflation rate. Given σ_r the steady state positive inflation rate that will ensure $R_t \geq 0$ with probability $1-p$ is given by

$$\pi^* = \max\left[\frac{z_0(p)\sigma_r - R}{1+R} \times 100, 0\right] \tag{68}$$

In our linear-quadratic framework we can write the intertemporal expected welfare loss at time $t = 0$ as the sum of stochastic and deterministic components, $\Omega_0 = \tilde{\Omega}_0 + \bar{\Omega}_0$. Note that $\bar{\Omega}_0$ incorporates in principle the new steady state values of all the variables; however the NK Phillips curve being almost vertical, the main extra term comes from a contribution from $(\pi^*)^2$. By increasing w_r we can lower σ_r thereby decreasing π^* and reducing the deterministic component, but at the expense of increasing the stochastic component of the welfare loss. By exploiting this trade-off, we then arrive at the optimal policy that, in the vicinity of the steady state, imposes the ZLB constraint, $r_t \geq 0$ with probability $1-p$.

Note that in our LQ framework, the zero interest rate bound is very occasionally hit. Then interest rate is allowed to become negative, possibly using a scheme proposed by Gesell (1934) and Keynes (1936). Our approach to the ZLB constraint (following Woodford (2003))[4] in effect replaces it with a nominal interest rate variability constraint which ensures the ZLB is hardly ever hit. By contrast the work of a number of authors

[4]We generalize the treatment of Woodford however by allowing the steady-state inflation rate to rise. Our policy prescription has recently been described as a dual mandate in which a central bank committed to a long-run inflation objective sufficiently high to avoid the ZLB constraint as well as a Taylor-type policy stabilization rule about such a rate - see Blanchard *et al.* (2010) and Gavin and Keen (2011).

including Adam and Billi (2007), Coenen and Wieland (2003), Eggertsson and Woodford (2003) and Eggertsson (2006) study optimal monetary policy with commitment in the face of a non-linear constraint $i_t \geq 0$ which allows for frequent episodes of liquidity traps in the form of $i_t = 0$.

A problem with the procedure described so far is that it shifts the steady state to a new one with a higher inflation, but continues to approximate the loss function and the dynamics about the original Ramsey steady state. We know from the work of Ascari and Ropele (2007a) and Ascari and Ropele (2007b) that the dynamic properties of the linearized model change significantly when the model is linearized about a non-zero inflation. This issue is addressed analytically in Coibion *et al.* (2011), but in a very simple NK model. We now propose a general solution and numerical procedure that can be used in any DSGE model.

1. Set up the Non-Linear Model in Dynare. Define a new parameter: p, the probability of hitting the ZLB, the weight w_r on the variance of the nominal net interest rate and a target steady state nominal interest rate $\hat{R}_n$.
2. Modify the single-period utility to $L_t = \Lambda_t - \frac{1}{2}w_r(R_{n,t} - \hat{R}_n)^2$.
3. In the first iteration let w_r to be low to get through OPT, say $w_r = 0.001$ and $\hat{R}_n = \frac{1}{\beta} - 1$, the no-growth zero-inflation steady-state nominal interest rate corresponding to the standard Ramsey problem with no ZLB considerations.
4. Perform the LQ approximation of the Ramsey optimization problem with modified loss function L_t. For standard problems the steady state nominal net inflation rate $\pi^{Ramsey} = 0$ and $R_n^{Ramsey} = \frac{1}{\beta} - 1$. In general, for $w_r > 0$, $R_n^{Ramsey} \neq \frac{1}{\beta} - 1$
5. Compute OPT or TCT or optimized simple rule SIM in Dynare-ACES
6. Extract $\sigma_r = \sigma_r(w_r)$.
7. Extract the minimized conditional (in the vicinity of the steady state, i.e. $z_0 = 0$ in ACES) stochastic loss function $\tilde{\Omega}_0(w_r)$
8. Compute $r_n^* = r_n^*(w_r)$ defined by $r_n^*(w_r) = \max\left[z_0(p)\sigma_r - R_n^{Ramsey} \times 100, 0\right]$, where in the first iteration $R_n^{Ramsey} = \frac{1}{\beta} - 1$ as noted above. This ensures that the ZLB is reached with a low probability p.
9. If $r_n^* < 0$, the ZLB constraint is not binding; if $r_n* > 0$ it is. Proceed in either case.
10. Define $\pi^* = \pi^{Ramsey} + r_n^*$.
11. Compute the steady state $\bar{\Omega}_0(\pi^*)$ at the steady state of the model with a shifted new inflation rate π^*. Then compute $\Delta\bar{\Omega}_0(r^*(w_r)) \equiv \bar{\Omega}_0(\pi^*) - \bar{\Omega}_0(\pi^{Ramsey})$

12. Compute the actual total stochastic plus deterministic loss function that hits the ZLB with a low probability p

$$\Omega_0(w_r) = \tilde{\Omega}_0^{actual}(w_r) + \Delta\bar{\Omega}_0(r^*(w_r)) \tag{69}$$

13. A good approximation for $\tilde{\Omega}_0(w_r)^{actual}$ is $\tilde{\Omega}_0(w_r)^{actual} \simeq \tilde{\Omega}_0(w_r) - \frac{1}{2}w_r\sigma_r^2$ provided the welfare loss is multiplied by $1-\beta$.

14. Finally minimize $\Omega_0(w_r)$ with respect to w_r. This imposes the ZLB constraint as in Figure 1.

15. What now changes is to reset $\hat{R}_n = \frac{1}{\beta} - 1 + \alpha\pi^*$ where $\alpha \in (0,1]$ is a relaxation parameter to experiment with, i.e., $(\hat{R}_n)^{new} = R^{Ramsey,old} + r_n^*$, $w_r^{new} = \text{argmin}\Omega_0(w_r)$ and return to the beginning. Iterate until $\hat{R}_n$ and w_r are unchanged.

9 Conclusions

We have provided novel algorithms for writing RE models in Blanchard-Kahn form, thereby enabling standard methods to be used for computing optimal policy, impulse response functions and second moments. We have also demonstrated how standard methods for controllability and observability need to be tailored for RE models. Finally we have described an algorithm with good convergence properties for computing policies - fully optimal, time consistent and optimized simple - that satisfy the ZLB for nominal interest rates.

Appendix

A Example of Non-observable Form

Consider the part of the NK Phillips Curve in non-linear form:

$$H_t - \xi\beta E_t[\Pi_t^{\zeta-1}H_{t+1}] = Y_t^{1-\sigma} \tag{70}$$

$$J_t - \xi\beta E_t[\Pi_t^{\zeta}J_{t+1}] = \alpha\left(\frac{Y_t}{A_t}\right)^{1-\sigma} \tag{71}$$

$$1 = \xi\Pi_t^{\zeta-1} + (1-\xi)\left(\frac{J_t}{H_t}\right)^{1-\zeta} \tag{72}$$

Linearization yields

$$h_t - \xi\beta((\zeta-1)\pi_{t+1} + h_{t+1}) = (1-\xi\beta)(1-\sigma)y_t \tag{73}$$

$$j_t - \xi\beta(\zeta\pi_{t+1} + j_{t+1}) = (1-\xi\beta)(1+\phi)(y_t - a_t) \tag{74}$$

Multiply (73) by ζ and (74) by $\zeta - 1$ and subtract, which gives

$$\zeta h_t - (\zeta - 1) j_t - \xi\beta(\zeta h_{t+1} - (\zeta - 1) j_{t+1}) = (1 - \xi\beta)[\zeta(1 - \sigma) y_t + (\zeta - 1)(1 + \phi)(y_t - a_t)] \quad (75)$$

This is an equation in $\zeta h_t - (\zeta - 1) j_t$ which is controllable, but which is not observable. Only $\pi_t = \frac{1-\xi}{\xi}(j_t - h_t)$ is observable.

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