A Note on Revenue Maximization and Efficiency in Multi-Object Auctions

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1. Introduction

We consider an auction with risk neutral agents having independent private valuations for several heterogenous objects. Most of the literature on revenue-maximizing auctions has focused on the sale of one good or on the sale of several identical units (thus yielding one-dimensional informational models). Two reasons for inefficiency in revenue-maximizing auctions have been identified 1) The (monopolist) seller can increase revenue by restricting supply. 2) A revenue maximizing seller will sell to bidders with the highest "virtual" valuations (see Myerson, 1981). Virtual valuations are adjusted valuations that take into account bidders' informational rents, and depend on the distribution of private information. Asymmetries among bidders (and possibly other properties of these distributions) drive a wedge between virtual and true valuations, leading to inefficiencies (see Ausubel and Cramton, 1998 for a recent discussion of these issues).

Our purpose here is to illustrate in the simplest possible way that a revenue-maximizing seller of several heterogenous objects has incentives to "misallocate" the sold objects even in symmetric settings, and no matter what the (symmetric) function governing the distribution of private information is. This inefficiency result should be contrasted with the efficiency result in Armstrong (1998) that applies only to some cases with discrete distributions of valuations.

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2. The Model

A seller has K heterogenous objects to sell, and there are N potential buyers. Agent $i, i = 0, 1, 2, ..n$ is characterized by a vector $\mathbf{v}^i = (v_1^i, v_2^i, ...v_K^i)$, where v_k^i is agent i's valuation for object k, and where agent zero denotes the seller. Let $\mathbf{P} = (P^0, P^1, ..., P^n)$ be a partition of the set of objects. Assume that buyer $i, i = 1, ..., n$ acquires the subset of objects P^i and he makes a payment x^i to the seller. Then i's utility is given by $\sum_{k \in P^i} v_k^i - x^i$, and the seller's utility is given by $\sum_{i=1}^n x^i - \sum_{k \in K \setminus P^0} v_k^0$. We consider a standard continuous "private independent values" model, where $f^i(\cdot) > 0$ denotes the density function underlying the distribution of $\mathbf{v}^i$ on the hypercube $[\underline{v}_1^i, \overline{v}_1^i] \times ...[\underline{v}_K^i, \overline{v}_K^i]$. We assume that the buyers are symmetric in the sense that for all $i, j,\ i \neq j$, it holds: 1) $\forall k$, $\underline{v}_k^i = \underline{v}_k^j = \underline{v}_k$ and $\overline{v}_k^i = \overline{v}_k^j = \overline{v}_k$; 2) $\forall \mathbf{v}$, $f^i(\mathbf{v}) = f^j(\mathbf{v}) = f(\mathbf{v})$;

3. Efficient Auctions

An auction is (ex-post) efficient if, for all possible realizations $(\mathbf{v}^1, ..., \mathbf{v}^n)$, the resulting equilibrium partition $\overline{\mathbf{P}}(\mathbf{v}^1, ..., \mathbf{v}^n)$ has the feature that,for all possible partitions $\mathbf{P}$, $\sum_{i=0}^n \sum_{k \in \overline{P}^i} v_k^i \geq \sum_{i=0}^n \sum_{k \in P^i} v_k^i$.

It is easy to construct a mechanism that allocates the goods efficiently: The seller conducts K separate auctions, such that at auction k good k is sold through a second-price sealed-bid auction with reserve price v_k^0. Since each good is allocated to the agent that values it most, efficiency is immediate. The seller's expected revenue is then given by $E(\sum_{k=1}^n (max(v_k^0, v_k^{(2)})))$ where for every k, $v_k^{(2)}$ denotes the second highest value among $v_k^1, ..., v_k^n$, and where E denotes the expectation operator.

4. Bundling and Revenue Maximization

In order to completely separate the misallocation effect from the supply restriction effect, we assume below that the seller is constrained to always sell the goods, and we focus on the efficiency of the allocation among buyers. We thus restrict attention to partitions $\mathbf{P} = (P^0, P^1, ..., P^n)$ such that $P^0 = \emptyset$. We say that an auction is (ex-post) efficient if, for all possible realizations $(\mathbf{v}^1, ..., \mathbf{v}^n)$, the resulting equilibrium partition $\overline{\mathbf{P}}(\mathbf{v}^1, ..., \mathbf{v}^n)$ has the feature that, for all partitions $\mathbf{P}$ with $P^0 = \emptyset$, $\sum_{i=1}^n \sum_{k \in \overline{P}^i} v_k^i \geq \sum_{i=1}^n \sum_{k \in P^i} v_k^i$.

We first focus on the two-buyers case. In this framework we can show that a revenue-maximizing auction is never ex-post efficient. If the seller conducts separate second-price auctions (without a reserve price), the allocation among buyers is efficient, and it yields an expected revenue of $E(\sum_{k=1}^K min(v_k^1, v_k^2))$. But the seller can also conduct a second-price sealed-bid auction (without a reserve price) for the bundle consisting of all k objects. In this case, the seller's expected revenue is given by $E(min(\sum_{k=1}^{K} v_k^1, \sum_{k=1}^{K} v_k^2))$. Note that for any realizations $\mathbf{v}^1 = (v_1^1, ..., v_K^1)$ and $\mathbf{v}^2 = (v_1^2, ..., v_K^2)$ it holds that $min(\sum_{k=1}^{K} v_k^1, \sum_{k=1}^{K} v_k^2) \geq \sum_{k=1}^{K} min(v_k^1, v_k^2)$, with strict inequality for a positive measure of realizations. Hence, the bundling auction surely achieves a strictly higher revenue than the separate auctions. This elegant and simple argument is due to Palfrey (1983).

To complete the argument, we use the celebrated Revenue Equivalence Theorem (see Myerson, 1981, and Engelbrecht-Wiggans, 1988), which asserts that in continuous frameworks the expected transfers to the seller in any incentive-compatible mechanism are determined, up to a constant, by the allocation rule. By an application of this theorem, any incentive-compatible, efficient allocation mechanism such that a buyer with valuation $\underline{v} = (\underline{v}_1, .., \underline{v}_K)$ gets an equilibrium payoff of zero, yields exactly the same expected revenue for the seller. In the class of efficient auctions, the revenue-maximizing procedure must obviously leave a buyer having type $\underline{\mathbf{v}} = (\underline{v}_1, .., \underline{v}_K)$ with a zero-payoff[1]. We thus obtain that the separate second-price auctions are revenue-maximizing among all efficient mechanisms. Hence, we have shown that the bundling auction achieves a strictly higher revenue than any efficient mechanism, and that the revenue maximizing auction (among all mechanisms) must necessarily involve some misallocation of goods among buyers.

For the case of more than two buyers, the result is less clear cut: the bundling auction is revenue superior to any efficient auction if the expectation of the second highest sum of valuations, i.e. $E((\sum_{k=1}^{K} v_k^i)^{(2)})$, is larger than the sum of the expectations of the second highest valuations, i.e. $\sum_{k=1}^{K} E(v_k^{(2)})$. This rather complex condition involving second-order statistics of convolutions may be fulfilled for some distributions of valuations, but may be violated for others. In any case, even if the condition is violated, there is no particular reason to expect the revenue-maximizing auction to be efficient (note also that this condition does not involve the hazard rates used to compute virtual valuations).

It is interesting to note that, if the number of buyers goes to infinity, then the expected revenue from the K separate, efficient auctions, $\sum_{k=1}^{K} E(v_k^{(2)})$, converges to $\sum_{k=1}^{K} \overline{v}_k$. Hence, in the limit, the seller can extract all available surplus[2], and the efficient procedure is revenue maximizing.

[1] If such a buyer has a positive expected payoff, the seller can increase revenue by imposing an entry fee, without affecting incentives.

[2] Monderer and Tenneholz (1999) derive a similar result in a more complex model where buyers are risk-averse.

5. Concluding Comments

It is very difficult to generally characterize the revenue maximizing auction for multi-object settings, since one needs to solve a maximization program under complex integrability constraints (see Jehiel, Moldovanu and Stacchetti, 1999). The difficulty is to locate the necessary inefficiencies. The nature of these inefficiencies is specific to multidimensional informational frameworks.

The focus on efficiency often yields an amenable analysis: In the independent private values model, efficient allocation mechanisms exist for arbitrary quasi-linear preferences, i.e. also for cases where the value of bundle P^i for agent i is not necessarily equal to $\sum_{k \in P^i} v_k^i$. An immediate application of the Revenue Equivalence Theorem shows that, in terms of expected transfers, any efficient mechanisms must be equivalent to some Clarke-Groves-Vickrey mechanism (see for example Williams, 1994). For the case of interdependent valuations, Jehiel and Moldovanu (1998) have shown that efficient, incentive compatible allocation mechanisms for several heterogenous objects need not even exist.

The study of efficient multi-object auctions is very important per-se, but it offers little insight for the revenue-maximization question.

6. References

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