MAXIMUM PRINCIPLES IN ANALYTICAL ECONOMICS

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The very name of my subject, economics, suggests economizing or maximizing. But Political Economy has gone a long way beyond home economics. Indeed, it is only in the last third of the century, within my own lifetime as a scholar, that economic theory has had many pretensions to being *itself* useful to the practical businessman or bureaucrat. I seem to recall that a great economist of the last generation, A. C. Pigou of Cambridge University, once asked the rhetorical question, "Who would ever think of employing an economist to run a brewery?" Well, today, under the guise of operational research and managerial economics, the fanciest of our economic tools are being utilized in enterprises both public and private.

So at the very foundations of our subject maximization is involved. My old teacher, Joseph Schumpeter, went much farther. Instead of being content to say economics must borrow from logic and rational empirical enquiry, Schumpeter made the remarkable claim that man's ability to operate as a logical animal capable of systematic empirical induction was itself the direct outcome of the Darwinian struggle for survival. Just as man's thumb evolved in the struggle to make a living - to meet his economic problem - so did man's brain evolve in response to the economic problem. Coming forty years before the latest findings in ethology by Konrad Lorenz and Nikolaas Tinbergen, this is a rather remarkable insight. It would take me away from my present subject to more than mention the further view enunciated by Schumpeter [1] in launching the new subject of econometrics. Quantity, he said, is studied by the physicist or other natural scientist at a fairly late and sophisticated stage of the subject. Since a quantitative approach is, so to speak, at the discretion of the investigator, all the more credit to the followers of Galileo and Newton for taking the mathematical approach. But in economics, said Schumpeter, the very subject matter presents itself in quantitative form: take away the numerical magnitude of price or barter exchange-ratio and you have nothing left. Accounting does not benefit from arithmetic; it is arithmetic - and in its early stages, according to Schumpeter, arithmetic is accounting, just as geometry in its early stages is surveying.

I must not leave you with the impression that analytical economics is concerned with maximization principles primarily in connection with providing vocational handbooks for the practising decision maker. Even back in the last generation, before economics had pretensions toward being itself useful to practitioners, we economists were occupied with maxima and minima. Alfred Marshall's *Principles of Economics*, the dominating treatise in the forty years after 1890, dealt much with optimal output at the point of maximum net profit. And long before Marshall, A. A. Cournot's *1838* classic, *Researches into the Mathematical Principles of the Theory of Wealth*, put the differential calculus to work in the study of maximum-profit output. Concern for minimization of cost goes back a good deal more than a century, at least back to the marginal productivity notions of von Thünen.

It is fashionable these days to speak of identity crises. One must not make the mistake attributed to Edward Gibbon when he wrote his *Decline and Fall of the Roman Empire*. Gibbon, it was said, sometimes confused himself and the Roman Empire. I know in these days of the living theater - and I ought to add on this occasion, of the theories of quantum mechanics - the distinction often becomes blurred between observing audience and acting players, between the observing scientist and the guinea pigs or atoms under observation. As I shall discuss in connection with the role of maximum principles in natural science, the plumb-line trajectory of a falling apple and the elliptical orbit of a wandering planet may be capable of being described by the optimizing solution for a specifiable programming problem. But no one will be tempted to fall into a reverse version of the Pathetic Fallacy and attribute to the apple or the planet freedom of choice and consciously deliberative minimizing. Nonetheless, to say "Galileo's ball rolls down the inclined plane as *if* to minimize the integral of action, or to minimize Hamilton's integral," does prove to be useful to the observing physicists, eager to formulate predictable uniformities of nature.

What is it that the scientist finds useful in being able to relate a positive description of behaviour to the solution of a maximizing problem? That is what a good deal of my own early work was about. From the time of my first papers on "Revealed Preference" [2] through the completion of *Foundations of Economic Analysis*, I found this a fascinating subject. The scientist, as with the housewife, finds his work is really never done. Just in these last weeks I have been working on the very difficult problem of understanding stochastic speculative price - e.g. how cocoa prices fluctuate on the London and New York exchanges. [3] When confronted with an unmanageable system of non-linear difference equalities and inequalities, I could have despaired of finding in the mathematical literature a proof of even the existence of a solution. But suddenly the problem became solvable in a flash, when out of the strata of memory, I dredged up the recollection that my positive descriptive relations could be interpreted as the necessary and sufficient conditions of a well defined maximum problem. But I run ahead of my story if I give you the impression that maximum principles are valuable merely as a convenience and crutch to the less-than-omniscient analyst.

Seventy years ago, when the Nobel Foundation was first established, the methodological views of Ernst Mach enjoyed a popularity they no longer possess.[1] Mach you will remember, said that what the scientist seeks is an "economical" description of nature. By this he did not mean that the navigation needs of traders decreed that Newton's system of the world had to get born. He meant rather that a good explanation is a simple one that is easy to remember and one which fits a great variety of the observable facts. It would be a Gibbonlike fallacy to illustrate this by the deistic view of Maupertuis that the laws of nature are the working out of a simple teleological purpose. Mach is not saying that Mother Nature is an economist; what he is saying is that the scientist who formulates laws of observed empirical phenomena is essentially an economist or economizer.

Nonetheless, I must point out that these distinct roles are, almost by coincidence so to speak, closely related. Often the physicist gets a better, a more economical, description of nature if he is able to formulate the observed laws by a maximum principle. Often the economist is able to get a better, a more economical, description of economic behaviour from the same device.

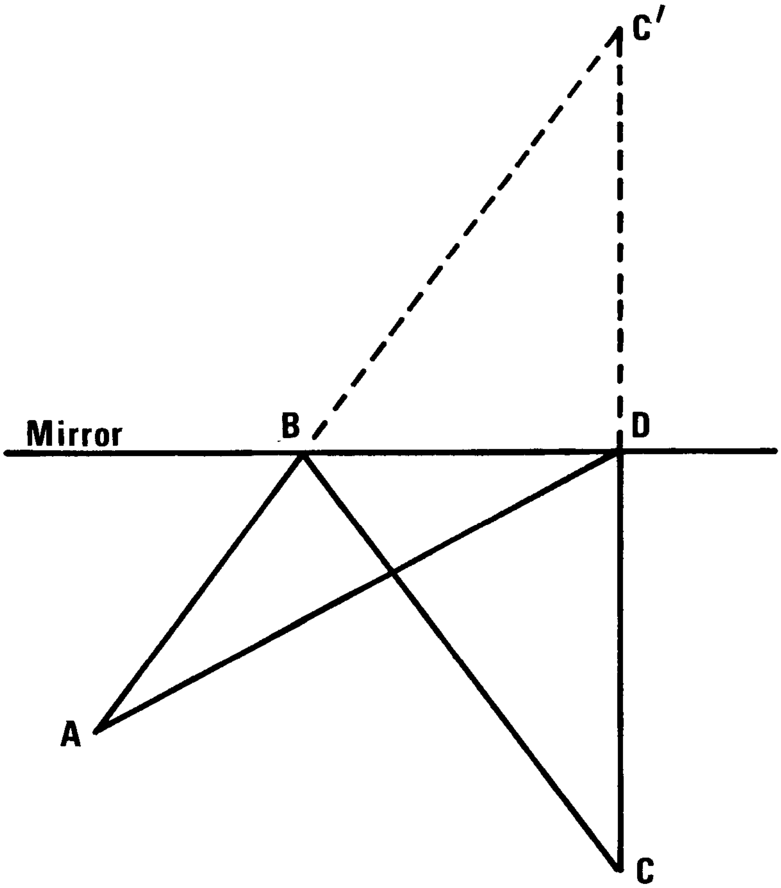
Let me illustrate this by some very simple examples. Newton's falling apple can be described in either of two ways: its acceleration toward the earth is a constant; or its position as a function of time follows that arc which minimizes the integral, taken from its moment of release to the terminal time at which it is observed, of an integrand which can be written as the square of its instantaneous velocity minus a linear function of its position. "What?" you will say, "can you seriously regard the second explanation as the simple one?" I will not argue the point, except to point out that simplicity is in the eye of the beholder, and that if I were to write out for the mathematical physicist the expression

$$\delta\int_{0}^{T}\left(\frac{1.2}{2}x - gx\right) dt = 0$$

he would not consider it less simple than $\ddot{x} = -g$; and he would know that the Hamilton principle formulation in variational terms has great mnemonic properties when it comes to transforming from one coordinate system to another.

Although I am not a physicist and do not suppose that many of my audience are either, let me give a clearer example of the usefulness of a minimum principle in physics. Light travels between two points in the air before me along a straight line. Alternatively like the apple's fall, this arc can be defined as the solution to a minimization problem in the calculus of variations. But now let us consider how light is reflected when it hits a mirror. You may observe and memorize the rule that the angle of reflection is equal to the angle of incidence. A neater way of understanding this fact is by the least-time principle of Fermat, which was already known to Hero and other Greek

[1]Whatever their ultimate worth, we must be grateful to Mach's concepts for their influence on the young Einstein's formulation of special relativity theory. Although an older Einstein rebelled against this same methodology, this cannot rob them of their just credit.



scientists. The accompanying diagram with its indicated similar triangles can be self-explanatory.

If ABC′ is clearly shorter in length than ADC′, it is evident that the similar ABC path is shorter and involves less time than any other path such as ADC.

You could validly argue that the minimum formulation is neat, but really no better than the other formulation. However, move from this lecture room to your bathtub and observe your big toe in the water. Your limbs no longer appear straight because the velocity of light in water differs from that in air. The least-time principle tells you how to formulate behaviour under such conditions and the memorizing of Snell's Law about angles does not. Who can doubt which is the better scientific explanation?

An Illustrative Economic Example

Let me illustrate the same thing in economics as a simplest imaginable case. Consider a profit-maximizing firm that sells its output along a demand curve in which the price received is a non-increasing function of the amount sold. Suppose further that output is producible by two, three, or ninety-nine different inputs. To keep the example simple, suppose that the production function relating outputs to inputs is smooth and concave.

As a positivistic scientist interested merely in cataloguing the observable facts, a Machian economist could in principle record on punch cards ninety-nine demand functions relating the quantity of each input bought by the firm to the ninety-nine variables depicting the input prices. What a colossal task it would be to store bits of information defining ninety-nine distinct surfaces in a one hundred dimensional space! But the ninety-nine surfaces are not really independent. In actuality, it is enough to have knowledge of a single parent surface in order to be able to calculate the exact information about the ninety-nine children. How is this tremendous economy of description possible? It is by virtue of the fact that the observed demand curves, which that great Swedish economist of the generation before last, Gustav Cassel, would have taken as the irreducible atoms of the economists' theory, are actually themselves solutions to a maximum-profit problem. Under simple regularity conditions of the calculus, they are the inverse functions of a family of partial derivatives of the Total Revenue function, where revenue is given by the output producible by any specified quantities of all inputs times the determinate demand price at which that output will sell. When smooth and strongly concave, this parent revenue function has as its children a ninety-nine by ninety-nine matrix of second partial derivatives which is symmetric and negative definite. It is an exercise in algebra to show that these functions can be uniquely inverted to form a new family of children with the same properties; and ninety-nine such children cannot fail to have a parent function which, so to speak, if it had never existed we should have to invent in a Pygmalion fashion.

Mathematically we have

$$\begin{aligned}\underset{\{v_i\}}{\mathrm{Max}}[R(v_1, \ldots, v_{99}) - \sum_1^{99} p_j v_j] &= R(v_1^*, \ldots, v_{99}^*) - \sum_1^{99} p_j v_j^* \\ &= -H(p_1, \ldots p_{99}), \qquad (1)\end{aligned}$$

where $$R(v_1, \ldots, v_{99}) = Q(v_1, \ldots, v_{99})\, P[Q(v_1, \ldots, v_{99})]$$

is a smooth, strongly concave "regular" revenue function. Necessary conditions for the maximum are

$$\partial R(v_1^*, \ldots, v_{99}^*)/\partial v_i = p_i, \quad (i = 1, \ldots, 99) \qquad (2)$$

If in addition the Hessian matrix of second partial derivatives, R_{99}, is negative definite, Equations (2) are sufficient for the maximum. This implies inverse relations that can be interpreted as partial derivatives of a Hotelling-Roy dual function, H, namely

$$v_i^* = \partial H(p_1, \ldots p_n)/\partial p_i, \quad (i = 1, \ldots, 99) \qquad (3)$$

It follows that for

$$\sum \Delta v_j^2 \neq 0 \neq \sum \Delta p_j^2,$$

our variables satisfy the inequality

$$\Delta p_1\, \Delta v_1 + \Delta p_2\, \Delta v_2 + \ldots + \Delta p_{99}\, \Delta v_{99} < 0 \qquad (4)$$

More can be said. Although my intuition is poor enough in three dimensional space, I can assert with confidence on the basis of the above that raising any input's price while holding all remaining inputs' prices constant will definitely reduce the amount demanded of that input by the firm - i.e. $\partial v_i/\partial p_i < 0$: Such a commonsense result might be expected by anyone who performed an act of empathetic introspection, "Suppose I were a jack-ass of an entrepreneur, what would I do to adjust to the dearness of an input in order to conserve as much profit as possible?"

Here the commonsense and advanced mathematics happen to agree. But we all know the Giffen pathology according to which an increase in the price of potatoes to Irish peasants, who must depend heavily on potatoes when they are poor, may itself impoverish them so as to force them into buying more rather than less potatoes. In this case common sense recognizes itself only under the search-light of mathematics.

With the assistance of mathematics, I can see a property of the ninety-nine dimensional surfaces hidden from the naked eye. If an increase in the price of fertilizer alone always increases the amount the firm buys of caviar, from that fact alone I can predict the answer to the following experiment which I have never seen performed and upon which I have no observations: an increase in the price of caviar alone will increase the amount the firm buys of fertilizer. In thermodynamics such reciprocity or integrability conditions are known as Maxwell Conditions; in economics they are known as Hotelling conditions in honor of Harold Hotelling's 1932 work [4] .

One of the pleasing things about science is that we do all climb towards the heavens on the shoulders of our predecessors. Economics, like physics has its heroes, and the letter "H" that I used in my mathematical equations was not there to honor Sir William Hamilton, but rather Harold Hotelling. For it was his work that I found so stimulating when I came on the scene, at about the same time that the late Henry Schultz [5] was trying by econometric methods to verify the empirical validity of the Hotelling integrability conditions.

There are still other predictable conditions of definiteness relating to how weak these "cross effects" must be in comparison with "own effects," but I will spare my audience discussion of them, except for mention of the condition that all principal minors have to oscillate in sign.

As a last illustration of the black magic by which a maximum formulation permits one to make clearcut inferences about a complicated system involving a large number of variables, let me recall the work I have done in formulating clearly and generalizing what is known in physics as LeChatelier's Principle [6]. This Principle was enunciated almost one hundred years ago by a French physicist interested in Gibbs-like thermodynamics. It is a vague principle. A third of a century ago when I thumbed through different physics treatises, my mathematical ear could not discern what tune was being played. If you pick up most physics books today, perhaps your luck will be no better. Usually the argument is obscurely teleological, reading something like the following: If you put an external constraint on an equilibrium system, the equilibrium shifts to "absorb" or "resist" or "adjust to" or "minimize" the change. I was struck by a remark made by an old teacher of mine at Harvard, Edwin Bidwell Wilson. Wilson was the last student of J. Willard Gibbs' at Yale and had worked creatively in many fields of mathematics and physics: his advanced calculus was a standard text for decades; his was the definitive writeup of Gibb's lectures on vectors; he wrote one of the earliest texts on aerodynamics; he was a friend of R. A. Fisher and an expert on mathematical statistics and demography; finally, he had become interested early in the work of Pareto and gave lectures in mathematical economics at Harvard. My earlier formulation of the inequality in equation (4) owed much to Wilson's lectures on thermodynamics. In particular I was struck by his statement that the fact that an increase in pressure is accompanied by a decrease in volume is not so much a theorem about a thermodynamic equilibrium system as it is a mathematical theorem about surfaces that are concave from below or about negative definite quadratic forms. Armed with this clue I set out to make sense of the LeChatelier Principle.

Let me now enunciate a valid formulation of that Principle. "Squeeze a balloon and its volume will contract. But compare how its volume contracts under two different experimental conditions. First, imagine that its surface is insulated from the rest of the world so that none of the so-called heat engendered can escape. In the second alternative administer the same increase in pressure in the balloon, but let it come into temperature equilibrium with the unchanged temperature of the room. Then according to LeChatelier Principle the increase in volume when the insulation constraint is placed on the system will be less than when the temperature is constrained to end up constant." The steeper light curve in Figure 2 (see a later page) shows the relationship between the pressure on the vertical axis and volume on the horizontal axis that prevails for the insulated increase. The less steep curve going through the same point "A" shows the pressure-volume relationship for an iso-thermal change. It is the essence of LeChatelier's Principle that the light curve must be more steep than the heavy curve or, in usual thermodynamic notation

$$(\partial v/\partial p)_t \leqslant (\partial v/\partial p)_s \leqslant 0 \tag{5}$$

where t stands for temperature held constant, and s stands for the insulated (or adiabatic or isentropic) change.

Now what in the world has all this to do with economics? There is really nothing more pathetic than to have an economist or a retired engineer try to force analogies between the concepts of physics and the concepts of economics. How many dreary papers have I had to referee in which the author is looking for something that corresponds to entropy or to one or another form of energy. Nonsensical laws, such as the law of conservation of purchasing power, represent spurious social science imitations of the important physical law of the conservation of energy; and when an economist makes reference to a Heisenberg Principle of indeterminacy in the social world, at best this must be regarded as a figure of speech or a play on words, rather than a valid application of the relations of quantum mechanics.

However, if you look upon the monopolistic firm hiring ninety-nine inputs as an example of a maximum system, you can connect up its structural relations with those that prevail for an entropy-maximizing thermodynamic system. Pressure and volume, and for that matter absolute temperature and entropy, have to each other the same conjugate or dualistic relation that the wage rate has to labor or the land rent has to acres of land. Figure 2 can now do double duty, depicting the economic relationships as well as the thermodynamic ones. Now on the vertical axis goes p_1, the price of the first input. On the horizontal axis goes v_1, its quantity. The story can be told of a ninety-nine variable system but I think you will forgive me if I discuss the simpler case of two variables, say labor and land.

As in the case of the balloon we perform an experiment under two alternative sets of specified conditions. In the first case, we raise p_1, the price of the first input labor, while holding constant the quantity of the second input, land or v_2 - as for example in the Marshallian short-run when only labor can be varied. The rise in p_1 must lower v_1 as shown by the negative slope of the light curve through A.

Now, in the second alternative experiment we raise p_1 by the same amount but hold the price of v_2, p_2, constant. Again, for a profit maximizing monopolist there can be but one qualitative answer: less of v_1 will now be bought, as shown by the negative slope of the heavy curve through *A*. Now one can state what perhaps might be called the LeChatelier-Samuelson Principle: The heavy curve of longer-run adjustment, with other price constant (and other quantity of course thereby itself adjusting *mutatis mutandis* to restore the maximum-profit equilibrium), must be less steep or more elastic than the light curve depicting the demand reaction when the other input is held constant. Mathematically now

$$(\partial v_1/\partial p_1)_{p_2} \leqslant (\partial v_1/\partial p_1)_{v_2} \leqslant 0 \tag{6}$$

I have included the equality signs to allow for the case where two inputs might be quite independent in production. What is remarkable about the relation is that the indicated inequalities will hold whether the two inputs are complements such as pumps and insecticides or substitutes such as organic and inorganic fertilizers. The interested listener might try to work out the intuitive verification of this in those opposite cases.

Not only in the theory of production but also in the general theory of constrained rationing does the LeChatelier Principle have various economic applications.

CONSUMER DEMAND THEORY

This brings me to the theory of consumer demand. Unlike the maximizing profit situation that has been discussed up to this time, now we have a budgetary constraint within which maximizing has to occur. Prior to the mid-1930's, utility theory showed signs of degenerating into a sterile tautology. Psychic utility or satisfaction could scarcely be defined, let alone be measured. Austrian economists would insist that people acted to maximize their utility, but when challenged as to what that was, they found themselves replying circularly that however people behaved, they would presumably not have done so unless it maximized their satisfaction. Just as we can cancel two from the ratio of even numbers, so one could use Occam's Razor to cut utility completely from the argument, ending one up with the fatuity: people do what they do.

I exaggerate only a little. It is true that the Russian Slutsky [7] had in 1915 gone beyond this, but his work, published in an Italian journal, was forgotten in the backwash of the First World War. The better known work of Pareto [8] lacked the mathematical technique of the Weierstrass theory of constrained extrema. Two dimensional analysis of indifference curves had been worked out by W. E. Johnson [9], a Cambridge logician who had studied with Marshall and Whitehead and who is thought to have influenced the probability researches of J. M. Keynes [10], Frank Ramsey [11], and Sir Harold Jeffreys [12]. However just before I arrived on the scene, when Sir Roy Allen and Sir John Hicks [13] at the London School, and Henry Schultz in Chicago, were pioneering the theory of consumers' behaviour, the contributions of Slutsky were unknown.

From the beginning I was concerned to find out what *refutable* hypotheses on the observable facts on price and quantity demanded were implied by the assumption that the consumer spends his limited income at given prices in order to maximize his ordinal utility (i.e., his better-or-worse situation without regard to any numerical indication of how much better or worse). To make a long story short, the flash of inspiration for "Revealed Preference" came to me in argument with one of my teachers, as so many of my best ideas have done. Having learned about indifference curves from Leontief, I put them to use next year in Haberler's international trade course. When he objected to my postulating convex indifference curves, I heard myself replying: "Well, if they are concave, then the Laspeyres-Paasche index-numbers of your doctoral thesis are no good."[2] Far from being *a reductio ad absurdum,* this proposition, upon reflection suggested how a scientific investigator could refute the hypothesis of maximizing behaviour by a test on two price-quantity observed situations. All that remained was to work out the details of the theory of revealed preference.

My early theory of revealed preference was by itself perfectly adequate to handle the problems of two consumption goods. I went on to conjecture that if we ruled out similar contradictions for choices of more than two situations,[3] then the phenomenon of "non-integrability" of the indifference field could be ruled out.

[2] In explanation, suppose you are maximizing the utility of your consumptions of (Qx,QY, . ..). at prices (P_x, P_y, . ..). spending positive income of $P_x Q_x + \ldots = \sum PQ$. Then in two situations, $(P^1, Q^1, \sum P^1Q^1)$ and (P", $Q^2, \sum P^2Q^2)$, is a contradiction to maximization of ordinal utility to be able to observe both $\sum P^1Q^2/\sum P^1Q^1 < 1$, and $\sum P^2Q^1/\sum P^2Q^2 < 1$. With variants of $\leqslant$ for $<$, denying this possibility is one form of the Weak Axiom of Revealed Preference.

[3] Using the notation of the previous situation, I conjectured that non-integrability could be ruled out by the axiom "$\sum P^iQ^i > \sum P^iQ^{i+1}$ for all of i = 1, ... n-l $\geqslant$ 1 rules out $\sum P^nQ^n > \sum P^nQ^1$" for n = 2, this merely repeats the Weak Axiom; for all n $\geqslant$ 2, it becomes the Strong Axiom of Houthakker.

Especially on occasions like this when one is only too likely to reminiscence about scientific victories, one ought to pause frequently along the way to express some lamentations over defeats and failures. Even with the aid of some of the world's leading mathematicians I was not able to verify and prove the truth of the previous footnote's conjecture, and I was persuaded to omit that material from the published version of "Revealed Preference". All the more credit therefore must go to Hendrik Houthakker [14] who on his maiden venture into economics formulated the Strong Axiom and proved that it did exclude nonintegrability.

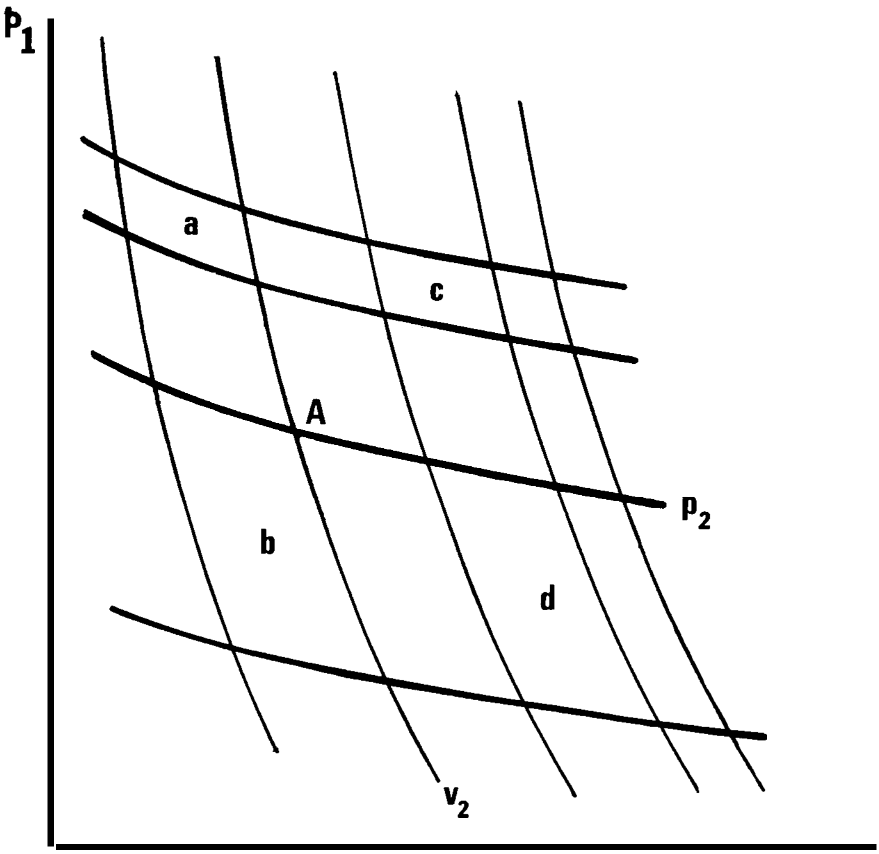
How shall I in a morning lecture explain in words what non-integrable indifference fields are all about? In 1950 I [15] gave a review of the integrability discussion, going back to Pareto in the early years of this century and before that to Irving Fisher's 1892 classic thesis [16], and even before that to resurrected work of the rather unknown Antonelli [17]. How obscure the status of the integrability problem was in the mid-1930's when I arrived on the scene can be indicated by the fact that two close collaborators already cited, Sir John Hicks and Sir Roy Allen, seemed actually to be at odds in their views on the subject. Now that the empirical implications of non-integrability are understood, most theorists are inclined to postulate integrability. How to make clear its meanings? My good friend Nicholas Georgescu-Roegen, from whose classic 1935 paper I gleaned so many insights into the integrability problem, would argue that it is impossible to state such complicated mathematical relations in mere words. I am on record with the contrary view, namely that mathematics is language and in principle what one fool can comprehend so can another. Let me therefore refer you to Figure 2 in which I am able to present an in-the-large interpretation of integrability conditions for our earlier profit maximizing firm with its hiring of ninety-nine inputs.

The steep curves in the diagram represent the demand functions for the first input v_1, in terms of its price p_1, when all other inputs are held constrained as in a Marshallian short-run. The lighter and less steep curves also represent demand functions for v_1 in terms of p_1, but with all other factor prices frozen. If someone challenged me to explain what the existence of integrability implies, but refused to let me use the language of partial derivatives, I could illustrate by an equi-proportional-area property in Fig. 2 that meaning of integability. I may say that the idea for this proposition in economics came to me in connection with some amateurish researches in the field of thermodynamics. While reading Clerk Maxwell's charming introduction to thermodynamics, I [19] found that his explanation of the existence of the same absolute temperature scale in every body could be true only if on the p-v diagram that I earlier referred to in connection with LeChatelier's Principle, the two families of curves - steep and light or less-steep and heavy - formed parallelograms like a, b, c, d in Fig. 2 which everywhere have the property area a/ area b = area c/ area d. And so it is with the two different economic curves. It is a consequence of the Hotelling integrability conditions which link together the ninety-nine different demand functions for factors that the areas shown have this proportionality property. In leaving this interesting result, let me mention that it holds even when - as in linear programming - the relevant surfaces have corners and edges along which unique partial derivatives are not defined. Finally, this illustrates that once we know one of the demand functions everywhere, the other function only needs to be known along one razor's edge in space in order for it to be determined everywhere.

I should not leave the analytics of maximizing functions without mentioning that all of this is not an idle exercise in logic and mathematics. Debates rage in economics as to whether corporations maximize their profits. Yet neither side of the debate pauses to ask what difference it ought to make for observables if there is or there is not some function that is being maximized. And if I depart from the narrow field of economics, I must confess that the writings of sociologists like Talcott Parsons [20] seem to me to be seriously empty because they never seem even to ask the question of what difference it makes to have social action part of a maximizing value system, or just what is implied by "functionalist" interpretations of the observed phenomena.

NON-MAXIMUM PROBLEMS

I must not be too imperialistic in making claims for the applicability of maximum principles in theoretical economics. There are plenty of areas in



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which they simply do not apply. Take for example my early paper dealing with the interactions of the accelerator and the multiplier [21]. This is an important topic in macroeconomic analysis. Indeed, as I have recorded elsewhere this paper brought me a disproportionate amount of reputation. True the topic was a fundamental one, and mathematical analysis of stability conditions was able to give it a neat solution at a level that could be understood both by the intelligent beginner and the virtuoso in mathematical economics. But the original specification of the model had been made by my Harvard teacher Alvin Hansen, and the works of Sir Roy Harrod [22] and Erik Lundberg [23] clearly pointed the way to the setting up of this model.

My point in bringing up the accelerator-multiplier here is that it provides a typical example of a dynamic system that can in no useful sense be related to a maximum problem. By examining the sick we learn something about those who are well; and by examining those who are well we may also learn something about the sick. The fact that the accelerator-multiplier *cannot be* related to maximizing takes its toll in terms of the intractability of the analysis. Thus when my colleague, Professor Richard Eckaus, was a younger man, he wrote a doctoral dissertation [24] under my direction on generalizing the accelerator-multiplier analysis to many sectors and countries. It was an excellent piece of scholarship; Dr. Eckaus, with great ingenuity and elegance, extracted everything from the model that could be extracted. Yet he would be the first to assert that, in a sense, the ratio of useful output to high grade input was somewhat disappointing. Few grand simplicities emerged. The conscientious investigator had to point out a great range of possibilities that could happen, and had to use up much of his intelligence in taxonomy and classification of those possibilities. To illustrate the intrinsic intractability of such a problem, let me recall to you a remarkable difficulty. Suppose Europe in 1970 is a seventeen sector multiplier-accelerator complex that is stable - i.e., we can show that all of its characteristic roots are damped and decaying rather than being anti-damped and explosive. Now go back in history to 1950. The coefficients of the Europe model will be somewhat different, but suppose again that they gave rise to a stable system. Now let me give you this exact bit of information. In 1960, which is a simple mean of 1950 and 1970, by miraculous coincidence it proved to be the case that the coefficients of the model were in each and every case the exact arithmetic mean of the 1950 and 1970 coefficients. What would you predict about the stability of the 1960 system?

If my asking the question had not alerted you to a paradox, I'm sure your first temptation would be to say that it is a stable system, being literally halfway between two stable systems. But that would not be consistent with Dr. Eckaus' findings. You can make the paradox evaporate when you realize that the determinantal conditions for stability of a system [25] do not define a stability region in terms of the coefficients of the system that is a convex region. Hence a point half-way between two points in the region may itself fall outside that region. This sort of thing does not arise in the case of well-behaved maximum systems.

I think I have said enough to demonstrate why perhaps the hardest part of my 1947 *Foundations of Economic Analysis* had to deal with the statics and dynamics of non-maximum systems.

DYNAMICS AND MAXIMIZING

Naturally this does not deny that there is a rich dynamics which can be related to maximizing. Thus consider the dynamic algorithm for finding the top of a mountain which consists of the "gradient method": this says to make your velocity in the direction of any coordinate proportional to the slope of the mountain in its direction. Such a method cannot be counted on to get you to the highest point in the Alps from any initial spot in Europe. But it is bound to converge to the maximum point of any concave surfaces that appear in the Santa Claus examples of the class-room textbooks.

Like the light rays of physics that I mentioned earlier, the optimal growth paths of the theories that have grown out of Frank Ramsey's pioneering work [26] of more than forty years ago, themselves provide a rich dynamics. Such a dynamics is quite different from that of say a positivistic accelerator-multiplier analysis. You may recall that Sir William Hamilton spent a great many years trying to generalize to more than two dimensions the notion of a complex number. The story is told that his family sympathized with his earnest quest for the quaternion, and each night his children would greet him on his return from the astronomical observatory with the question: "Poppa, can you multiply your quaternions?" - only to be sadly told, "I can make my quaternions add but I can't make them multiply." Back in the 1930's if Lloyd Metzler and I had had any children, they would have asked each night: "Did all your characteristic roots turn out nicely stable?" For in those days, impressed by the stubborness of the American Depression and its resistance to transitory pump-priming, we more or less embraced the dogma of stability.

How different were my preoccupations during the 1950's when I was on the fruitless search for a proof of the so-called "Turnpike Theorem" [27]. Here one does deal with a maximizing model, at least in the sense of intertemporal efficiency. When you study a von Neumann input-output model, it becomes the case of a min-max, or saddle-point problem like that of von Neumann's theory of games; and this destroys the possibility that your dynamic characteristic roots could all be damped. So, if my children did not treat my scholarly work with what can only be called "benign neglect," in the 1950's they would have had to ask me: "Daddy, did your characteristic roots come in reciprocal or opposite-signed pairs, as befits a catenary motion around a saddlepoint turnpike?"

May I crave your indulgence to digress and tell an anecdote? I do so with some trepidation because when I was invited to give this lecture I was warned by Professor Lundberg that it must be a serious one. Although it is said I was a brash young man, I had only one encounter with the formidable John von Neumann, who of course was a giant of modern mathematics and who in addition proved himself to be a genius in his work on the hydrogen bomb, game theory, and the foundations of quantum mechanics. To illustrate his stature, I will defy Professor Lundberg even more shamelessly and tell an anecdote within an anecdote. Someone once asked Yale's great mathematician, Kakutani: "Are you a great mathematician?" Kakutani modestly replied, "Oh, not at all. I am a nothing, a mediocre plodder after truth." "Well if you're not a great mathematician, who would you name as one?" he was asked.

Kakutani thought and he thought and he thought, and then according to the story he finally said - "Johnny von Neumann."

This sets the stage for my encounter with Goliath. Sometime around 1945 von Neumann gave a lecture at Harvard on his model of general equilibrium. He asserted that it involved new kinds of mathematics which had no relation to the conventional mathematics of physics and maximization. I piped up from the back of the room that I thought it was not all that different from the concept we have in economics of the opportunity-cost-frontier, in which for specified amounts of all inputs and all but one output society seeks the maximum of the remaining output. Von Neumann replied at that lightning speed which was characteristic of him: "Would you bet a cigar on that?" I am ashamed to report that for once little David retired from the field with his tail between his legs. And yet some day when I pass through Saint Peter's Gates I do think I have half a cigar still coming to me - only half because von Neumann also had a valid point.

A glance through modern journals and texts will show that, whereas the student of classical mechanics deals often with vibrations around an equilibrium, as in the case of a pendulum, the student of economics deals more often with motions around a saddlepoint of catenary shape: i.e., just as a rope suspended between two nails will hang in the shape of a catenary, leaning toward the groundlevel, so will the economic motions hang in the shape of a catenary toward the turnpike. I might mention how the turnpike got its name. All Americans are used to the notion that in going from Boston to Los Angeles, the fastest way is to move quickly to a major highway and only at the end of your voyage depart to your local goal. So in economics: to develop a country most efficiently, under certain circumstances it should proceed rather quickly toward the configuration of maximum balanced growth, catch a ride so to speak on this fast turnpike, and then at the end of the twenty year plan move off to its final goal. An interesting triple limit is involved: as the horizon becomes *large*, you spend an indefinitely *large* fraction of your time within a *small* distance of the turnpike. I shall not spell out this tongue-twister further.

FINALE

I have not been able in one lecture even to scratch the surface of the role of maximum principles in analytic economics. Nor have I even been able to present a representative sample of my own research interests in economics, or for that matter in the narrower area of maximization theory. Thus, one of my abiding concerns over the years has been the field of *welfare economics*.

Along with my close friend, Abram Bergson of Harvard, I have tried to understand what it is that Adam Smith's "invisible hand" is supposed to be maximizing. Thus, consider the concept which we today call Pareto-optimality - and which might with equal propriety be called Bergson-optimality, since it was Bergson [28] who, back in 1938, read sense into what Pareto was groping to say and who related that narrow concept to the broader concept of social norms and a welfare function. Just recently I was reading an article by a writer of the New Left. It was written in blank verse, which turns out to be an extremely inefficient medium for communication but which a dedicated scholar must be prepared to struggle through in the interest of science. The writer was scathing on the notion of Pareto-optimality. Yet as I digested his message, it seemed to me that precisely in a society grown affluent, where dissident groups are called toward a way of life of their own, there arises an especial importance to the notion of giving people what *they* want. An Old Left writer dealing with a socialist economy on the verge of subsistence has surely less need for the concept of Pareto-optimality than does the modern social observer in the United States or Sweden.

Moreover, it has been a special source of satisfaction to me that the calculus of modern welfare economics [29] was able to elucidate the old problem of Knut Wicksell [30] and Erik Lindahl [31], the analysis of public goods.

An American economist of two generations ago, H. J. Davenport, who was the best friend Thorstein Veblen ever had (Veblen actually lived for a time in Davenport's coal cellar) once said: "There is no reason why theoretical economics should be a monopoly of the reactionaries." All my life I have tried to take this warning to heart, and I dare call it to your favorable attention.

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