Efficient Design with Interdependent Valuations

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Abstract

We study efficient, Bayes-Nash incentive compatible mechanisms in a social choice setting that allows for informational and allocative externalities. We show that such mechanisms exist only if a congruence condition relating private and social rates of information substitution is satisfied. If signals are multidimensional, the congruence condition is determined by an integrability constraint, and it can hold only in non-generic cases such as the private value case or the symmetric case. If signals are one-dimensional, the congruence condition reduces to a monotonicity constraint and it can be generically satisfied.

We apply the results to the study of multi-object auctions, and we discuss why such auctions cannot be reduced to one-dimensional models without loss of generality.

1. Introduction

There exists an extensive literature on efficient auctions and mechanism design. A lot of attention has been devoted to the case where each agent i has a quasi-linear utility function that depends on the chosen social alternative, on information (or signal) privately known to i, and on a monetary transfer, but does not depend

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on information available to other agents. In this framework, a prominent role is played by the Clarke-Groves-Vickrey (CGV) mechanisms (see Clarke, 1971, Groves, 1973, Vickrey, 1961). These are mechanisms that ensure both that an efficient decision is taken and that truthful revelation of privately held information is a dominant strategy for each agent. This result holds for arbitrary dimensions of signal spaces and for arbitrary signals' distributions[1].

In this paper we study the case where each agent has a quasi-linear utility function having as arguments the signals received by *all* agents and the chosen social alternative. Hence, besides allocative externalities, we allow for informational externalities, and we speak of "interdependent valuations". Signals may be multi-dimensional, but we assume that they are independently drawn across agents. (Signal independence is the most seriously restrictive assumption; observe though that this assumption is not required for the result in the one-dimensional case of Section 5.)

Interdependent valuations naturally appear in many (two-stage) games studied in applications. In those applications the role of the social alternatives is played by possible allocations of property rights (such as licenses, patents, control rights over firms, etc...) at stage one. These allocations influence then the interaction among agents at stage two. For example, consider an oligopoly model with n firms producing an homogenous good. Each firm i is characterized by a vector of parameters c_0^i, which usually contains (possibly private) information about fixed costs, marginal costs, etc... The profit of each firm is given by a function $\Pi^i(c_0^i, c_0^{-i})$. Assume now that an innovation appears such that a firm licensed to use the innovation will be characterized by a new vector c^i, which is private information. A social alternative can be described by the set L of licensed firms. The valuation of firm i for alternative L is given by the change in profits relative to status-quo: $\Pi^i((c^j)_{j\in L}, (c_0^j)_{j\notin L}) - \Pi^i(c_0^i, c_0^{-i})$. Note how firm i's valuation depends both on who else is licensed (allocative externalities), and on information available to other firms[2].

Our model can be applied to the study of multi-object auctions. There are many auction papers that go beyond the private values case (e.g., the literature following Milgrom and Weber, 1982), but almost all of them restrict attention to

[1]It is well known that, generally, CGV mechanisms cannot simultaneously satisfy conditions such as budget-balancedness and individual rationality (for example, Myerson and Satterthwaite's (1983) impossibility result can be obtained as a corollary of this fact).

[2]The private information held by each firm is typically multidimensional, since it includes information about fixed costs, marginal costs, etc... Since fixed and marginal costs do not affect competition in the same way, they cannot be reduced to a one-dimensional parameter without loss of generality. If several types of licenses were sold, the private information would include cost parameters for each type of license thus increasing the dimensionality of the signal space even further.

situations where there is one object (or there are several identical units), signals are one-dimensional, agents are ex-ante symmetric and do not care about what other agents receive at the auction[3]. Applications of the present model to auctions allow for several heterogenous objects, asymmetries among bidders, and both allocative and informational externalities[4].

In the social choice framework considered here, Williams and Radner (1988) have shown that, in general, no efficient, dominant-strategy incentive compatible mechanisms exist[5]. Important insights about auctions with informationally interdependent valuations (but without allocative externalities) can be found in Maskin (1992) and Dasgupta and Maskin (1998). Maskin (1992) considers an auction for an indivisible object and observes that no efficient, incentive-compatible auction exists if a buyer's valuation for that object depends on a multi-dimensional signal (see further comments on this result in Section 4 below). Dasgupta and Maskin (1998) study multi-object auctions where agents have one-dimensional signals and where there are no allocative externalities. They construct a mechanism that achieves efficient allocations (under appropriate conditions on marginal valuations). Ausubel (1997) and Perry and Reny (1998) present specific bidding procedures that achieve efficient allocations for a one-dimensional model with M identical units and no allocative externalities. Ausubel assumes symmetry among bidders and constant marginal valuations. Perry and Reny drop symmetry and allow for decreasing marginal valuations.

This paper is organized as follows: In Section 2 we present the social choice model. In Section 3 we obtain a characterization theorem for Bayesian incentive compatible direct mechanisms. In Section 4 we exhibit impossibility results about efficient, Bayesian incentive compatible mechanisms. We only require value maximization, and we completely ignore budget-balancedness and any other properties. Hence, we show that providing incentives for truthful revelation of privately held

[3]Auction models emphasizing the role of allocative externalities in a one-object setup are discussed in Jehiel and Moldovanu (1996) and Jehiel, Moldovanu and Stacchetti (1996, 1999).

[4]These features will, in general, give rise to multidimensional signal spaces, since the payoff-relevant part of the signal varies with the chosen alternative (e.g., with the acquired bundle or with the entire distribution of objects among agents).

[5]Cremer and McLean (1985,1988) and McAfee and Reny (1992) have given conditions under which a principal can extract the full surplus available when types are correlated. Full extraction mechanisms are, in particular, efficient. Neeman (1998) shows that these results do not hold in a model that can be interpreted as one where agents have multidimensional signals, and signals have some private and some common components. Aoyagi (1998) presents a general existence result of efficient, budget balanced and incentive compatible mechanisms when agents have finitely many correlated types. None of the above papers covers the present framework (i.e., a continuum of mutually payoff relevant multidimensional types), but we suspect that correlation among types allows some possibility results. On the other hand, the mechanisms displayed in the literature above are not very intuitive and require potentially unlimited transfers.

information is not compatible even with a very weak efficiency requirement.

The logic behind the impossibility results is as follows: An incentive compatible mechanism generates for agent i a (convex) equilibrium expected utility function $V_i(\cdot) : S^i \to \Re$, where S^i is the multidimensional type space of that agent. By a well-known calculus result (Schwarz's Theorem), the cross-derivatives of such functions are equal[6]. This requirement implies several equalities involving the conditional expected probabilities with which the various alternatives must be chosen in incentive compatible mechanisms (these expected probabilities form the gradient of $V_i(\cdot)$)[7]. The impossibility results follow by showing that the conditional expected probabilities generated by efficient mechanisms satisfy the required equalities only under very restrictive conditions.

The first result is obtained for situations where incentive compatibility implies that an informational variable has a zero marginal effect on some of the conditional expected probabilities, while this variable is relevant for efficiency considerations. Theorem 4.2 shows impossibility for the case where there is at least one agent possessing essential information that affects other agents, but does not directly affect the owner of that information. A similar argument is used in Example 4.3 which shows that efficient, incentive compatible mechanisms may not exist if there exist an alternative k and an agent i such that agent i ´s signal affecting her valuation for alternative k is multidimensional (this corresponds to Maskin's (1992) example).

Our main impossibility result is Theorem 4.4 (which is significantly different from Maskin's example and from other impossibility theorems identified so far). We consider there a framework where each agent i has a $K-$dimensional signal s^i. The coordinate s^i_k is a **one-dimensional** signal affecting the valuations of **all** agents for alternative k. This framework is critical since, a-priori, all informational variables may have a non-zero marginal effect on the conditional expected probabilities generated by incentive compatible mechanisms, and we cannot use the method sketched above. The argument showing impossibility is now more refined: the conditional expected probabilities generated by an efficient mechanism satisfy the conditions implied by the equality of the cross derivatives only if a congruence condition relating private and social rates of informational substitution is satisfied. The congruence condition holds only for a closed, zero-measure set of parameters[8].

[6] This is the mathematical statement of the pretty obvious fact that the net height covered by climbing a mountain is independent of the path of ascent.

[7] A very similar phenomenon appears in the classical demand theory for several goods (see Chapter 3 in Mas-Colell, Whinston and Green, 1995): the matrix of price derivatives for a demand function arising from utility maximization must be *symmetric*.

[8] We show that the congruence condition is satisfied in situations where either symmetry, or the private values assumption hold.

Since the constraints imposed by Schwarz's Theorem apply as soon as signals are multidimensional, results similar to Theorem 4.4 hold as soon as there is at least one agent whose signal is of dimension $d \geq 2$. In Section 5 we study the remaining case where the signal spaces of all agents are one-dimensional. For that case we construct a mechanism that is efficient and incentive compatible if a monotonicity condition on marginal valuations is satisfied. Our treatment is based on the idea (which can be traced back to Pigou) that transfers should stand for the cumulative effect of one's action (here a signal report) on all other agents. The first illustration of this idea in an auction context with interdependent valuations appears in Dasgupta and Maskin (1998).

The expected equilibrium utility functions $V_i(\cdot)$ depend here on a real-valued signal, and there are no cross-derivatives to consider. The implementability condition reduces to a monotonicity constraint that can be satisfied in non-trivial cases.

Concluding comments are gathered in Section 6. In particular, we comment on the difficulty of finding constrained efficient (i.e., second-best) mechanisms.

2. The Model

There are K social alternatives, indexed by $k = 1, ...K$ and there are N agents, indexed by $i = 1, .., N$. Each agent i has a signal (or type) s^i which is drawn from a space $S^i \subseteq \Re^{K \times N}$ according to a continuous density $f_i(s^i)$, independently of other agents' signals. Each agent i knows s^i, and the densities $\{f_j(\cdot)\}_{j=1}^N$ are common knowledge. The idea is that the coordinate s^i_{kj} of s^i influences the utility of agent j in alternative k[9]. We assume that the signal spaces S^i are bounded and convex[10].

If alternative k is chosen, and if i obtains a transfer x_i, then $i's$ utility is given by $V^i_k(s^1_{ki}, ..., s^n_{ki}) + x_i$, where $V^i_k(s^1_{ki}, ..., s^n_{ki}) = \sum_{j=1}^n a^j_{ki} s^j_{ki}$, and where the scalar parameters[11] $\{a^j_{ki}\}_{1 \leq k \leq K, 1 \leq j, i \leq N}$ are common knowledge. We assume throughout the paper that $\forall i, \forall k$, $a^i_{ki} \geq 0$.

[9] We address below (see Example 4.3) situations where the signal of an agent i affecting the utility of agent j in alternative k is itself multidimensional.

[10] Convexity is assumed for convenience. If S^i is simply connected all results go through unchanged.

[11] The analysis directly extends to the case where the valuation functions include also a constant, i.e., $V^i_k(s^1_{ki}, ..., s^n_{ki}) = \sum_{j=1}^n a^j_{ki} s^j_{ki} + b^i_k$ (because such constants do not affect incentives).

2.1. An Application to Auctions

Consider an auction where a set M of heterogenous objects is divided among $n+1$ agents (agent zero is the seller, the rest are potential buyers). An alternative is a partition μ of M, $\mu = \{M_i\}_{i=0}^{N}$, where M_i is the set of objects allocated to bidder i, $i = 1, 2, ...N$ and M_0 is the set of unsold objects. Bidder i's piece of information $s^i_{\mu j}$ summarizes, from the point of view of i, the important aspects for j (say, attributes of the objects in M_j) given partition μ.

This framework allows for informational and allocative externalities and for asymmetric bidders. Particularly simple cases are: 1) The private values case where $V\mu^i(\cdot)$ is only a function of $s^i_{\mu i}$; 2) The private values case without allocative externalities where $V\mu^i(\cdot)$ is only a function of $s^i_{\mu i}$, and $V\mu^i(\cdot) = V^i_{\mu'}(\cdot)$ for all partitions μ and μ' such that i receives the same set of objects , etc... Even these simple cases require, in general, multidimensional signals. In our introductory licensing example both informational and allocative externalities emerge naturally.

3. Direct Revelation Mechanisms

By the revelation principle, we can restrict attention to direct, incentive compatible mechanisms. We first define Direct Revelation Mechanisms and then turn to incentive compatibility.

Let S denote the Cartesian product $\prod_{i=1}^{N} S^i$, with generic element s. Define S^{-i} as the type space of agents other than i, with s^{-i} as generic element.

A function $p : S \to \Re^K$ such that $\forall k, s$, $0 \le p_k(s) \le 1$ and $\forall s$, $\sum_{k=1}^{K} p_k(s) = 1$ is called a *social choice rule.* A social choice rule (SCR) is said to be *efficient* if

$$\forall s,\ p_q(s) \neq 0\ \Rightarrow q \in \arg\max_k \sum_{i=1}^{N} V^i_k(s^1, ..s^N) = \arg\max_k \sum_{i=1}^{N}\sum_{j=1}^{N} a^j_{ki} s^j_{ki}.$$

A *direct revelation mechanism* (DRM) is defined by a pair (p, x) where p is a social choice rule, and $x : S \to \Re^N$ is a payment scheme. The term $p_k(s)$ is the probability that alternative k is chosen if the agents report signals $s = (s^1, ..., s^N)$, and $x_i(s)$ is the transfer to agent i if the agents report signals s. A DRM is *efficient* if the associated social choice rule is efficient[12].

Given a payment scheme x and a social choice rule p, we now define for each agent i the conditional expected payment function $y_i : S^i \to \Re$ and the conditional expected probability assignment functions $q^i : S^i \to \Re^K$ associated with x and p :

[12] We ignore here (as in the CGV approach) the (ex post) "budget balancedness" condition, which imposes $\sum_i x_i(s) \le 0$, $\forall s$. In other words, we abstract from efficiency losses due to potential external subsidies.

$$y_i(t^i) = \int_{S^{-i}} x_i(t^i, s^{-i}) f_{-i}(s^{-i}) ds^{-i}$$

$$q_k^i(t^i) = \int_{S^{-i}} p_k(t^i, s^{-i}) f_{-i}(s^{-i}) ds^{-i}.$$

Assume that agent i believes that all other agents report truthfully and assume that i reports type t^i when his true type is s^i. Then, $i's$ expected utility is given by:

$$U_i(t^i, s^i) = \int_{S^{-i}} [\sum_k (p_k(t^i, s^{-i}) \cdot \sum_{j=1}^{N} a_{ki}^j s_{ki}^j)] f_{-i}(s^{-i}) ds^{-i} + y_i(t^i) =$$
$$\sum_k a_{ki}^i s_{ki}^i q_k^i(t^i) + \sum_k \int_{S^{-i}} [(p_k(t^i, s^{-i}) \cdot \sum_{j \neq i} a_{ki}^j s_{ki}^j)] f_{-i}(s^{-i}) ds^{-i} + y_i(t^i). \quad (3.1)$$

Define also

$$V_i(s^i) = U_i(s^i, s^i).$$

3.1. Incentive Compatible Mechanisms

A DRM is (Bayes-Nash) *incentive compatible* if:

$$\forall i, \forall s^i, t^i \in S^i, \ U_i(s^i, s^i) \geq U_i(t^i, s^i).$$

For the characterization of incentive compatible mechanisms we need two definitions.

A (possibly multivalued) mapping $\Psi : S^i \rightarrow \Re^{K \times N}$ is *monotone* if $(x_0 - x_1) \cdot (x_0^* - x_1^*) \geq 0$ for any set of pairs (x_i, x_i^*), $i = 0, 1$, such that $x_i^* \in \Psi(x_i)$.[13]

A vector-field $\Omega : S^i \rightarrow \Re^{K \times N}$ is *conservative* if $\int \gamma \Omega = 0$ for every closed curve γ in S^i. Conservativeness is a necessary and sufficient condition for Ω to be the gradient of a function $\omega : S^i \rightarrow \Re$.

Theorem 3.1. *Let (p, x) be a DRM, and let $\{q^i(\cdot)\}_{i=1}^n$ be the associated conditional probability assignments. For each agent i, let $Q^i(s^i) : \Re^{K \times N} \rightarrow \Re^{K \times N}$ be the vector field, where, for each alternative k, the ki^{th} coordinate is given by $a_{ki}^i q_k^i(s^i)$ and the kj^{th} coordinate, $j \neq i$, is zero. Then (p, x) is incentive compatible if and only if the following conditions hold:*

[13] Note the analogy with the classical "law of demand".

1. $\forall i$, the vector field $Q^i(\cdot)$ is monotone and conservative.

2. $\forall i$, $\forall s^i, t^i \in S^i, V_i(s^i) = V_i(t^i) + \int_{t^i}^{s^i} Q^i(\tau^i)d\tau^i$ [14][15]

Proof. See Appendix.

4. Impossibility Results

In an incentive compatible mechanism (p, x) we have $V_i(s^i) = \max_{t^i} U_i(t^i, s^i)$. The function $V_i(\cdot)$ is convex (see the proof of Theorem 3.1), and hence twice differentiable almost everywhere. Assuming that $V_i(\cdot)$ is differentiable at s^i we obtain by the Envelope Theorem that:

$$\forall k,\ \frac{\partial V_i}{\partial s^i_{ki}}(s^i) = a^i_{ki} q^i_k(s^i) \tag{4.1}$$

$$\forall k, \forall j \neq i,\ \frac{\partial V_i}{\partial s^i_{kj}}(s^i) = 0 \tag{4.2}$$

Assuming that $V_i(\cdot)$ is twice continuously differentiable at s^i, we obtain by Schwarz's Theorem that the cross-derivatives at s^i must be equal. This implies :

$$\forall k, k',\ a^i_{ki}\frac{\partial q^i_k(s^i)}{\partial s^i_{k'i}} = \frac{\partial V_i}{\partial s^i_{k'i}\partial s^i_{ki}}(s^i) = \frac{\partial V_i}{\partial s^i_{ki}\partial s^i_{k'i}}(s^i) = a^i_{k'i}\frac{\partial q^i_{k'}(s^i)}{\partial s^i_{ki}}\ ; \tag{4.3}$$

$$\forall k, k', \forall j \neq i,\ a^i_{ki}\frac{\partial q^i_k(s^i)}{\partial s^i_{k'j}} = \frac{\partial V_i}{\partial s^i_{k'j}\partial s^i_{ki}}(s^i) = \frac{\partial V_i}{\partial s^i_{ki}\partial s^i_{k'j}}(s^i) = 0\ . \tag{4.4}$$

The idea behind the following impossibility results is to check whether efficient mechanisms yield conditional expected probability assignment functions that satisfy conditions 4.3 and 4.4.

Note that an efficient SCR is piece-wise constant. Hence, for efficient mechanisms we obtain that the associated functions $\{q^i(\cdot)\}_{i=1}^n$ are everywhere continuously differentiable by assuming, for example, that the (convex) type spaces have

[14] The integral can be defined on any path connecting t^i and s^i since the vector field $Q^i(\cdot)$ is conservative. For example, we can choose a straight line, to obtain
$\int_{t^i}^{s^i} Q^i(\tau^i)d\tau^i = \int_0^1 Q^i((1-\alpha)t^i + \alpha s^i)) \cdot (s^i - t^i).d\alpha$

[15] Note that the Theorem implies a "Revenue Equivalence" result. The conditional expected payment of agent i in any incentive compatible mechanism is solely a function of the associated expected probability assignment, and of the expected utility of an arbitrary type. Any two incentive compatible mechanisms with the same probability assignment yield, up to a constant, the same conditional expected payments.

a non-empty interior and a piece-wise smooth boundary, and that for all i and all $s^i \in S^i$, $f_i(s^i) > 0$.

We first focus on the simpler condition 4.4.

Definition 4.1. *Let $\hat{p}$ be an efficient SCR, and let $\{\hat{q}^i(\cdot)\}_{i=1}^N$ be the associated conditional expected probability assignments. The variable $\hat{s}^i_{kj}$ is said to be* essential *if there exist $s^i, t^i \in S^i$ such that:*

1. $s^i_{k'j'} = t^i_{k'j'}$ for all $(k', j') \neq (k, j)$.
2. $s^i_{kj} \neq t^i_{kj}$.
3. $\hat{q}^i_k(s^i) \neq \hat{q}^i_k(t^i)$.

Note that unless alternative k is always welfare-dominated (or always welfare superior) or the density $f_i(\cdot)$ is degenerate (i.e., does not have full-dimensionality), *all* variables $\hat{s}^i_{kj}$ such that $a^i_{kj} \neq 0$ are essential[16].

Theorem 4.2. *Assume that i, j, k exist such that $i \neq j$, $a^i_{ki} \neq 0$, and $\hat{s}^i_{kj}$ is essential. Then efficient, incentive compatible DRMs do not exist.*

Proof. Let s^i, t^i satisfy the conditions in Definition 4.1, let (p, x) be an efficient, incentive-compatible DRM with associated conditional expected probability assignments $\{q^i(\cdot)\}_{i=1}^N$. By efficiency, we must have $q^i(u^i) = \hat{q}^i(u^i)$ for all $u^i \in S^i$. Since $a^i_{ki} \neq 0$, we obtain by equation 4.4 and by the construction of s^i, t^i, that $q^i(s^i) = q^i(t^i)$. Since, by definition, $\hat{q}^i(s^i) \neq \hat{q}^i(t^i)$, we obtain a contradiction. ■

We next show that the simple phenomenon displayed in Theorem 4.2 has a deeper consequence. So far we have assumed that $\hat{s}^i_{kj}$, agent i' s piece of information affecting the utility of agent j in alternative k, is one-dimensional. We next look at an example where this requirement is not satisfied. An impossibility result in such situations has been observed by Maskin (1992). What we show here is that Maskin's result is a consequence of the phenomenon displayed in Theorem 4.2.

Example 4.3.

There are two agents $i = 1, 2$ and two alternatives $k = A, B$. Signals are two-dimensional, $s^i = (s^i_1, s^i_2)$, $i = 1, 2$. Valuations are given by: $V^1_A(s^1, s^2) = s^1_1 + a(s^1_2 + s^2_2)$, $V^1_B(s^1, s^2) = 0$, $V^2_A(s^1, s^2) = 0$, $V^2_B(s^1, s^2) = s^2_1 + a(s^1_2 + s^2_2)$.

[16] Since an efficient SCR is uniquely defined almost everywhere, the definition of essentiality does not depend on the specific SCR $\hat{p}$ which is used.

(Imagine an auction for an indivisible good where the components s_1^i , $i = 1, 2$, are the private parts of the signals while the components s_2^i are common parts).

Consider the change of variables:

$$t^i = (t_1^i, t_2^i) = (s_1^i + as_2^i, s_2^i)$$

In the t^i type space we obtain: $V_A^1(t^1, t^2) = t_1^1 + at_2^2$, $V_B^1(t^1, t^2) = 0$, $V_B^2(t^1, t^2) = t_1^2 + at_2^1$, $V_A^2(t^1, t^2) = 0$.

Hence, agent 1 has a signal t_2^1 which does not affect her utility (in particular it does not affect her utility in alternative A), but affects the utility of agent 2 in alternative B. In incentive compatible mechanisms we obtain by condition 4.4 that agent 1's interim expected probability for alternative A is independent of t_2^1 , while t_2^1 is essential for the determination of ex-post efficiency. The impossibility result follows as in Theorem 4.2.

The example[17] can be extended to the case where $V_A^1(s^1, s^2) = s_1^1 + as_2^1 + bs_2^2$ and $V_B^2(s^1, s^2) = s_1^2 + as_2^2 + bs_2^1$. Even when the dependence of an agent's valuation on the signal of another agent is very small (i.e., b is very close to zero), efficiency cannot be attained. ∎

Our results so far suggest that, in order to obtain generic existence of efficient and incentive compatible mechanisms, it is necessary that $\forall i, j, i \neq j$, $\forall k$, s_{kj}^i is a function of the signals $s_{k'i}^i$, $k' = 1, ...K$, and that each $s_{k'i}^i$ is one-dimensional. Since we want to remain in the linear framework, we consider the case where $\forall i, j, i \neq j$, $\forall k$, s_{kj}^i is a linear function of $s_{k'i}^i$, $k' = 1, ...K$. In order to make the argument as transparent as possible, we simplify further by assuming below that $\forall i, j, i \neq j$, $\forall k$, $s_{kj}^i = s_{ki}^i$.

Hence, we now look at $K-$ dimensional type-spaces, and we denote by s_k^i agent i's one-dimensional piece of information affecting (possibly in different ways) the utility of *all* agents in alternative k.

In this setup, the impossibility of efficient, incentive compatible mechanisms is less immediate. The question is whether the conditional expected probability assignment functions generated by efficient mechanisms satisfy the more complex condition 4.3.

To be precise, recall that we have derived conditions 4.3 and 4.4 for signals of dimension $K \times N$. For each K-dimensional signal $\widetilde{t^i}$, define $\widetilde{V}_i(\widetilde{t^i}) \equiv V_i(t^i)$ and $\widetilde{q}_k^i(\widetilde{t^i}) \equiv q_k^i(t^i)$, where t^i is the $K \times N-$dimensional signal such that $t_{kj}^i = \widetilde{t}_k^i$ for all k, j. Assuming that $V_i(t^i)$ is differentiable at t^i, we obtain by conditions 4.3 and 4.4 that:

$$\forall k, \frac{\partial \widetilde{V}_i}{\partial \widetilde{t}_k^i}(\widetilde{t^i}) = \sum_j \frac{\partial V_i}{\partial t_{kj}^i}(t^i) = a_{ki}^i q_k^i(t^i) = a_{ki}^i \widetilde{q}_k^i(\widetilde{t^i}).$$

[17]Compte and Jehiel (1998) look at related examples in order to study the value of competition in standard auctions.

The equality of cross-derivatives implies that :

$$a^i_{ki}\frac{\partial \widetilde{q}^i_k(\widetilde{t}^i)}{\partial \widetilde{t}^i_{k'}} = a^i_{k'i}\frac{\partial \widetilde{q}^i_{k'}(\widetilde{t}^i)}{\partial \widetilde{t}^i_k} \tag{4.5}$$

In order to simplify notation, we drop from now on the "tilde" and denote by $s^i = (s^i_1, ...s^i_K)$ a $K-$dimensional signal of agent i, yielding expected probability assignments $\{q^i_k(\cdot)\}_{k=1}^K$, and equilibrium utility $V_i(\cdot)$.

Theorem 4.4. *Assume that (p, x) is an efficient DRM that is incentive compatible for agent i. Let k, k' be any pair of alternatives such that: 1) $a^i_{k'i} \neq 0$; 2) There exists a type t^i such that $q^i_k(s^i) \neq 0$, $q^i_{k'}(s^i) \neq 0$ for all s^i in a neighborhood of t^i* [18]*. Then it must be the case that*

$$\frac{a^i_{ki}}{a^i_{k'i}} = \frac{\sum_{j=1}^N a^i_{kj}}{\sum_{j=1}^N a^i_{k'j}}. \tag{4.6}$$

Proof. See Appendix[19].

Condition 4.6 is a congruence requirement between private and social rates of information substitution (see Example below for more intuition about these terms). The implied algebraic relations among parameters cannot be generically satisfied[20]. Note that condition 4.6 is trivially satisfied in two interesting and much studied cases: the private values case where $\forall i, j, i \neq j, \forall k$, $a^i_{kj} = 0$, and the symmetric case where $\forall i, j, k$, $a^i_{kj} = a^i_{ki}$.

Example 4.5. *4 There are two agents $i = 1, 2$ and two alternatives $k = A, B$. Signals are two dimensional, $s^i = (s^i_A, s^i_B)$, $i = 1, 2$. For $i = 1, 2$ let $-i$ denote the agent other than i. Valuations are given by:*

$$V^i_k(s^i, s^{-i}) = a^i_{ki}s^i_k + a^{-i}_{ki}s^{-i}_k,\ i = 1, 2, ; k = A, B$$

Assume that an efficient, incentive compatible DRM exists, and denote it by (p, x). Let $q^i_k(\cdot)$ denote i's interim expected probability that the mechanism chooses alternative k.

We will first show that, as a consequence of equation 4.5, incentive compatible mechanisms must yield the same vector of conditional expected probability

[18] Note that $q^i_k(t^i) \neq 0$, $q^i_{k'}(t^i) \neq 0$ imply that $\sum_{j=1}^N a^i_{k'j} \neq 0$ and that $\sum_{j=1}^N a^i_{kj} \neq 0$.

[19] The Theorem has also converse: If condition 4.6 is satisfied, and if an efficient SCR p yields, for each agent i a monotone vector field $Q^i(\cdot)$ then there exists a payment schedule $x_i(\cdot)$ such that (p, x) is incentive compatible for i.

[20] i.e., the set of parameters satisfying the condition is closed and has Lebesgue-measure zero.

assignments for types of agent $i, i = 1, 2$, lying on lines with slope $\frac{a^i_{Ai}}{a^i_{Bi}}$. We next show that efficient mechanism yield the same vector of conditional expected probability assignments for types lying on lines with slope $\frac{a^i_{Ai}+a^i_{A-i}}{a^i_{Bi}+a^i_{B-i}}$. Hence, incentive compatibility can be consistent with efficiency only if these two slopes are equal.

We know that

$$\forall i, \forall s^i, q^i_A(s^i) + q^i_B(s^i) = 1. \tag{4.7}$$

Consider agent 1. Equation 4.5 yields

$$a^1_{A1} \frac{\partial q^1_A(s^1)}{\partial s^1_B} = a^1_{B1} \frac{\partial q^1_B(s^1)}{\partial s^1_A}. \tag{4.8}$$

By taking the derivative with respect to s^1_A in identity 4.7, we get

$$\frac{\partial q^1_B(s^1)}{\partial s^1_A} = -\frac{\partial q^1_A(s^1)}{\partial s^1_A}$$

By equation 4.8, we get:

$$a^1_{A1} \frac{\partial q^1_A(s^1)}{\partial s^1_B} + a^1_{B1} \frac{\partial q^1_A(s^1)}{\partial s^1_A} = 0. \tag{4.9}$$

Fix now $t^1 = (t^1_A, t^1_B)$ such that the assumptions in the Theorem are satisfied, and consider a line in the type space of agent 1 having the form $s^1 = s^1(z) = (t^1_A + z, t^1_B + \frac{a^1_{A1}}{a^1_{B1}} \cdot z)$. By equation 4.9 we have:

$$\frac{dq^1_A(t^1_A + z, t^1_B + \frac{a^1_{A1}}{a^1_{B1}} \cdot z)}{dz} = \frac{\partial q^1_A(s^1)}{\partial s^1_A} + \frac{a^1_{A1}}{a^1_{B1}} \frac{\partial q^1_A(s^1)}{\partial s^1_B} = 0. \tag{4.10}$$

Hence, in incentive compatible mechanisms the function $q^1_A(\cdot)$ is constant along lines having the form $(t^1_A + z, t^1_B + \frac{a^1_{A1}}{a^1_{B1}} \cdot z)$ (by equation 4.7 the same is of course true for the function $q^1_B(\cdot)$).

We now turn to the consequences of efficiency. Alternative A is chosen by an efficient DRM at reports (s^1, s^2) iff

$$\sum_{i=1}^{2}\sum_{j=1}^{2} a^j_{Ai} s^j_A \geq \sum_{i=1}^{2}\sum_{j=1}^{2} a^j_{Bi} s^j_B$$

This is equivalent to:

$$(a^1_{A1} + a^1_{A2})s^1_A - (a^1_{B1} + a^1_{B2})s^1_B \geq (a^2_{B1} + a^2_{B2})s^2_B - (a^2_{A1} + a^2_{A2})s^2_A \tag{4.11}$$

Efficiency implies that:

$$q_A^1(s^1) = \int_{\Delta(s^1)} f_2(s^2) ds^2$$

where $\Delta(s^1) = \{s^2$ such that condition 4.11 is satisfied$\}$.

Consider a line in agent 1's type space having the form $s^1 = s^1(z) = (t_A^1 + z, t_B^1 + \frac{a_{A1}^1 + a_{A2}^1}{a_{B1}^1 + a_{B2}^1} z)$. For any two signals θ^1, τ^1, on this line, we have $\Delta(\theta^1) = \Delta(\tau^1)$. Therefore $q_A^1(s^1(z))$ does not depend on z. Taking the derivative with respect to z, and multiplying by $(a_{B1}^1 + a_{B2}^1) \neq 0$, this yields :

$$(a_{B1}^1 + a_{B2}^1) \frac{\partial q_A^1(s^1)}{\partial s_A^1} + (a_{A1}^1 + a_{A2}^1) \frac{\partial q_A^1(s^1)}{\partial s_B^1} = 0 \tag{4.12}$$

Equations 4.10 and 4.12 yield together:

$$\frac{a_{A1}^1}{a_{B1}^1} = \frac{a_{A1}^1 + a_{A2}^1}{a_{B1}^1 + a_{B2}^1}. \tag{4.13}$$

The same reasoning yields an analogous condition for $i = 2$. ■

Two remarks regarding Theorem 4.4 follow.

Remark 1.

Technically, Theorem 4.4 applies to the case where the dimensionality of signal spaces coincides with the number of alternatives $K \geq 2$. But it should be clear that the same type of results can be obtained whenever the integrability constraint expressed by the equality of cross-derivatives bites (i.e., whenever, for at least one agent, the dimensionality of the signal space is greater than one.) For signal spaces of any dimension d, $1 < d \leq K$, efficiency and incentive compatibility imply together algebraic conditions on the parameters (analogous to condition 4.13) that cannot be generically satisfied. An illustration is offered in Example 7.1 in the Appendix. ■

Remark 2.

Dasgupta and Maskin (1998) suggest that the "second best" mechanism for a multidimensional model can be analyzed by performing a reduction to a one-dimensional model for which an efficient mechanism can be sometimes constructed (see next Section). The constructed mechanism is then "constrained efficient" for the original multidimensional model. Simple dimension reductions are indeed available in two cases: 1) The only integrability constraints are of the form given by condition 4.4, which implies that incentive compatible mechanisms cannot condition on a variable $\hat{s}_{kj}^i$, $j \neq i$, if such a variable moves independently of $(\hat{s}_{k'i}^i)_{k'}$

. 2) There are only two alternatives. If the alternatives are k and k' , then $q_k^i(\cdot) = 1 - q_{k'}^i(\cdot)$, and the integrability conditions expressed in equation 4.3 can be written in terms of a unique function. In Example 4 we have exhibited the lines along which conditional expected probability assignments in an incentive compatible mechanism must be constant (and hence we have exhibited the appropriate reduction to one dimension[21]).

If at least one agent perceives more than two payoff relevant alternatives[22], the constraints expressed by conditions 4.3 simultaneously affect several functions, and further dimension reductions become endogenous and impossible to perform *a-priori*. ■

The above analysis sheds some light on the outcome of a multi-object auction where the objects and the agents are heterogenous in a non-trivial way. If there are informational externalities, and if signals are independent, whatever sale mechanism is considered (including mechanisms that allow for "combinatorial" bidding), efficiency cannot be achieved.

5. One-Dimensional Signals

We now assume that agents have one-dimensional signals. Agent i's payoff in alternative k is given by

$$V_k^i(s^i, s^{-i}) = \sum_{j=1}^{N} a_{ki}^j s^j$$

where $s^j \in [\underline{s}^j, \overline{s}^j]$ denotes the one-dimensional signal of agent j. Signals need not be independently distributed, and the result below does not depend on the signals' distribution functions.

In order to avoid a tedious case differentiation, we assume that, for each agent i, there are no alternatives k , k', $k' \neq k$, such that $a_{ki}^i = a_{k'i}^i$. Our result will rely on the following assumption:

$$\forall i, \forall k, k', \; a_{ki}^i > a_{k'i}^i \Rightarrow \sum_{j=1}^{n} a_{kj}^i > \sum_{j=1}^{n} a_{k'j}^i \tag{5.1}$$

Condition 5.1 (referred below as the weak congruence condition) requires that the sequence of alternatives obtained by ordering (in terms of magnitude) the impacts

[21] Similar reductions can be performed in models where there are possibly more than two alternatives, but each agent perceives only two outcomes as payoff relevant. For example, in an auction for one unit of an indivisible good without allocative externalities, an agent cares only about "winning" or "losing".

[22] This is the general case in auctions for several heterogenous objects or in auctions for one object with allocative externalities.

of i's signal on i's payoff is the same as the sequence obtained by ordering the impacts of i's signal on total welfare. Note the analogy with condition 4.6, but note also the gained slack in the one-dimensional framework. This slack (i.e., required inequalities instead of equalities) allows the condition to be satisfied for an open set of parameters' values.

Proposition 5.1. *Assume that the weak congruence condition 5.1 is satisfied. Then there exists an efficient, Bayesian incentive compatible mechanism. Moreover, the associated transfers do not depend on the distribution of signals*[23].

Proof. See Appendix.

6. Conclusions

We have shown that efficient, incentive compatible mechanisms can exist only if a congruence condition relating private and social rates of information substitution is satisfied. If signals are multi-dimensional, the congruence condition is determined by an integrability constraint, and it can be satisfied only in non-generic cases such as the private value case or the symmetric case. If signals are one-dimensional, the congruence condition reduces to a monotonicity constraint and it can be generically satisfied.

Our impossibility theorems can be extended to more general specifications of quasi-linear valuation functions - the integrability constraints expressed by the equality of cross-derivatives will not generally agree with the requirements imposed by efficiency. We have chosen the linear formulation for ease of exposition, and because it yields nice properties of equilibrium utility functions without further assumptions on the used mechanisms (see Section 3).

The impossibility results in the multi-dimensional case suggest a quest for the second-best (or constrained efficient) mechanisms. It is straightforward to construct second-best mechanisms if the inefficiency is purely due to the fact that some informational variables must have a zero marginal effect on the expected probability assignment in incentive compatible mechanisms. It is then possible to reduce the dimensionality of the model (without loss of efficiency) by eliminating such variables. If, after performing these reductions, it is still the case that the

[23] A similar result appears in Dasgupta and Maskin (1998), who were the first to exhibit the basic intuition behind the construction. Technically, our result is not a special case of theirs because Dasgupta and Maskin's framework is, specifically, one of multi-object auctions (without allocative externalities), while we study a general social choice problem. Dasgupta and Maskin's mechanism is more complex since it also elicits reports about valuation functions.

The general condition allowing implementation (condition 5.1) was first identified in an earlier version of this paper.

payoff-relevant information depends in a non-trivial way on the chosen alternative (as it is the case, say, in a general multi-object auction), we are left in a framework covered by Theorem 4.4 and further dimension reductions become endogenous. The construction of a second-best mechanism is then equivalent to the difficult problem of finding a monotone and conservative vector field that maximizes the (expected) welfare functional[24]. This will be the subject of future work.

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Appendix

Proof of Theorem 3.1

a) Assume first that a DRM (p, x) satisfies the conditions in the Theorem. Choose any agent i. We must show that $\forall s^i, t^i$, $U_i(s^i, s^i) - U_i(t^i, s^i) \geq 0$. We obtain the following chain of equalities:

$$\begin{aligned} U_i(s^i, s^i) - U_i(t^i, s^i) &= V_i(s^i) - V_i(t^i) - Q^i(t^i) \cdot (s^i - t^i) \\ &= \int_{t^i}^{s^i} Q^i(\tau^i) \cdot d\tau^i - Q^i(t^i) \cdot (s^i - t^i) \\ &= \int_0^1 [Q^i((1-\alpha)t^i + \alpha s^i)) - Q^i(t^i)] \cdot (s^i - t^i) d\alpha \end{aligned}$$

The first equality follows by equation 3.1 and by the definition of $V_i(\cdot)$. The second equality follows by assumption. The last equality follows by choosing to perform the integration on the straight line connecting t^i and s^i.

The condition $\forall s^i, t^i$, $U_i(s^i, s^i) - U_i(t^i, s^i) \geq 0$ is therefore equivalent to the condition

$$\forall s^i, t^i, \int_0^1 [Q^i((1-\alpha)t^i + \alpha s^i)) - Q^i(t^i)] \cdot (s^i - t^i) d\alpha \geq 0.$$

It is enough to show that the integrand is non-negative for any α, $0 \leq \alpha \leq 1$. For $\alpha = 0$, the claim is obvious. Assume that $\alpha > 0$. We can write:

$$(s^i - t^i) = \frac{1}{\alpha}((1-\alpha)t^i + \alpha s^i - t^i).$$

We now obtain:

$$\begin{aligned}[Q^i((1-\alpha)t^i+\alpha s^i))-Q^i(t^i)]\cdot(s^i-t^i) &= \\ \frac{1}{\alpha}[Q^i((1-\alpha)t^i+\alpha s^i))-Q^i(t^i)]\cdot((1-\alpha)t^i+\alpha s^i-t^i) &\geq 0\end{aligned}$$

The last inequality follows from the monotonicity of $Q^i(\cdot)$.

b) For the converse, assume that the DRM (p,x) is incentive compatible. This implies that $V_i(s^i)=U_i(s^i,s^i)=\max_{t^i}U_i(t^i,s^i)$. The function $V_i(\cdot)$ is the supremum of a collection of affine functions and it must be convex. Convex functions are twice differentiable almost everywhere[25]. The convexity of $V_i(\cdot)$ implies the monotonicity of the subdifferential map $\partial V_i(s^i)$. At all points where $V_i(\cdot)$ is differentiable (i.e., a.e.) the subdifferential $\partial V_i(\cdot)$ consists of a unique point, the gradient $\nabla V_i(\cdot)$. Hence, the function $\nabla V_i(\cdot)$ is well-defined, monotone and differentiable a.e. Assuming that $V_i(\cdot)$ is differentiable at s^i we obtain by expression 3.1 and by the Envelope Theorem that:

$$\forall k,\ \frac{\partial V_i}{\partial s^i_{ki}}(s^i)=\frac{\partial U_i}{\partial s^i_{ki}}(t^i,s^i)\mid_{t^i=s^i}=a^i_{ki}q^i_k(s^i) \tag{7.1}$$

$$\forall k,\forall j\neq i,\ \frac{\partial V_i}{\partial s^i_{kj}}(s^i)=\frac{\partial U_i}{\partial s^i_{kj}}(t^i,s^i)\mid_{t^i=s^i}=0 \tag{7.2}$$

Hence, we obtain $\nabla V_i(s^i)=Q^i(s^i)$ whenever the gradient is well-defined (a.e.). The integral representation is immediately obtained from the fundamental theorem of calculus if $V_i(\cdot)$ is everywhere differentiable. Otherwise, the result follows by noting that a convex function is (up to a constant) uniquely determined by its subdifferential (see Rockafellar 1997, Theorem 24.9), and that it can be recovered (up to a constant) by integrating any measurable selection from its subdifferential map (see Krishna and Maenner, 1999). ■

Proof of Theorem 4.4: Let (p,x) be an efficient, incentive compatible DRM, and let $(q^i_k(\cdot))^K_{k=1}$ be the associated vector field of interim expected probabilities for agent i. Consider a type t^i and two alternatives k and k' such that $q^i_k(s^i)\neq 0$ and $q^i_{k'}(s^i)\neq 0$ for all s^i in a neighborhood of t^i. We consider below signals s^i in that neighborhood.

Since (p,x) is incentive compatible, the associated indirect utility function $V_i(\cdot)$ is twice-differentiable a.e. Since (p,x) is efficient, the associated functions $(q^i_k(\cdot))^K_{k=1}$ are continuously differentiable.

By equation 4.5 we obtain for almost all s^i:

[25] This and all following properties of convex functions are listed in the classical text of Rockafellar, 1997.

$$\forall k, k^{'}, \; a^i_{ki} \frac{\partial q^i_k(s^i)}{\partial s^i_{k^{'}}} = a^i_{k^{'}i} \frac{\partial q^i_{k^{'}}(s^i)}{\partial s^i_k} \tag{7.3}$$

Since p is efficient, we obtain:

$$\begin{aligned} q^i_k(s^i) &= \text{Prob } \{\sum_{j=1}^{N}\sum_{g=1}^{N} a^j_{kg}s^j_k = \max_{k^*}\sum_{j=1}^{N}\sum_{g=1}^{N} a^j_{k^*g}s^j_k\} = \\ & \int_{\Delta_k(s^i)} f_{-i}(s^{-i})ds^{-i} \end{aligned} \tag{7.4}$$

where

$$\Delta_k(s^i) = \{s^{-i} \mid \sum_{j=1}^{N}\sum_{g=1}^{N} a^j_{kg}s^j_k = \max_{k^*}\sum_{j=1}^{N}\sum_{g=1}^{N} a^j_{k^*g}s^j_k\} \tag{7.5}$$

An analogous expression holds for $q^i_{k'}(s^i)$. Define now the set

$$\Omega_{k,k'}(s^i) = \{s^{-i} \; \mid \sum_{j=1}^{N}\sum_{g=1}^{N} a^j_{kg}s^j_k = \sum_{j=1}^{N}\sum_{g=1}^{N} a^j_{k'g}s^j_{k'} = \max_{k^*}\sum_{j=1}^{N}\sum_{g=1}^{N} a^j_{k^*g}s^j_k\} \tag{7.6}$$

We will show that the derivative $\frac{\partial q^i_k(s^i)}{\partial s^i_{k'}}$ involves only an integral over $\Omega_{k,k'}(s^i)$ multiplied by the "rate of change" of this set with respect to $s^i_{k'}$, which is given by $-(\sum_{g=1}^N a^i_{k'g})$.

To see this, consider an affine, bijective change of variable in the space S^{-i}, where $x_0 = \sum_{j\neq i}\sum_{g=1}^N a^j_{kg}s^j_k - \sum_{j\neq i}\sum_{g=1}^N a^j_{k'g}s^j_{k'}$ is one of the new variables, and s^{-i,x_0} denotes the set of the other variables. Such a change of variables exists because x_0 is not identically equal to zero (since $q^i_k(t^i) \neq 0$ and $q^i_{k'}(t^i) \neq 0$). The explicit change of variable may depend on the coefficients.

To fix ideas, suppose that the coefficients are such that for all alternatives k'' there exists an agent $j(k'') \neq i$, such that $a^{j(k'')}_{k''j(k'')} \neq 0$. Consider then the mapping $\{s^j_{k''}\}_{j\neq i,k''} \rightarrow \{x^j_{k''}\}_{j\neq i,k''}$ where: 1) For $k'' \neq k$, $j = j(k'')$, $x^{j(k'')}_{k''} = \sum_{j\neq i}\sum_{g=1}^N a^j_{kg}s^j_k - \sum_{j\neq i}\sum_{g=1}^N a^j_{k''g}s^j_{k''}$ (observe that $x^{j(k')}_{k'} = x_0$) ; 2) For all (j, k'') such that $k'' = k$ or $j \neq j(k'')$, $x^j_{k''} = s^j_{k''}$.

Denote by $J(s^{-i})$ the Jacobian induced by this change of variable. Recalling expression 7.5, observe that

$$\begin{aligned} \Delta_k(s^i) &= \{s^{-i} \mid x_0 \geq -(\sum_{g=1}^{N} a^i_{kg})s^i_k + (\sum_{g=1}^{N} a^i_{k'g})s^i_{k'} \wedge \\ \sum_{j=1}^{N}\sum_{g=1}^{N} a^j_{kg}s^j_k &\geq \sum_{j=1}^{N}\sum_{g=1}^{N} a^j_{k''g}s^j_{k''} \text{ for } k^{''} \neq k^{'}\} \end{aligned} \tag{7.7}$$

Note that variables x_0 and $s^i_{k'}$ appear only in the first inequality defining $\Delta_k(s^i)$. Moreover, the area in $\Delta_k(s^i)$ where marginal variations of $s^i_{k'}$ are relevant (i.e., where $x_0 = -(\sum_{g=1}^N a^i_{kg})s^i_k + (\sum_{g=1}^N a^i_{k'g})s^i_{k'}$) is precisely $\Omega_{k,k'}(s^i)$. Hence, recalling expression 7.4, we obtain:

$$\frac{\partial q^i_k(s^i)}{\partial s^i_{k'}} = -(\sum_{g=1}^N a^i_{k'g}) \int_{\Omega_{k,k'}(s^i)} f_{-i}(s^{-i})J(s^{-i})ds^{-i,x_0}. \tag{7.8}$$

The term $\frac{\partial q^i_{k'}(s^i)}{\partial s^i_k}$ is analogously computed (since the area in $\Delta_{k'}(s^i)$ where marginal variations of s^i_k are relevant is also $\Omega_{k,k'}(s^i)$) :

$$\frac{\partial q^i_{k'}(s^i)}{\partial s^i_k} = -(\sum_{g=1}^N a^i_{kg}) \int_{\Omega_{k,k'}(s^i)} f_{-i}(s^{-i})J(s^{-i})ds^{-i,x_0}. \tag{7.9}$$

Combining equations 7.8 and 7.9 , we obtain that:

$$\frac{\partial q^i_k(s^i)}{\partial s^i_{k'}}(\sum_{g=1}^N a^i_{kg}) = \frac{\partial q^i_{k'}(s^i)}{\partial s^i_k}(\sum_{g=1}^N a^i_{k'g}) \tag{7.10}$$

Equations 7.3 and 7.10 yield together the wished result. ■

Example 7.1.

There are N agents and three alternatives denoted A_1, A_2 and B. The only additional assumption (compared to those in Theorem 4.4) is that the signal of one agent, say agent i, is always the same in alternatives A_1 and A_2, i.e. $s^i_{A_1} = s^i_{A_2}$ .We denote by s^i_A this common signal. Let $s^i = (s^i_A, s^i_B)$ denote the two-dimensional signal of agent i. For an, incentive compatible DRM (p, x) define $V_i(s^i)$ and $q^i_k(\cdot)$, $k = A_1, A_2, B$, in the usual way. At a type s^i where $V^i(\cdot)$ is twice differentiable, we have:

$$\frac{\partial V_i}{\partial s^i_A}(s^i) = a^i_{A_1 i}q^i_{A_1}(s^i) + a^i_{A_2 i}q^i_{A_2}(s^i) \tag{7.11}$$

$$\frac{\partial V_i}{\partial s^i_B}(s^i) = a^i_{Bi}q^i_B(s^i) \tag{7.12}$$

By Schwarz's Theorem we obtain:

$$a^i_{A_1 i}\frac{\partial q^i_{A_1}(s^i)}{\partial s^i_B} + a^i_{A_2 i}\frac{\partial q^i_{A_2}(s^i)}{\partial s^i_B} = a^i_{Bi}\frac{\partial q^i_B(s^i)}{\partial s^i_A}. \tag{7.13}$$

We now turn to the consequences of efficiency. Define the sets $\Delta_k(s^i)$, $\Omega_{k,k'}(s^i)$ as in the proof of Theorem 4.4. The derivative $\frac{\partial q^i_{A_r}(s^i)}{\partial s^i_B}$, $r = 1, 2$, is computed as before, i.e.,

$$\frac{\partial q^i_{A_r}(s^i)}{\partial s^i_B} = (-\sum_{g=1}^{N} a^i_{Bg}) \int_{\Omega_{B,A_r}(s^i)} f_{-i}(s^{-i}) J_r(s^{-i}) ds^{-i,x_r}$$

where $x_r \equiv \sum_{j\neq i}\sum_{g=1}^{N} a^j_{A_rg}s^j_{A_r} - \sum_{j\neq i}\sum_{g=1}^{N} a^j_{Bg}s^j_B$ and $J_r(s^{-i})$ stands for the Jacobian of the change of variable in the S^{-i} space where x_r is one of the new variables and s^{-i,x_r} the other ones. The derivative $\frac{\partial q^i_B(s^i)}{\partial s^i_A}$ is different, since it is now composed of two parts:

$$\frac{\partial q^i_B(s^i)}{\partial s^i_A} = \sum_{r=1,2}[(-\sum_{g=1}^{N} a^i_{A_rg}) \int_{\Omega_{B,A_r}(s^i)} f_{-i}(s^{-i}) J_r(s^{-i}) ds^{-i,x_r}]$$

To see this observe that $\Delta_B(s^i) = \{s^{-i} \mid x_r \leq -(\sum_{g=1}^{N} a^i_{A_rg})s^i_A + (\sum_{g=1}^{N} a^i_{Bg})s^i_B \text{ , } r = 1, 2\}$. The formula follows because, when s^i_A varies, the two relevant boundaries of $\Delta_B(s^i)$ are those where $x_r = -(\sum_{g=1}^{N} a^i_{A_rg})s^i_A + (\sum_{g=1}^{N} a^i_{Bg})s^i_B$, $r = 1, 2$ (they corresponds to $\Omega_{A_r,B}(s^i)$, $r = 1, 2$, respectively) Combining the above expressions, we conclude that efficiency implies:

$$\sum_{g=1}^{N} a^i_{A_1g}\frac{\partial q^i_{A_1}(s^i)}{\partial s^i_B} + \sum_{g=1}^{N} a^i_{A_2g}\frac{\partial q^i_{A_2}(s^i)}{\partial s^i_B} = \sum_{g=1}^{N} a^i_{Bg}\frac{\partial q^i_B(s^i)}{\partial s^i_A}. \tag{7.14}$$

Finally, note that conditions 7.13and 7.14 are, in general, inconsistent. ■

Proof of Theorem 5.1: Since all a^i_{ki} are assumed to be different, we can re-order the alternatives so that the sequence $(a^i_{ki})_k$ is strictly increasing, i.e. $a^i_{(k+1)i} > a^i_{ki}$ for $k = 1, .., K - 1$. Condition 5.1 implies then that the sequence $(\sum_{j=1}^{n} a^i_{kj})_k$ is also strictly increasing.

We construct an efficient, incentive compatible, DRM. For any reported signals the mechanism chooses an efficient alternative given those reports. To specify transfers, we proceed as follows. For fixed reports s^{-i} and i's report t^i,denote by $k^*(t^i)$ the efficient alternative chosen as a function of t^i, i.e.

$$k^*(t^i) \in \arg\max_k \sum_{j=1}^{n} V^j_k(t^i, s^{-i}).$$

Because the sequence $(\sum_{j=1}^{n} a^i_{kj})_k$ is also strictly increasing , we can define for every vector s^{-i}, a **non-decreasing** sequence of agent i's signals $(\overline{s}^{i,k}(s^{-i}))_k$ with the property that, for any $t^i \in (\overline{s}^{i,k}(s^{-i}), \overline{s}^{i,k+1}(s^{-i}))$, the efficient alternative is $k^*(t^i) = k$.

For each vector s^{-i} we inductively define a sequence of transfers, $\{\overline{x}_i^k(s^{-i})\}_k$, as follows: $\overline{x}_i^1(s^{-i}) \in \Re$ is an arbitrary constant, and for all k, $1 < k \leq K-1$,

$$\overline{x}_i^{k+1}(s^{-i}) - \overline{x}_i^k(s^{-i}) = \sum_{j, j \neq i} [V_{k+1}^j(\overline{s}^{i,k+1}(s^{-i}), s^{-i}) - V_k^j(\overline{s}^{i,k+1}(s^{-i}), s^{-i})] \quad (7.15)$$

If the vector of reports is (t^i, s^{-i}), then i's transfer is defined to be $x_i^*(t^i, s^{-i}) = \overline{x}_i^{k^*(t^i)}(s^{-i})$[26].

The logic underlying the specification of the transfers is as follows. Fix a vector of reports s^{-i}. Suppose that both intervals $(\overline{s}^{i,k}(s^{-i}), \overline{s}^{i,k+1}(s^{-i}))$ and $(\overline{s}^{i,k+1}(s^{-i}), \overline{s}^{i,k+2}(s^{-i}))$ are non-empty. For s^i slightly above $\overline{s}^{i,k+1}(s^{-i})$ the only efficient alternative is $k+1$. For s^i slightly below $\overline{s}^{i,k+1}(s^{-i})$ the only efficient alternative is k. At $s^i = \overline{s}^{i,k+1}(s^{-i})$ both alternatives are efficient. The transfers are adjusted so that, given s^{-i}, agent i with type $\overline{s}^{i,k+1}(s^{-i})$ is made indifferent between alternative k with transfer $\overline{x}_i^k(s^{-i})$ and alternative $k+1$ with transfer $\overline{x}_i^{k+1}(s^{-i})$.

We now show that it is optimal for agent i to report truthfully if all other agents report truthfully.

Fix s^{-i} the (truthfully) reported signal of all agents other than i. In order to have a more transparent notation, we omit below the dependence of $\overline{s}^{i,k}(\cdot)$ and $\overline{x}_i^k(\cdot)$ on the fixed s^{-i}.

Suppose without loss of generality that agent i's true type s^i lies in $\left[\overline{s}^{i,k}, \overline{s}^{i,k+1}\right)$. If agent i reports truthfully $t^i = s^i$, his payoff is

$$U_i(s^i, s^{-i}) = V_k^i(s^i, s^{-i}) + \overline{x}_i^k.$$

For any report $t^i \in \left[\overline{s}^{i,k}, \overline{s}^{i,k+1}\right)$, agent i gets the same payoff. Suppose that agent i makes a report $t^i \in \left[\overline{s}^{i,k+r}, \overline{s}^{i,k+r+1}\right)$ with $r > 0$. This non-truthful report yields for agent i a payoff of

$$U_i(t^i, s^{-i}) = V_{k+r}^i(s^i, s^{-i}) + \overline{x}_i^{k+r}.$$

[26] To avoid a cumbersome case differentiation, we have assumed that, given s^{-i}, the set $\{k^*(t^i)\}_{t^i \in S^i}$ includes the entire set of alternatives. If this is not the case, then some of the intervals $(\overline{s}^{i,k}(s^{-i}), \overline{s}^{i,k+1}(s^{-i}))$ may be empty. Transfers are then defined up to the arbitrary value of the transfer in the first non-empty interval. Furthermore, if a signal $\overline{s}^{i,k+1}(s^{-i})$ hits the upper bound of agent i's signal interval, then the transfer for all reports $t^i > \overline{s}^{i,k}(s^{-i})$ is set to be equal to $\overline{x}_i^k(s^{-i})$.

Noting that $\overline{x}_i^{k+r} = \sum_{l=1}^{r} (\overline{x}_i^{k+l} - \overline{x}_i^{k+l-1}) + \overline{x}_i^k$ and using expression 7.15, we obtain:

$$\begin{aligned} U_i(s^i, s^{-i}) - U_i(t^i, s^{-i}) &= V_k^i(s^i, s^{-i}) - V_{k+r}^i(s^i, s^{-i}) \\ &\quad - \sum_{l=1}^{r} \left(\sum_{j,j\neq i} [V_{k+l}^j(\overline{s}^{i,k+l}, s^{-i}) - V_{k+l-1}^j(\overline{s}^{i,k+l}, s^{-i})] \right). \end{aligned}$$

By the definition of $\overline{s}^{i,k+l}$ (at which both alternatives $k+l-1$ and $k+l$ are efficient), we obtain:

$$\sum_{j,j\neq i} [V_{k+l}^j(\overline{s}^{i,k+l}, s^{-i}) - V_{k+l-1}^j(\overline{s}^{i,k+l}, s^{-i})] = -[V_{k+l}^i(\overline{s}^{i,k+l}, s^{-i}) - V_{k+l-1}^i(\overline{s}^{i,k+l}, s^{-i})]$$

Finally, we obtain that:

$$\begin{aligned} & U_i(s^i, s^{-i}) - U_i(t^i, s^{-i}) = V_k^i(s^i, s^{-i}) - V_k^i(\overline{s}^{i,k+1}, s^{-i}) \\ & + \sum_{l=1}^{r-1} [V_{k+l}^i(\overline{s}^{i,k+l}, s^{-i}) - V_{k+l}^i(\overline{s}^{i,k+l+1}, s^{-i})] + V_{k+r}^i(\overline{s}^{i,k+r}, s^{-i}) - V_{k+r}^i(s^i, s^{-i}) \quad = \\ & a_{ki}^i \left(s^i - \overline{s}^{i,k+1}\right) + \sum_{l=1}^{r-1} [a_{(k+l)i}^i \left(\overline{s}^{i,k+l} - \overline{s}^{i,k+l+1}\right)] + a_{(k+r)i}^i \left(\overline{s}^{i,k+r} - s^i\right) \quad = \\ & \sum_{l=1}^{r} \left(a_{(k+l-1)i}^i - a_{(k+l)i}^i\right) \left(s^i - \overline{s}^{i,k+l}\right) \quad \geq \quad 0 \end{aligned}$$

The last inequality follows because each of the terms in the sum is non-negative: by the assumption on the sequence $(a_{ki}^i)_k$, we have $a_{(k+l-1)i}^i - a_{(k+l)i}^i < 0$; because s^i lies in $\left[\overline{s}^{i,k}, \overline{s}^{i,k+1}\right)$, and because the sequence $\overline{s}^{i,k}$ is non-decreasing, we have $s^i - \overline{s}^{i,k+l} \leq 0$.

The proof for a report $t^i \in \left[\overline{s}^{i,k+r}, \overline{s}^{i,k+r+1}\right)$ with $r < 0$ is completely analogous.

Note that the transfers defined above do not depend on the distribution of signals, and our mechanism implements the efficient social choice rule no matter how the signals of the various agents are distributed[27] ■

[27] In other words, truth-telling constitutes an *ex-post equilibrium*.