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On the Risks of Stocks in the Long Run: A Probabilistic Approach Based on Measures of Shortfall Risk

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Peter Albrecht, Raimond Maurer and Ulla Ruckpaul

1. Introduction

The impact of the time horizon on the risk of stock investments is still a subject of intense and controversial debate within the academic and investment communities.[1] A number of closely related questions are connected with this subject such as e.g.

- What is an adequate way to conceptualise the long term risks of a stock investment?

- In what way does the risk of a stock investment depend on the time horizon (are stocks more risky or less risky over the long term compared to the short, or is the time horizon irrelevant in this case?)

- To what extent and in what way does the "optimal" proportion of stocks in an (typically simply structured) investment portfolio depend on the time horizon?

- Does, in the long run, an investment in stocks always outperform investments (e.g. bonds) which are less risky over the short term? (Do stocks always beat bonds?[2] And: are bonds redundant assets on a long term view or are they only relevant as a temporary substitute (safe haven) in times of stock crashes?[3])

- What are the implications for the structure of "optimal" portfolios of long term investors, as e.g. pension funds or within the scope of private provision for old age retirement?[4] (Could even a 100% stock investment be adequate in this case?[5])

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1 For topical surveys - with different subjects of main emphasis - cf. e.g. *Albrecht* (1999), *Ammann/Zimmermann* (1997), *Kramer/Weber* (1999), *Kritzman/Rich* (1998) and *Rosen* (1999).

2 For a current overview see especially *Rosen* (1999) and *Stehle* (1998, 1999). A main point in this context is the different tax treatment of investments in stocks and bonds respectively.

3 Cf. *Navon* (1998) for a discussion of this question.

4 Cf. e.g. *Bodie/Crane* (1998) or *Albrecht/Maurer* (2000).

5 Cf. for a discussion of this question *Thaler/Williamson* (1994), *Asness* (1996) and *Biermann* (1998).

In addition there is a close connection to the question of the level of the empirical risk premium of stock investments (compared to bonds) and whether this premium is appropriate from a theoretical point of view (equity premium puzzle[6]).

With respect to the contributions to the above questions one can basically distinguish between a purely historical approach on the one hand and on the other, an approach where the empirical data are analysed on the basis of a stochastic model for the development of stocks and bonds respectively. An approach in which historical data are used exclusively and ex post-performance over different time periods[7] is compared, implies the problem[8] that the 10-, 15- or 20-year periods used are strongly overlapping and the resulting roll-over returns have a high degree of correlation. This results in a serious estimation bias. A high degree of statistical significance requires independent returns based on non-overlapping 10-, 15- or 20-year periods. The existing horizon of experience, however, is too short to obtain enough data of this kind. *Navon* (1998, p. 67) writes very appropriately: „We mere mortals live only one life. And we haven't had sufficient ten-, twenty- or thirty-year periods that are independent of another to derive statistically significant conclusions from history.“

The solution to this dilemma is to base the analysis on an appropriate stochastic model, usually a version of a random walk[9] or a process with mean reversion[10], to estimate the parameters using independent observations (over shorter time periods) and finally to obtain the values desired analytically or by means of simulation.

A second basic distinction of the contributions to the literature in this field is the way in which the (long term) distributions of prices or returns are evaluated. Here we can distinguish between approaches where the utility function of the investor is used in an explicit way for the analysis[11] and approaches where the utility function is only used implicitly or not at all. Examples for the latter class are approaches which analyse the (long term) risk of a stock in-

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6 Cf. for the fundamental contribution by *Mehra/Prescott* (1985) as well as *Albrecht* (1999) for a more topical review.

7 Cf. e.g. *Thaler/Williamson* (1994) or *Stehle* (1998, 1999).

8 *Bernstein* (1996) and topically *Löffler* (2000) refer to this problem.

9 In the literature the time-continuous model of the geometric Brownian motion (geometric Wiener-process) serves as standard reference model which, from a time-discrete view, implies independent and logarithmically normally distributed price increments.

10 A current example for this can be found in *Löffler* (2000, S. 355).

11 Cf. for an overview e.g. *Kritzman/Rich* (1998) or *Albrecht* (1999, chapter 4). Contributions using stochastical dominance, cf. e.g. *Levy/Cohen* (1998) and *Albrecht* (1999, section 3.3) belong to these approaches, too.

vestment in the framework of the theory of option pricing[12] or on the basis of the concept of shortfall risk.[13] Since the utility function of the investor is not explicitly considered in the analysis, both approaches can be refered to as preference-free. *Ammann/Zimmermann* (1997, 2000), however, argue that because the utility function is implicit in any floor specification the assumption of a preference-free determination of risk with the help of option pricing theory is a fallacy. The same argument holds true for the shortfall approach as here, too, different target returns are considered. Moreover, there are explicit connections between utility theory and the shortfall approach.[14]

In this study we analytically quantify the long-term risks of a stock investment within a shortfall framework with respect to deterministic (real) target returns as well as with respect to a bond investment. One central conclusion will be that under a worst-case perspective, formalised by the measure of conditional shortfall expectation (or: mean excess loss), an investment in stocks exhibits an increasing and substantial risk and that this characteristic is the true danger of a long-term stock investment.

The primary focus of this study is not to answer the question of what ultimately is the correct measure of risk but to work out different dimensions of risk in the sense of purely probabilistic characteristics of stock price developments. Utility theoretical or behavioral approaches[15] which consider these dimensions of risk for evaluation would be the logical next step to be taken.[16]

2. Design of the Analysis

Within the contributions to the time-diversification subject which are based on the shortfall approach, the shortfall probability has been the main focus of consideration. Formally, the shortfall probability of the return R of an investment with respect to a (deterministic or stochastic) benchmark z - the target return or a desired minimum return - is given by:

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12 Cf. *Bodie* (1995), *Albrecht* (1999, section 3.2.1) or *Ammann/Zimmermann* (2000).

13 Cf. e.g. *Leibowitz/Krasker* (1998), *Zimmermann* (1991), *Albrecht* (1999, section 3.1) and *Albrecht/Maurer* (2000).

14 Cf. *Fishburn* (1977) or *Albrecht/Maurer/Möller* (1999).

15 Cf. e.g. *Benartzi/Thaler* (1999) or *Kramer/Weber* (1999).

$$\mathrm{SP}(z) = \mathrm{P}(R < z). \tag{1}$$

The shortfall probability only evaluates the probability of possible shortfalls with respect to the target but does not evaluate the potential extent of this shortfall.

A risk measure allowing consideration not only of the probability but also the extent of the shortfall of an investment with respect to the target return is the measure of shortfall expectation, formally given by:

$$\mathrm{SE}(z) = \mathrm{E}[\max(z - R, 0)]. \tag{2}$$

An additional main risk measure to be used in the subsequent analysis is the mean excess loss (MEL), formally given by:

$$\mathrm{MEL}(z) = \mathrm{E}[z - R \mid R < z]. \tag{3}$$

This measure intuitively considers the mean amount of a shortfall with respect to the benchmark *z under the condition* that a shortfall occurs.

The risk measure MEL, the conditional shortfall expectation, is the suitable version for the purpose of the present contribution of the mean excess- respectively mean excess loss-function[17] $\mathrm{E}(X - u \mid X > u)$ considered in extreme value theory.[18] In addition, there is a close connection of the MEL to the Tail Conditional Expectation (TCE)[19], defined by $\mathrm{TCE}(z) = \mathrm{E}(R \mid R < z)$. This connection is given by:

$$\mathrm{TCE}(z) = z - \mathrm{MEL}(z).$$

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16 We share the view of an implicit association to a utility-theoretical analysis, explicitly made clear by *Ammann/Zimmermann* (1997, 2000), however, this will not be pursued further in the present contribution.

17 Cf. especially *Embrechts/Klüppelberg/Mikosch* (1997, p. 160 ff.), *Borkovic/Klüppelberg* (2000, p. 228f.), *Embrechts/Resnick/Samorodnitsky* (1999, p. 35 f.) and *Wirch* (1999, p. 110).

18 Cf. for the methods of extreme-value theory generally *Embrechts/Klüppelberg/Mikosch* (1997). For the use of the extreme-value theory for questions concerning financial risk management cf. *Borkovic/Klüppelberg* (2000) and *Embrechts/Resnick/Samorodnitsky* (1999).

19 Cf. e.g. *Artzner et al.* (1999, p. 223), *Wirch/Hardy* (1999, p. 339) or *Barth* (2000, p. 126 f.). *Barth* (2000) uses the above general version of TCE. *Artzner et al.* (1999) and *Wirch/Hardy* (1999) consider the special case, where the benchmark quantity z is identical to the value at risk of the investment considered. In this connection other authors, e.g. *Embrechts/Resnick/Samorodnitsky* (1999, p. 40) use the alternative term “conditional value at risk”.

MEL and TCE are connected by a simple (deterministic) translation.

The risk measure TCE has been receiving increasing attention in connection with the analysis and management of financial risk, e.g. generalised value-at-risk analysis of market risk or credit risk.[20] In the sense of the system of axioms of *Artzner et al*. (1999) the TCE is - in contrast to the traditional value at risk - a *coherent* measure of risk.[21] *Barth* characterises the TCE as a worst-case risk measure[22] which is very sensitive with respect to realisations at the tail of the distribution, i.e. large-scale shortfalls, however, with a very small probability. With considering MEL instead of TCE the property of coherency is lost but the characterisation as a worst case risk measure is preserved, which is sufficient for this contribution.

Between the shortfall measures introduced so far the following relation, which is rather intuitive, is valid[23]:

$$\mathrm{SE}(z) = \mathrm{MEL}(z) \cdot \mathrm{SP}(z). \tag{4}$$

The mean shortfall level is simply the product of the mean level of shortfall *given the occurrence* of a shortfall and the probability of this occurrence.

The shortfall measures introduced above will be used in the following analysis. The benchmark-returns will be specified later on.

The following evaluation begins with an analysis of the risks (in the sense defined above) of an investment that is representative for the German stock market relative to the development of a risk-free (deterministic) benchmark depending on the time horizon. The DAX usually serves as the standard representative for the development of German blue chip stocks. However, it specifically represents the viewpoint of a representative investor with a marginal tax rate of 36% (resp. of 30% since 1994). Therefore, we have not used the original time series of

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20 Cf. e.g. *Artzner et al.* (1999), *Barth* (2000), *Embrechts et al*. (1999), *Wirch* (1999) and *Wirch/Hardy* (1999).

21 Cf. *Barth* (2000, S. 127) as well as *Wirch/Hardy* (1999, S. 330).

22 Cf. *Barth* (2000, S. 127).

23 Cf. in general *Maurer* (2000, S. 63).

the DAX for our analysis but instead a corrected time series that assumes a domestic investor with a tax rate of 0%, in the following referred to as DAX/0 (DAX according to *Stehle*[24]).

In addition to the correction of the tax effects, the DAX time series is inflation-adjusted, which means all returns we have used are returns in real terms. Thus we assume for our study rational investors that are free of money illusion, i.e. changes in price level that leave all real terms unchanged do not lead to any changes in investment behavior.

To generate a representative distribution of real and tax-adjusted DAX returns the standard model of the geometric Brownian motion has been assumed for the development of the DAX. In a time-discrete context of successive one-year periods this assumption implies[25] independent and lognormally distributed price changes or equivalently independent and normally distributed continuous rates of return. To generate the parameters of a representative distribution of this kind, DAX returns of the evaluation period 1980 to 1999 have been applied. This time period represents a rather positive development compared to other periods (average nominal return of 15.47% and average real return of 12.88% from the viewpoint of an investor with a tax-rate of 0%). To illustrate the sensitivity of our results regarding the chosen empirical period of observation, an alternative 14-year evaluation period from 1986 - 1999 has also been applied[26] (average nominal return of 12.08% and average real return of 9.99% from the viewpoint of an investors with a tax-rate of 0%).

Therefore not a historical time series but a probability distribution consistent to the empirically observed data has been taken as the basis of our evaluation. The representative density functions of the time-continuous one year returns of the periods 1980 - 1999 and 1986 - 1999 respectively identified according to the above-mentioned method are shown in figure 1.

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[24] The DAX-adjustment mentioned, as well as others, are taken from publications of Richard *Stehle*, cf. e.g. *Stehle* (1998, 1999). Updated versions of these DAX-adjustments can be found on the homepage of his institute: http://www.wiwi.hu-berlin.de/finance.

[25] Cf. e.g. *Hull* (1993, p. 210 ff.).

[26] This alternative evaluation period eliminates especially the outlier return of 1985 of 86.35% in nominal terms and 83.42% respectively in real terms from the viewpoint of an investor with a tax-rate of 0%.

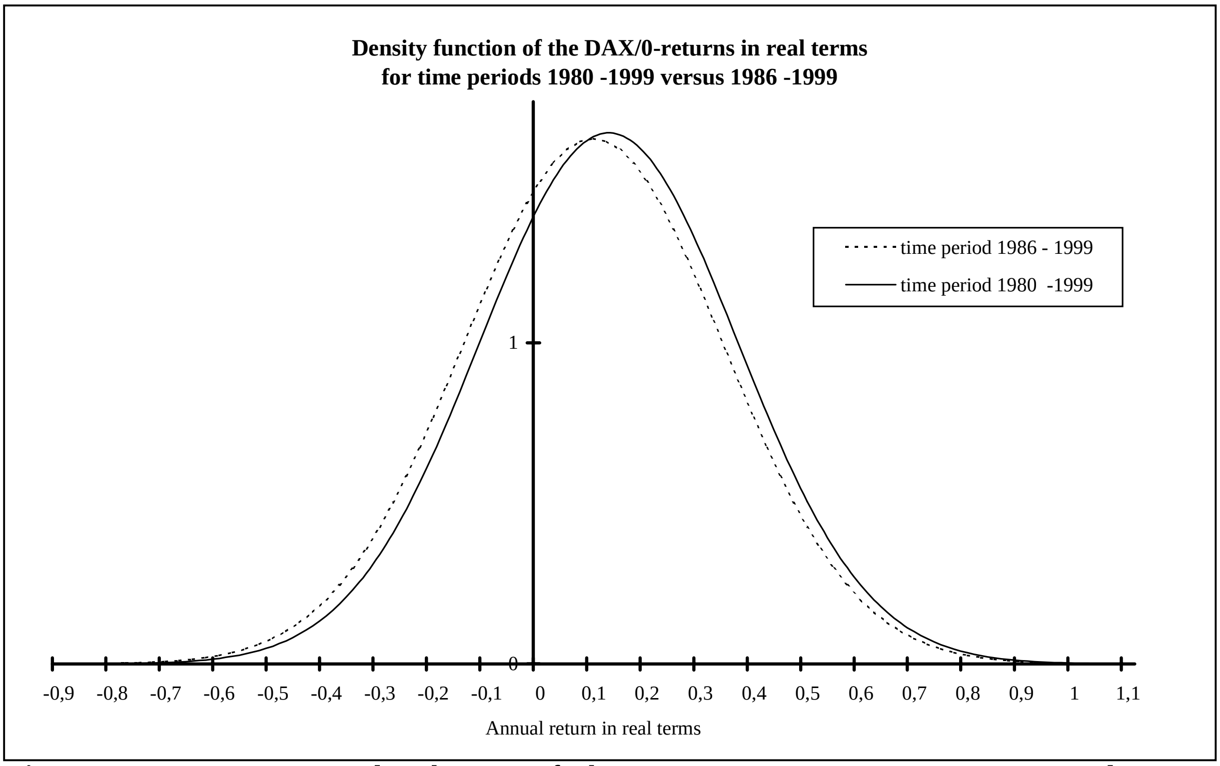

Figure 1: Representative distribution of the continuous DAX/0 returns in real terms, evaluation periods 1980 - 1999 and 1986 - 1999 respectively

The return distributions presented in the above figure are the basis for the evaluation of the shortfall-risks of a stock investment depending on the time horizon compared to a risk-free investment with the alternative benchmark returns of 0%, 2% and 4% in real terms. To ensure an analytical solution, a one-time investment in stocks has been applied.[27]

The second part of the evaluation consists of an analysis of the risks (in the sense defined above) of a representative investment in the German stock market compared to a representative investment in the German bond market. The construction of the representative return-distributions remains unchanged. For the German bond market the German bond performance index REXP has been used as representative. A tax-adjustment is not necessary in the case of the REXP since it is originally constructed from the viewpoint of an investor with a tax-rate of 0%. To maintain comparability, the time series of the REXP[28] is inflation-adjusted too, and alternative distributions of the REXP-returns have been identified on the basis of the time periods 1980 - 1999 and 1986 - 1999 respectively on the assumption of a geometric Brownian

27 In the case of a recurrent investment in stocks the evaluation has to be undertaken by means of a stochastic simulation. This will be the subject of a subsequent study.

28 We wish to thank Deutsche Börse AG for providing the necessary data.

motion.[29] Figure 2 compares the density function of the time-continuous REXP-returns in real terms to those of the DAX/0 for the evaluation period 1980 - 1999. Furthermore, it visualises the different risk/return profiles of both investments.[30]

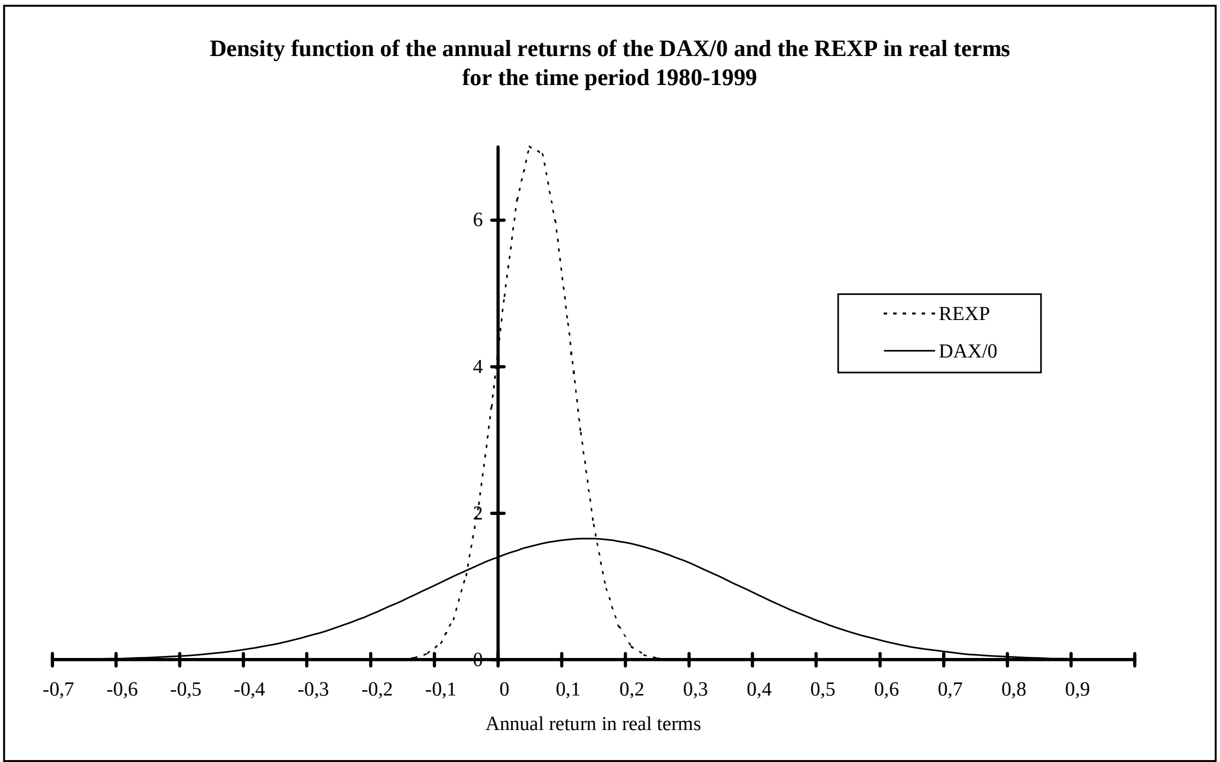

Figure 2: Representative distributions of the continuous DAX/0- and REXP-return in real terms, evaluation period 1980 - 1999

Within the scope of the definitions (1) - (4) of the shortfall-risk measures used, the REXP-return distribution represents a stochastic benchmark *z*. For all technical details of the specification and identification of the return distributions as well as the determination of the analytical expressions for the respective shortfall-risks, we refer to the appendix.

3. Long-term risks of stock investments: Deterministic benchmark-developments

Regarding the presentation of our evaluation, we begin with the results for the development over time of the shortfall probability of the DAX/0 in real terms under the assumption of a

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29 To be more precise, in order to be able to consider the correlation between stock and bond development a two-dimensional geometric Brownian motion resp. bivariate normally distributed continuous rates of return have been used. For technical details we refer to the appendix.

30 For the concrete numerical specification of the return, the risk and the correlation, we refer to the appendix.

one-time investment. The developments of the risk-free benchmarks are based on the deterministic target returns of 0%, 2% and 4%. Figures 3 and 4 illustrate the respective results for the evaluation periods of both 1980 - 1999 and 1986 - 1999.

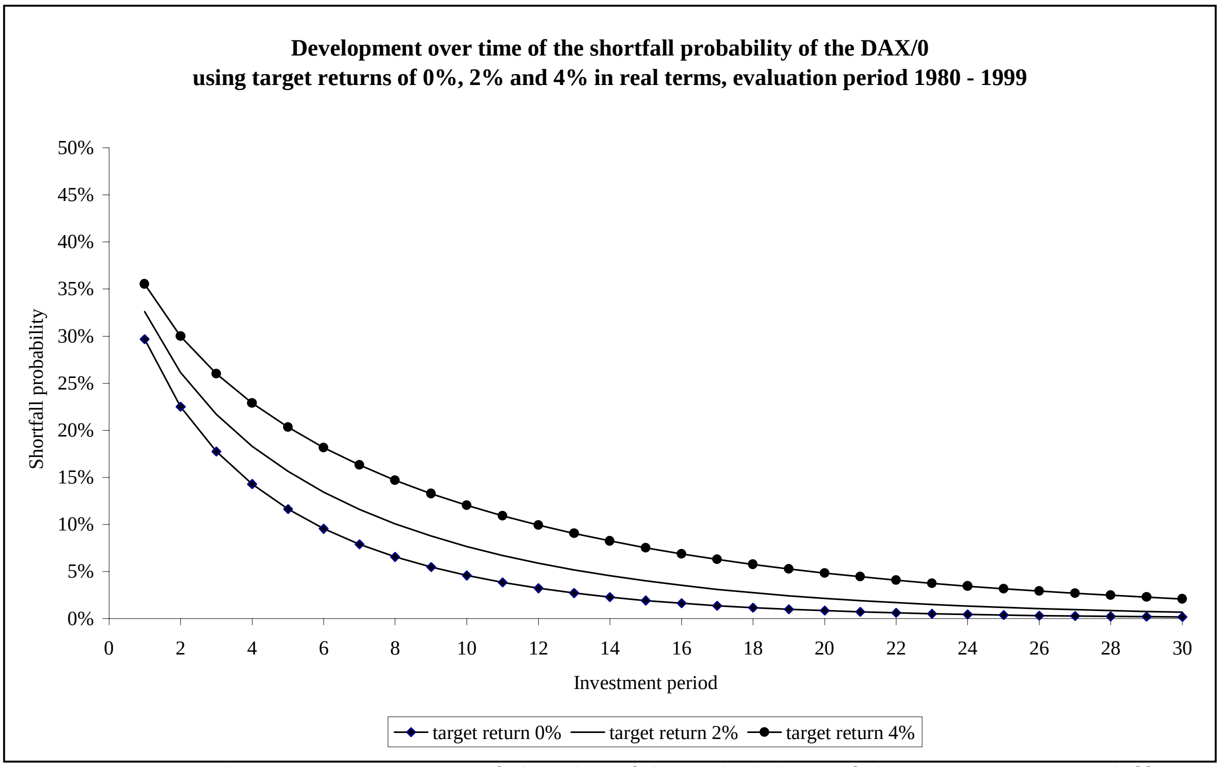

Figure 3: Development over time of the shortfall probability of the DAX/0 using different target returns, evaluation period 1980 - 1999

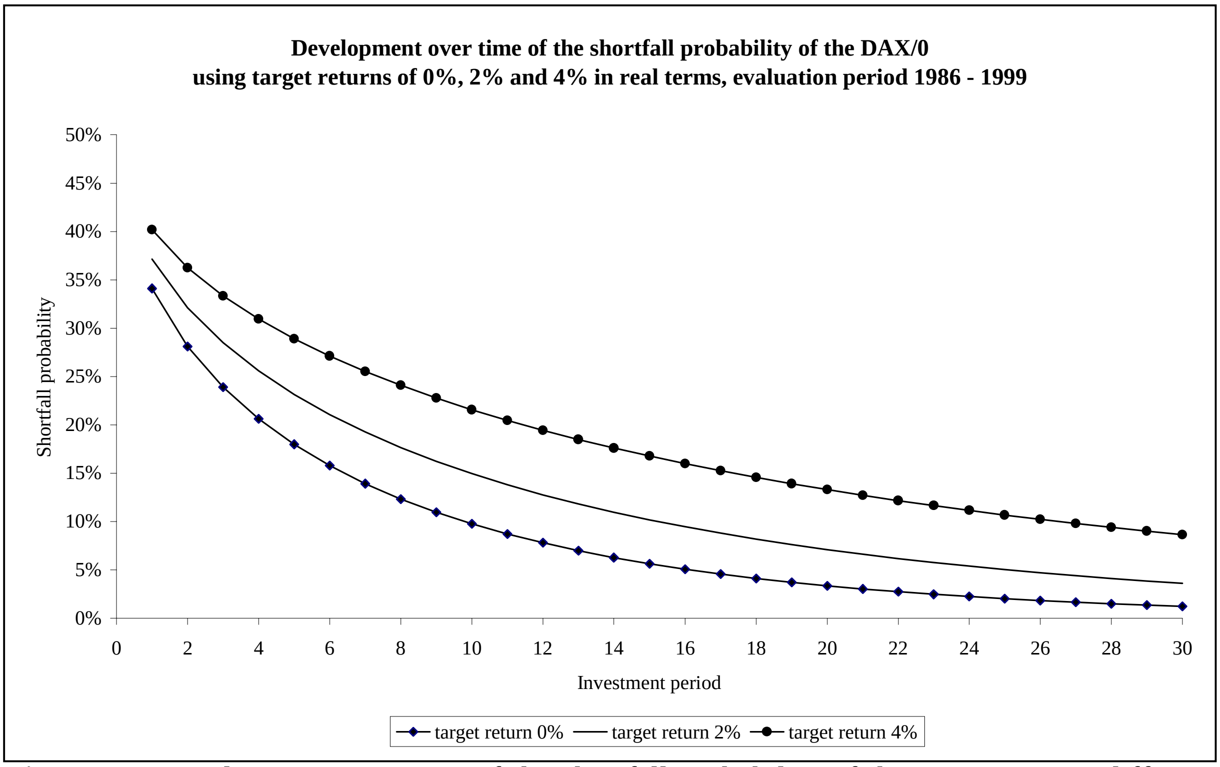

Figure 4: Development over time of the shortfall probability of the DAX/0 using different target returns, evaluation period 1986 - 1999

In principle, these results confirm a characteristic which *Leibowitz/Krasker* (1988) called *persistence of risk* and which indicates that the shortfall probability of a stock investment does not converge rapidly towards zero depending on time - as historical studies suggest - but rather that this convergence is rather slow. Thus, even for very long time horizons, the shortfall probability remains at a substantially high level. The level itself depends both on the target return chosen and on the assumed distribution of the stock returns.

Figures 5 and 6 show the corresponding results regarding the development over time of the shortfall expectation. To ease the interpretation of the results, the development of the shortfall expectation has been put in relation to the development of the corresponding risk-free benchmark and is thus shown as a percentage.

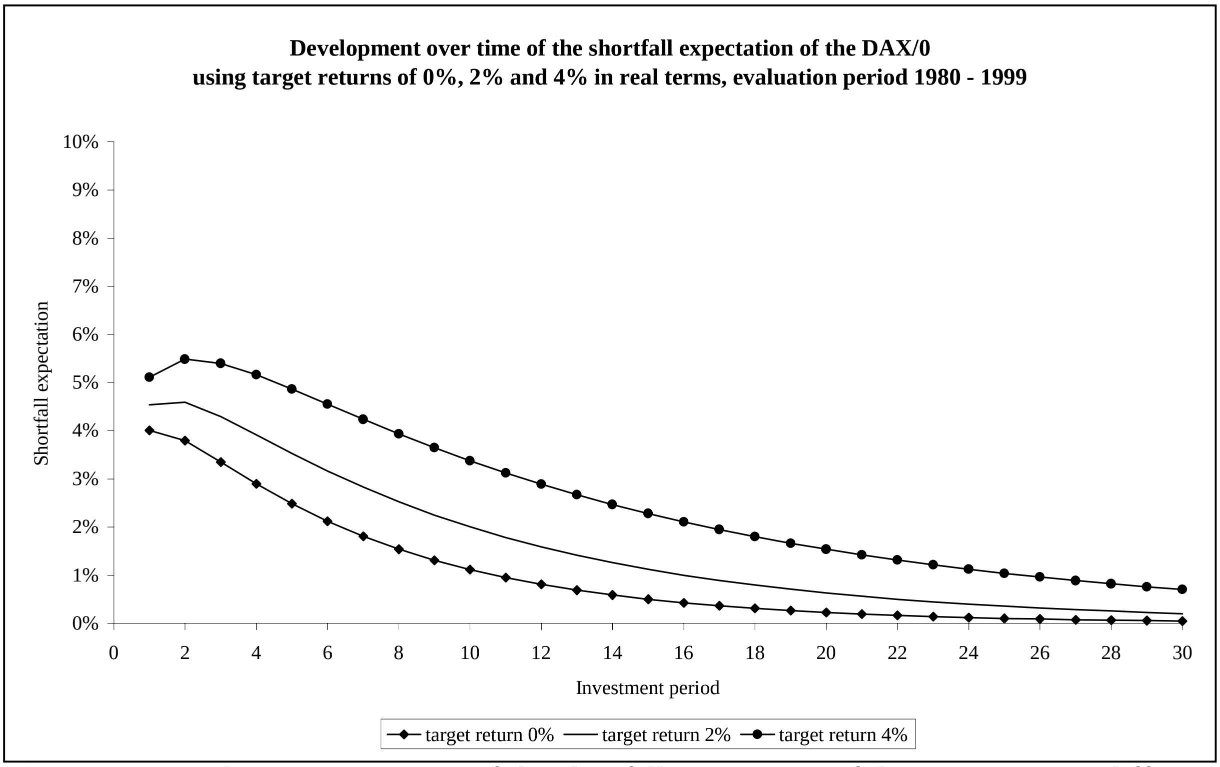

Figure 5: Development over time of the shortfall expectation of the DAX/0 using different target returns, evaluation period 1980 - 1999

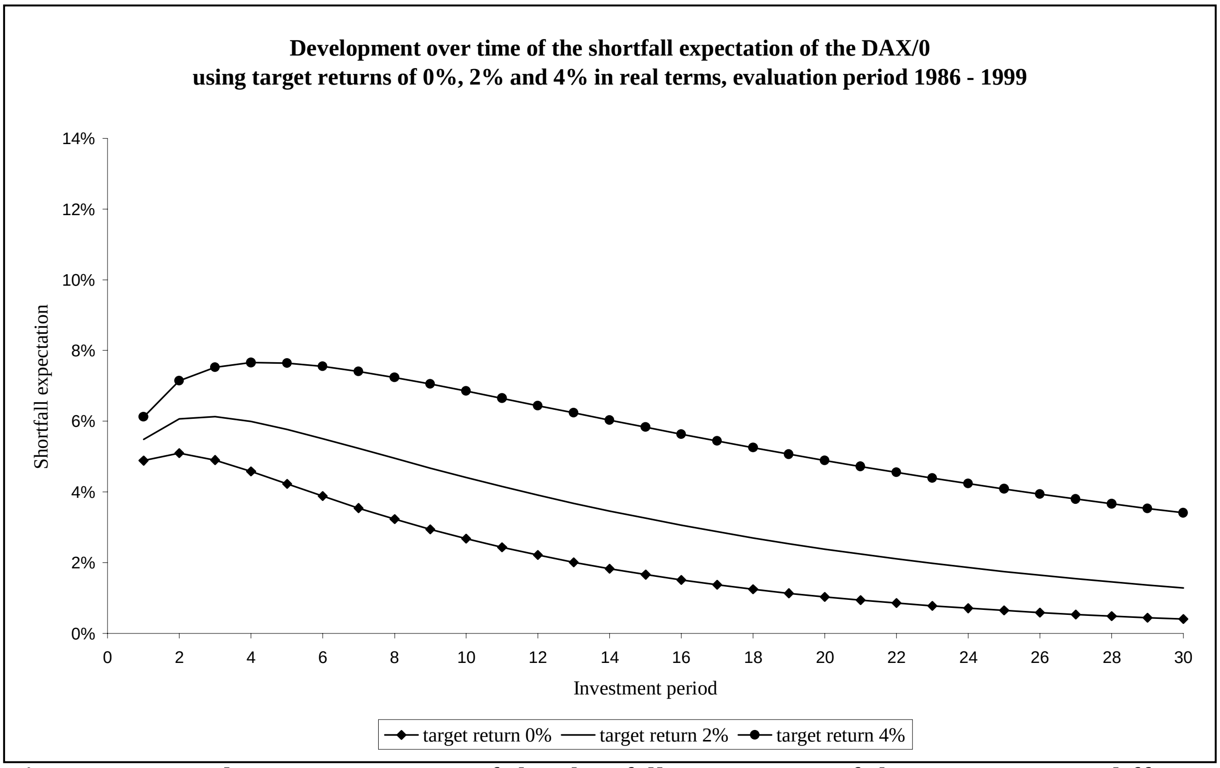

Figure 6: Development over time of the shortfall expectation of the DAX/0 using different target returns, evaluation period 1986 - 1999

Basically, after a phase of increasing risk at the beginning of the investment period, the shortfall expectation of a stock investment shows a monotonously decreasing development. In addition, the shortfall expectation has, like the shortfall probability, a persistence-characteristic. As with the level of the shortfall probability, the level of the remaining risk measured with the shortfall expectation depends on the target return chosen and - even more obviously than in the case of the shortfall probability - on the assumed representative distribution of the stock returns.

The corresponding results for the risk measure mean excess loss are presented in the following two figures 7 and 8. Again, the development of the mean excess loss has been put in relation to the development of the corresponding risk-free benchmark returns and thus is presented as a percentage.

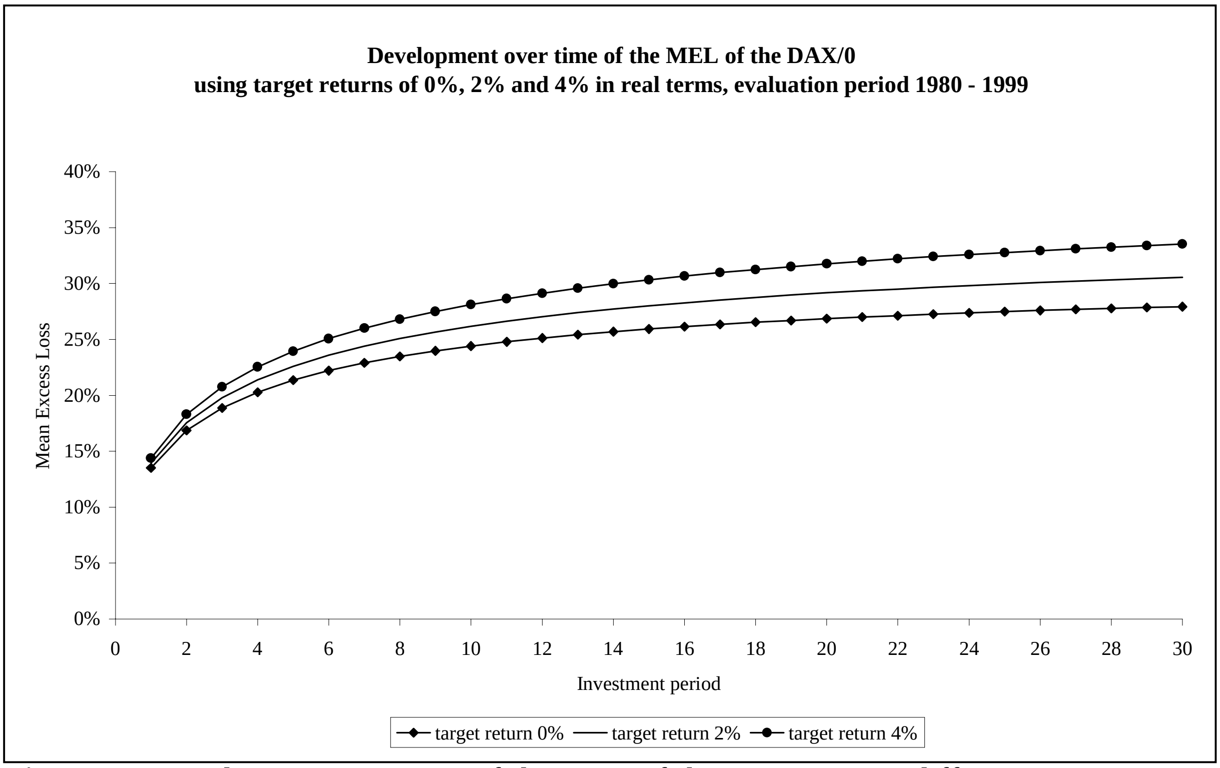

Figure 7: Development over time of the MEL of the DAX/0 using different target returns, evaluation period 1980 - 1999

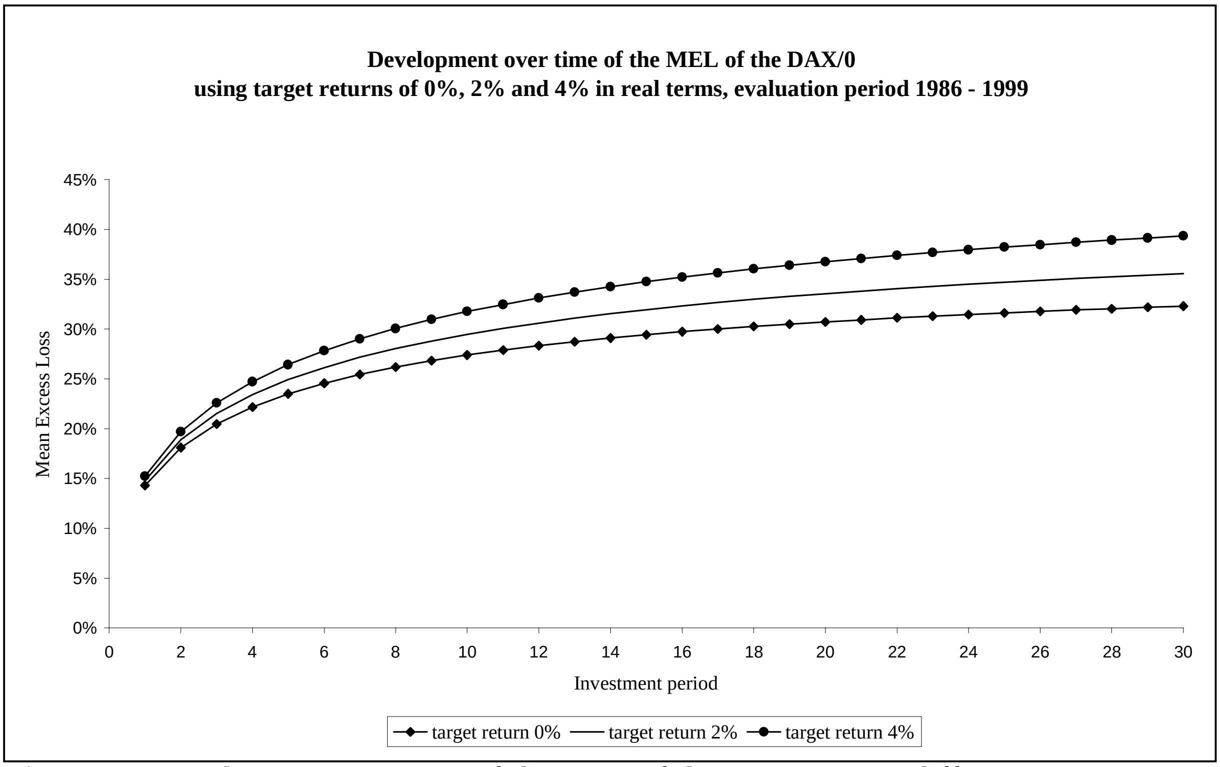

Figure 8: Development over time of the MEL of the DAX/0 using different target returns, evaluation period 1986 - 1999

The analysis of the (relative) mean excess loss reveals a new type of structural phenomenon: the expected conditional shortfall-level is monotonously increasing over time. This phenomenon is independent of both the benchmark return chosen and the return distribution chosen, which only determines the level of the expected conditional shortfall. Therefore, from a worst-case perspective the analysis of the worst-case risk measure MEL reveals that the risk of a stock investment increases and thus shows the true risk of a stock investment.

Taking a shortfall relative to a benchmark of 0 % return in real terms, the level of mean excess loss after 30 years is - depending on the supposed distribution of stock returns - in average about 28% - 32% of the corresponding value. This is a substantially high shortfall level.

Tables 1 and 2 illustrate exact figures of the shortfall risk measures for chosen time horizons, depending on the supposed distribution of stock returns.

Investment period	1 year	5 years	10 years	15 years	20 years	25 years	30 years
Target return 0% p.a.							
SP	29,68	11,63	4,57	1,94	0,85	0,38	0,17
SE	4,01	2,48	1,12	0,50	0,23	0,10	0,05
MEL	13,52	21,36	24,41	25,93	26,85	27,48	27,93
Target return 2% p.a.							
SP	32,58	15,63	7,66	4,01	2,17	1,20	0,67
SE	4,54	3,53	2,01	1,12	0,63	0,36	0,20
MEL	13,95	22,60	26,17	28,01	29,16	29,95	30,54
Target return 4% p.a.							
SP	35,53	20,33	12,02	7,53	4,85	3,17	2,10
SE	5,11	4,87	3,38	2,28	1,54	1,04	0,70
MEL	14,38	23,94	28,11	30,33	31,75	32,76	33,52

Table 1: Shortfall risk (in %) of a one-time investment in the DAX/0 in real terms for various target returns and chosen time horizons on the basis of the representative return-distribution 1980 - 1999

Investment period	1 year	5 years	10 years	15 years	20 years	25 years	30 years
Target return 0% p.a.							
SP	34,11	18,00	9,77	5,64	3,36	2,03	1,25
SE	4,88	4,23	2,68	1,66	1,03	0,64	0,40
MEL	14,32	23,50	27,39	29,43	30,72	31,63	32,30
Target return 2% p.a.							
SP	37,14	23,15	14,96	10,18	7,10	5,04	3,61
SE	5,49	5,77	4,41	3,25	2,38	1,75	1,28
MEL	14,77	24,92	29,47	31,94	33,56	34,71	35,58
Target return 4% p.a.							
SP	40,18	28,91	21,58	16,77	13,30	10,68	8,66
SE	6,12	7,64	6,85	5,83	4,89	4,08	3,41
MEL	15,24	26,44	31,76	34,75	36,76	38,22	39,35

Table 2: Shortfall risk (in %) of a one-time investment in the DAX/0 in real terms for various target returns and chosen time horizons on the basis of the representative return-distribution 1986 - 1999

Kritzman (1994, S. 15) provides the following very intuitive explanation of *Samuelson's* (1963) classical result concerning the irrelevance of the time horizon for the level of the stock-ratio of an investment: „The growing improbability of a loss is offset by the increasing magnitude of potential losses."

Thus, the main argument is that the occurrence of a loss from a stock investment becomes more and more improbable, but at the same time goes hand in hand with an increasing level of loss. In addition, these results make clear that the use of the shortfall probability alone is insufficient for the assessment of the risk of stock investments in the long run. Consequently, not only the probability of the occurrence of a loss or a shortfall, but also the possible extent of loss, has to be taken into consideration.

By means of equation (4) and the results presented above the cited intuitive explanation of *Kritzman* can be put on a theoretically precise basis. In addition it can be generalised with respect to shortfall-events (compared to pure loss-events). Indeed, the probability of a loss or a shortfall decreases with the length of the time horizon. However, the average level of the loss or the shortfall respectively, given a loss or a shortfall has occured, increases. But, at least[31] at the level of the purely statistical relation (4), the balance of these two effects is not compensatory. The shortfall probability over-compensates the mean excess loss to a certain extent. The worst-case aspect of a long time investment in stocks is partly hidden by only taking the shortfall probability into consideration. Thus, the elucidation of the worst-case risk immanent in a long time investment in stocks represents an additional piece of information that might be essential for investors.

All in all, the judgement of the long term risk of a stock investment is in a decisive way dependent on the risk measure used and with it the perspective of risk-assessment. This study is not meant to ascertain once and for all the "correct" risk measure but instead seeks greater transparency with regard to the various facets of risk.

4. Long term risks of a stock investment relative to an investment in bonds

As pointed out in section 2, the second part of the evaluation applies to a stochastic benchmark on the basis of a representative distribution of REXP-returns in real terms. The following figures 9 - 11 illustrate the development over time of the risk measures shortfall-probability, shortfall expectation and mean excess loss of the DAX/0 relative to the REXP. The representative distribution has been identified on the basis of the evaluation periods 1980 - 1999 and alternatively 1986 - 1999. In addition, table 3 contains the corresponding numerical results for the chosen time horizons.

[31] A trade-off concerning the utility of these two components might arrive at different results. However, at the moment a methodical approach is lacking to explicate this trade-off on a utility theoretical basis.

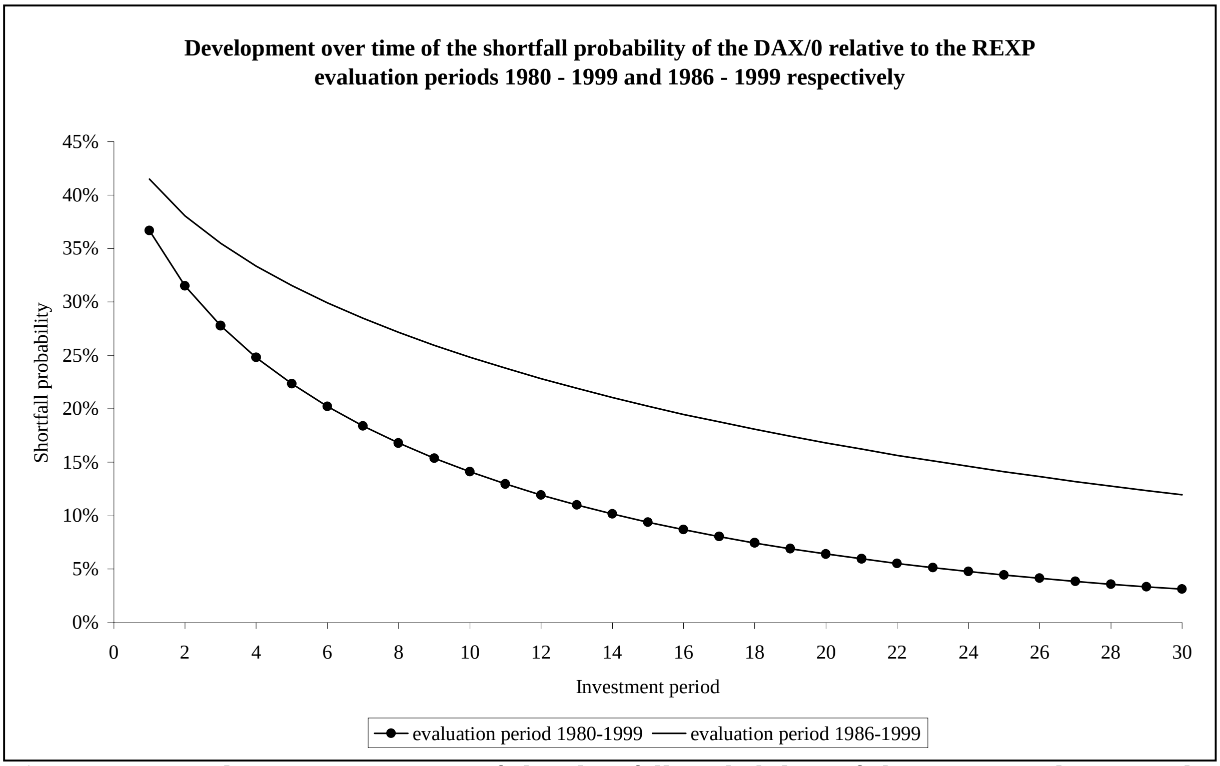

Figure 9: Development over time of the shortfall probability of the DAX/0 relative to the REXP on the basis of representative return distributions 1980 - 1999 and 1986 - 1999 respectively

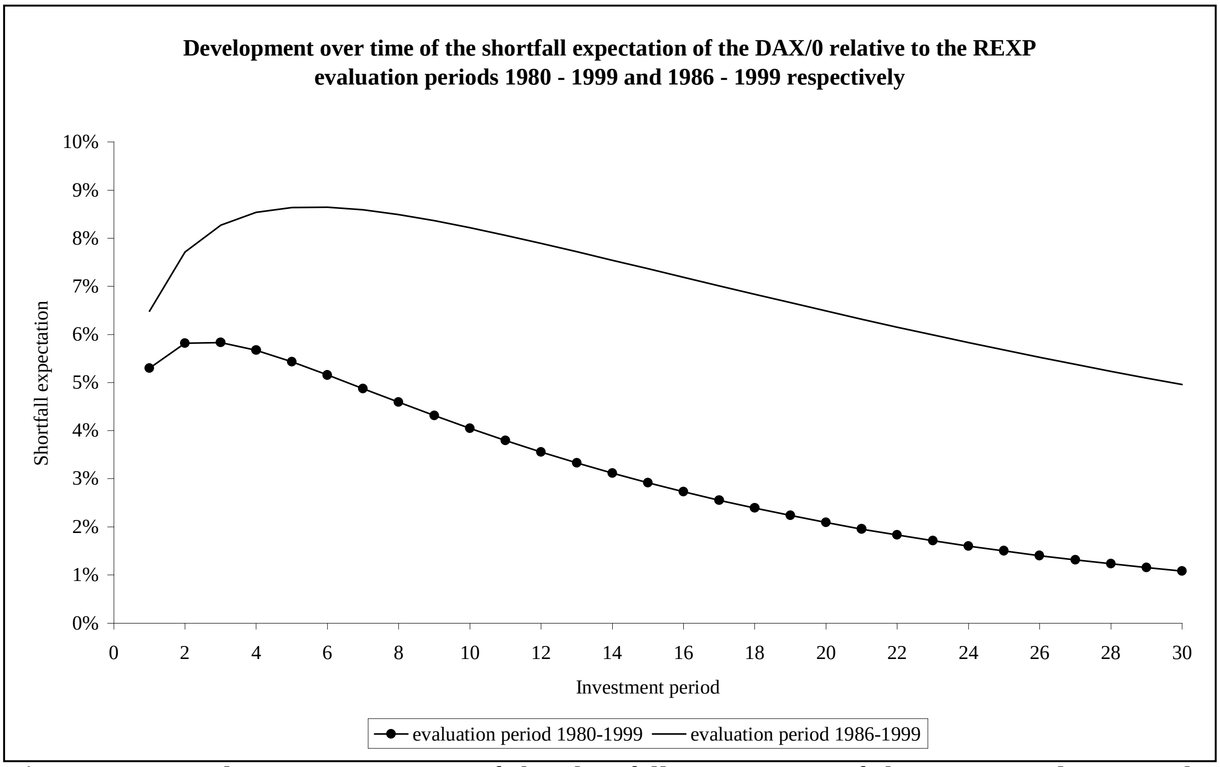

Figure 10: Development over time of the shortfall expectation of the DAX/0 relative to the REXP on the basis of representative return distributions 1980 - 1999 and 1986 - 1999 respectively

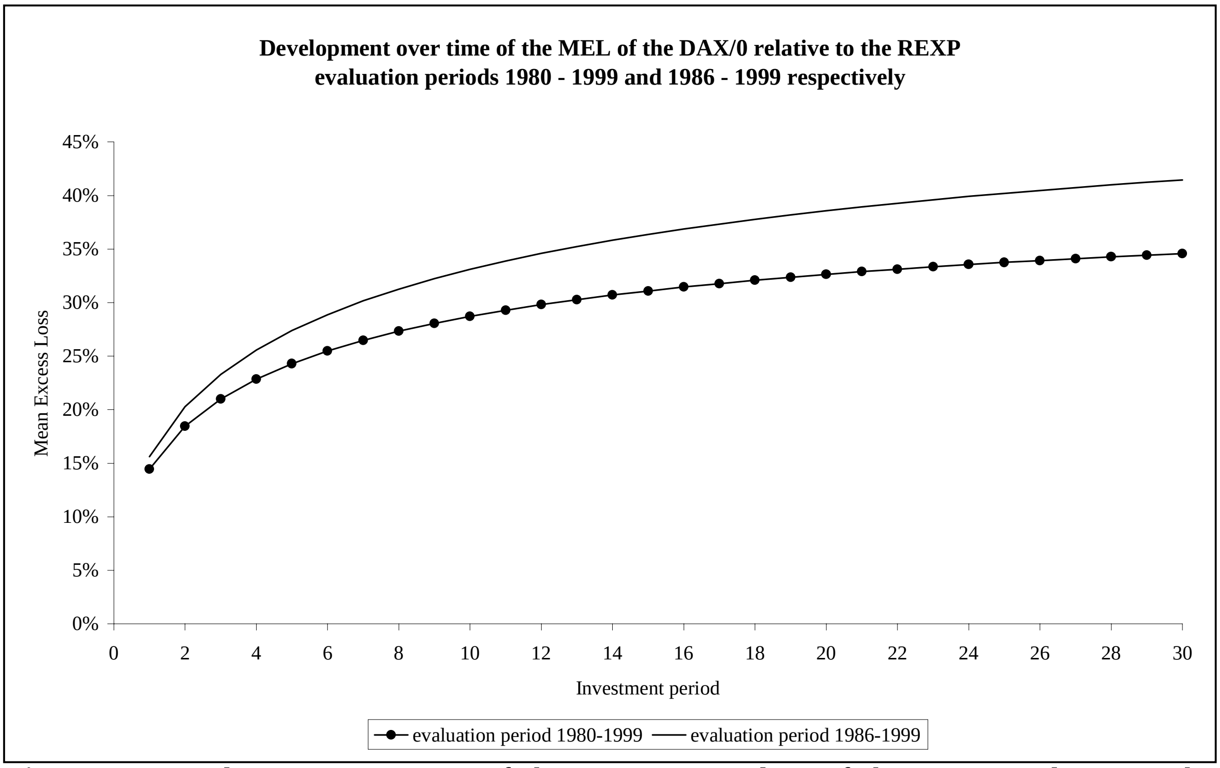

Figure 11: Development over time of the mean excess loss of the DAX/0 relative to the REXP on the basis of representative return distributions 1980 - 1999 and 1986 - 1999 respectively

Investment period	1 year	5 years	10 years	15 years	20 years	25 years	30 years
Representative distribution 1980 - 1999							
SP	36,69	22,34	14,10	9,38	6,41	4,45	3,12
SE	5,30	5,43	4,05	2,92	2,09	1,50	1,08
MEL	14,44	24,30	28,71	31,09	32,64	33,75	34,58
Representative distribution 1986 - 1999							
SP	41,49	31,53	24,83	20,25	16,81	14,12	11,95
SE	6,48	8,64	8,22	7,36	6,49	5,68	4,95
MEL	15,62	27,39	33,11	36,37	38,58	40,21	41,47

Tab. 3: Shortfall risk (in %) of a one-time investment in the DAX/0 in real terms relative to the REXP for chosen time horizons on the basis of representative return-distributions 1980 - 1999 and 1986 - 1999

From a structural point of view, the results in this section completely back up the results for deterministic benchmarks of section 3. Here, the dependence of the results on the evaluation period chosen is even more obvious. The shortfall probability of the DAX/0 relative to the REXP in principle is monotonous depending on the time horizon. Additionally, it shows a persistence-characteristic which, for the evaluation period 1986 - 1999, is at a substantially high level. The shortfall expectation of the DAX/0 relative to the REXP - as like in section 3 - increases at the beginning[32] but decreases with the length of the investment period. The shortfall expectation, too, has a persistence-characteristic which is substantially high for the evaluation period 1986 - 1999. Finally, the development of the mean excess loss over time of the DAX/0 relative to the REXP is monotonously increasing. Depending on the representative distribution the evaluation is based on, the average level of the shortfall, given that the development of the DAX/0 underperforms the development of the REXP, is 37% - 47%, which again is a substantially high level.

5. Résumé

In this study we have examined the long-term risks of a representative one-time investment in German stocks (DAX/0) in real terms relative to various risk-free investments (returns of 0%, 2% and 4% in real terms) as well as relative to a representative investment in German bonds (REXP). As underlying risk measures the shortfall probability, the mean excess loss (conditional shortfall expectation) as well as the product of these two measures, the shortfall expectation have been used. From a structural point of view, both the shortfall probability and the shortfall expectation show a monotonously decreasing development over time. The shortfall expectation shows a phase of increasing value only at the beginning. However, both risk measures have a persistence-characteristic, which means that the corresponding risk measure does not converge rapidly but rather slowly against zero and that even for very long time horizons (30 years) the risk remains at a substantially high level.

In our opinion, the analysis of the mean excess loss reveals the true danger of a long term investment in stocks. From a worst-case perspective the risk of a stock investment increases with the investment period and reaches substantial levels. This effect is concealed if only the shortfall expectation is analysed since it is over-compensated by the higher convergence rate (in relative terms) of the shortfall probability.

The structural pattern of the mean excess loss delivers the investor important information about the long term risk of a stock investment. In fact, the probability of a shortfall decreases

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32 After all, the time until the shortfall expectation sinks below its initial level is in the case of the evaluation period 1986 - 1999 about 20 years.

with an increasing time horizon. But if a shortfall occurs it may underperform the benchmark at a substantial level which does not disappear with an increasing investment horizon but rather grows.

This worst-case characteristic of a stock investment is not subject to a diversification effect over time. According to the thesis that a stock investment in the long run (e.g. for pension plans) has - at least partly - to be analysed from a worst-case perspective, the risk of a long term investment in stocks must be seen in a different light.

Methodical Appendix

Let in the following $\{D(t);\, t \geq 0\}$ denote the development of the DAX and $\{R(t);\, t \geq 0\}$ correspondingly the development of the REXP over time t. Let's assume the process $\{(D(t), R(t));\, t \geq 0\}$ is following a two-dimensional geometric Brownian motion.

Considering the continuous one-period returns

$$I_D(t) := \ln\{D(t) / D(t\text{-}1)\}$$

respectively

$$I_R(t) := \ln\{R(t) / R(t\text{-}1)\}$$

for a series of time points $t = 1, ..., T$, then $(I_D(t), I_R(t))$ are stochastically independent and normally distributed two-dimensional random variables. Thus the empirical observations $(i_D(t), i_R(t))$, $t = 1, ..., T$, can be seen as samples of a bivariate normal-distribution with expectation-vector $u = \begin{pmatrix} u_D \\ u_R \end{pmatrix}$ and variance-covariance-matrix $\Sigma = \begin{pmatrix} \sigma_D^2 & \rho\sigma_D\sigma_R \\ \rho\sigma_D\sigma_R & \sigma_R^2 \end{pmatrix}$. Correspondingly, the parameters in question can be estimated by their sample counterparts sample mean, (adjusted) variance of the sample or standard deviation of the sample respectively and correlation coefficient of the sample. In the case of the evaluation period 1980 - 1999 this leads to the figures $u_D = 0{,}1288$, $s_D = 0{,}2413$, $u_R = 0{,}0475$, $s_R = 0{,}054$ and $\rho = 0{,}1545$. In the case of the evaluation period 1986 - 1999 this leads to the figures $u_D = 0{,}0999$, $s_D = 0{,}2440$, $u_R = 0{,}0467$, $s_R = 0{,}0562$ and $\rho = 0{,}057$.

Letting $R(t)$ be a stochastic benchmark, relative to which the shortfall of $D(t)$ is measured, we obtain: $D(t) < R(t) \Leftrightarrow \dfrac{D(t)}{R(t)} < 1$, i.e. we equivalently examine the shortfall of $\dfrac{D(t)}{R(t)}$ relative to 1. Consequently we have the case of a deterministic benchmark.

Since $\ln\left(\dfrac{D(t)}{R(t)}\right) = \sum_{\tau=1}^{t} I_D(\tau) - \sum_{\tau=1}^{t} I_R(\tau)$ and thus is a sum of normally distributed random variables, the quotient $D(t)/R(t)$ is at any moment logarithmically normally distributed with the parameters m_t and v_t, i.e.

$$\ln\left(\frac{\mathrm{D}(t)}{\mathrm{R}(t)}\right) \sim \mathrm{N}(m_t, v_t).$$

Especially

$$m_t = t(u_D - u_R) + \ln(\mathrm{D}_0/\mathrm{R}_0)$$
$$v_t^2 = t[\sigma_\mathrm{D}^2 + \sigma_\mathrm{R}^2 - 2\rho\sigma_\mathrm{D}\sigma_\mathrm{R}].$$

holds true.

Assuming now that in $t = 0$ the initial investments in the DAX and the REXP have the same amount, i.e. $\mathrm{D}_0 = \mathrm{R}_0$, we furthermore obtain:

$$m_t = t(u_\mathrm{D} - u_\mathrm{R}).$$

The shortfall risk measures of the DAX relative to the REXP result in:

$$\mathrm{SE}_{\mathrm{D/R}}(1) = \mathrm{E}\left[\max\left(1 - \frac{\mathrm{D}(t)}{\mathrm{R}(t)}, 0\right)\right],$$
$$\mathrm{SP}_{\mathrm{D/R}}(1) = \mathrm{P}\left(1 - \frac{\mathrm{D}(t)}{\mathrm{R}(t)} < 1\right)$$

and

$$\mathrm{MEL}_{\mathrm{D/R}}(1) = \mathrm{E}\left[1 - \frac{\mathrm{D}(t)}{\mathrm{R}(t)} \middle| \frac{\mathrm{D}(t)}{\mathrm{R}(t)} < 1\right]$$

$$= \frac{\mathrm{SE}_{\mathrm{D/R}}(1)}{\mathrm{SP}_{\mathrm{D/R}}(1)}.$$

If, e.g. the $\mathrm{MEL}_{\mathrm{D/R}}(1) = 0{,}25$, this result is to be interpreted as follows: If $\mathrm{D}(t) < \mathrm{R}(t)$, then $\mathrm{D}(t)$ in average is 25% below $\mathrm{R}(t)$.

The analytically closed evaluation of the $\mathrm{SP}_{\mathrm{D/R}}$, $\mathrm{SW}_{\mathrm{D/R}}$ and with it the $\mathrm{MEL}_{\mathrm{D/R}}$ is based on $\mathrm{D}(t)/\mathrm{R}(t) \sim \mathrm{LN}(m_t, v_t)$, e.g. according to the results of *Maurer* (2000, S. 73) where $z = 1$ and q_t

$= (\ln z - m_t) / v_t = -m_t / v_t$ and $SP_z(t) = \phi(q_t)$ as well as $SE_z(t) = z\phi(q_t) - \exp(m_t + ½v_t^2)\, \phi(q_t - v_t)$, and $\phi(x)$ denotes the distribution function of the standard normal distribution.

With $R(t) = D(0)e^{rt}$ we obtain the case of deterministic targets, especially $\sigma_R = \rho = 0$ and $u_R = r$ hold true. First of all we obtain:

$$E[1 - \frac{D(t)}{D(0)e^{rt}} \; \Big| D(t) < D(0)e^{rt}]$$

$$= \frac{E[D(0)e^{rt} - D(t) \; \big| D(t) < D(0)e^{rt}]}{D(0)e^{rt}}$$

$$= \frac{MEL_{D(t)}\,(D(0)e^{rt})}{D(0)e^{rt}}.$$

This is equivalent to the MEL of D(t) concerning $z = D(0)e^{rt}$ relative to the development of the benchmark $D(0)e^{rt}$.

By analogy we obtain:

$$\begin{aligned} SE_{D/R}(1) &= E[\max(1 - \frac{D(t)}{D(0)e^{rt}}, 0)] \\ &= \frac{E[\max(D(0)e^{rt} - D(t), 0)]}{D(0)e^{rt}}. \end{aligned}$$

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