Tijdschrift voor Economie en Management
Vol. LII, 2, 2007

Comonotonicity

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ABSTRACT

In an actuarial or financial context one often encounters the calculation of risk measures of random variables of the type $S = \sum_{i=1}^{n} X_i$. In many applications, the individual risks X_i are not mutually independent, for example because their outcomes are all influenced by the same economic or physical environment. Comonotonicity, which is an extremal form of positive dependence, can be used to determine easy to compute and accurate upper and lower bounds for the distribution of S, and hence, also for risk measures related to S.

* Jan Dhaene, Marc Goovaerts and Steven Vanduffel acknowledge the financial support of the Onderzoeksfonds K.U. Leuven (GOA/02: Actuariële, financiële en statistische aspecten van afhankelijkheden in verzekeringsen financiële portefeuilles) and the financial support of Fortis (Fortis Chair in Financial and Actuarial Risk Management). A shorter version of this paper will appear in the Wiley Encyclopedia of Quantitative Risk Assessment.

I. AGGREGATING NON-INDEPENDENT RISKS

In an actuarial or financial context one often encounters a random variable (r.v.) S of the type

$$S = \sum_{i=1}^{n} X_i. \tag{1}$$

For example, for an insurer the different X_i may represent the claims from individual policies over a specified time horizon and S represents the aggregate risk related to the entire insurance portfolio. In another context, the X_i denote the risks of a particular business line and S is then the aggregate risk across all business lines. In a pension fund context, random variables of this type appear when determining provisions and related optimal investment strategies. Another field of application concerns personal finance problems where a decision maker faces a series of future consumptions and looks for optimal saving and investment strategies. In option pricing random variables of this type appear to describe the pay-offs of Asian and basket options. Finally, they also appear in a capital allocation or capital aggregation context. Roughly speaking, these applications amount to the evaluation of risk measures related to the cumulative distribution function (cdf) $F_S(x) = \Pr[S \leq x]$ of the random variable S. We refer the interested reader to (Dhaene e.a. (2002); (2005); (2006) and Simon e.a. (2000)) for more details on these applications.

It is well-known that Monte Carlo simulations may be helpful in the evaluation of S but since these are often computationally intensive, there is space for analytical (approximate) solutions as well. For example, financial institutions evaluate the 'fair value' of their balance sheet which involves the use of so-called 'risk neutral probabilities' and then project how this value can evolve stochastically over a given time frame (often one year) requiring 'physical probabilities' (i.e. the probabilities in the real world). In this case Monte Carlo simulations require the combination of risk neutral and physical scenarios which will dramatically increase the number of scenarios that are needed to obtain accurate answers. Even modern computers will often not be able to handle this efficiently. In contrast, comonotonicity can be used to evaluate the part that involves 'risk neutral scenarios' in conjunction with Monte Carlo simulations for the 'physical scenarios'.

In order to avoid technical complications we will assume that the expectations of the X_i exist. We denote the random vector

$(X_1, X_2, \ldots, X_n)$ by $\underline{X}$. Let $\underline{U} = (U_1, U_2, \ldots, U_n)$ be a random vector of uniformly (0,1) distributed random variables U_i such that:

$$\underline{X} \stackrel{d}{=} \left(F_{X_1}^{-1}(U_1), F_{X_2}^{-1}(U_2), \ldots, F_{X_n}^{-1}(U_n)\right). \tag{2}$$

Here, $F_{X_i}^{-1}$ denotes the quantile function of the r.v. X_i and '$\stackrel{d}{=}$' stands for 'equality in distribution'. Hence,

$$F_{\underline{X}}(\underline{x}) = F_{\underline{U}}\left(F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_n}(x_n)\right), \tag{3}$$

which means that the cdf $F_{\underline{X}}$ of $\underline{X} = (X_1, X_2, \ldots, X_n)$ is completely specified by the marginal cdf's F_{X_i} of the X_i and by the cdf $F_{\underline{U}}$ of $\underline{U}$. The function $F_{\underline{U}}$ is called a 'copula function'. For more details on this decomposition of a multivariate distribution into its marginal distributions and a copula function, see for example (Nelsen (1999)).

From (1) and (2), we find that the distribution of S can be characterized as follows:

$$S \stackrel{d}{=} \sum_{i=1}^{n} F_{X_i}^{-1}(U_i). \tag{4}$$

It is convenient to assume that the random variables U_i are mutually independent, as in this case the distribution of S can be computed using the technique of convolution. Powerful and accurate exact or approximate recursive computation methods such as De Pril's recursion and Panjer's recursion can also be applied in this case. We refer to (Panjer (1981)), (De Pril (1989)) and (Dhaene e.a. (2006)). When S represents the aggregate claims of an insurance portfolio the assumption of independence is sometimes realistic. Moreover, the existence of an insurance industry, where risks are pooled between a large number of insureds, is mainly based on the fact that the risks X_i associated with the individual policies can be assumed to be mutually independent.

However, in many other actuarial and financial applications the individual risks X_i in the sums S cannot be assumed to be mutually independent, for instance because all X_i are influenced by the same economic or physical environment. The independence assumption is then violated and as a consequence it is not straightforward to determine the cdf of S. In the case of non-independent risks the problem of determining the cdf of S is often further complicated by the fact that the copula connecting the marginals F_{X_i} is unknown or too cumbersome to work with.

A sum S of non-independent risks may occur for instance when considering the aggregate claims amount of a non-life insurance risk portfolio or a credit portfolio where the insured risks are subject to some common factors such as geography or economic environment. Another example concerns the aggregate payments of a pension fund when the insured parties are working in the same company. These people work at the same location and may use the same transport facilities which will result in some positive dependency between their mortality rates.

II. COMONOTONICITY

Let us consider the situation where the individual risks X_i of the random vector $\underline{X}$ are subject to the same claim generating mechanism in the sense that

$$\underline{X} \stackrel{d}{=} (g_1(Z), g_2(Z), \ldots, g_n(Z)), \tag{5}$$

for some common random variable Z and non-decreasing functions g_i. In this case, the random vector $\underline{X}$ is said to be 'comonotonic' and the distribution of $\underline{X}$ is called the 'comonotonic distribution'. Notice that all $g_i(Z)$ are monotonic increasing functions of the random variable Z, which explains the word comonotonic (common monotonic).

Intuitively, it is clear that comonotonicity corresponds to an extreme form of positive dependency between the individual risks involved. Indeed, increasing the outcome z of the common source of risk Z is tied to a simultaneous increase in the different outcomes $g_i(z)$.

One can prove that the comonotonicity of $\underline{X}$ can also be characterized by

$$\underline{X} \stackrel{d}{=} \left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\right), \tag{6}$$

which means that the representation (2) for the distribution function of $\underline{X}$ holds true with $U_1 \equiv U_2 \equiv \ldots U_n \equiv U$. Hence, the n-dimensional stochastic nature of a general random vector $\underline{X}$ reduces to a single dimension in the case of comonotonicity. This aspect of comonotonicity implies that simulating outcomes of a comonotonic random vector reduces to simulating outcomes of a univariate uniform (0, 1) r.v. U.

It is straightforward to prove that comonotonicity of $\underline{X}$ is equivalent to

$$F_{\underline{X}}(\underline{x}) = \min\left[F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_n}(x_n)\right]. \tag{7}$$

It is known since Hoeffding (Hoeffding (1940)) and Fréchet (Fréchet (1951)) that the function $\left[F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_n}(x_n)\right]$ is the multivariate cdf of a random vector which has the same marginal distributions as the random vector $\underline{X}$.

Let us denote the sum of the components of the comonotonic random vector $\left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\right)$ by S^c:

$$S^c = \sum_{i=1}^{n} F_{X_i}^{-1}(U). \tag{8}$$

Comonotonicity of $\underline{X}$ implies that $S = \sum_{i=1}^{n} X_i \stackrel{d}{=} S^c$.

Several important actuarial quantities of S^c such as quantiles and stop-loss premiums exhibit an additivity property in the sense that they can be expressed as a sum of corresponding quantities of the marginals involved. For the quantiles, we have that

$$F_{S^c}^{-1}(p) = \sum_{i=1}^{n} F_{X_i}^{-1}(p), \qquad 0 < p < 1. \tag{9}$$

Let us now assume that the marginal cdf's F_{X_i} are strictly increasing. In this case, one can prove that

$$\left[S^c - d\right]_+ = \sum_{i=1}^{n} \left[F_{X_i}^{-1}(U) - d_i^*\right]_+ \tag{10}$$

for any d such that $0 < F_{S^c}(d) < 1$, and with the d_i^* given by

$$d_i^* = F_{X_i}^{-1}\left(F_{S^c}(d)\right). \tag{11}$$

Notice that $\sum_{i=1}^{n} d_i^* = d$. Taking expectations of both sides of (10) leads to the following additive relation for the stop-loss premiums of S^c:

$$E\left[S^c - d\right]_+ = \sum_{i=1}^{n} E\left[X_i - d_i^*\right]_+. \tag{12}$$

The expressions (10), (11) and (12) can be generalized to the case of general distribution functions, see (Dhaene e.a. (2000)) and (Kaas e.a. (2000)) for more details. Expressions similar to (10) and (12) can also be found in (Jamshidian (1989)) where it is proven that in the Vasicek (Vasicek (1977)) model, a European call option on a portfolio of zero coupon bonds (in particular, an option on a single coupon paying bond) decomposes into a portfolio of European call options on the individual zero coupon bonds in the portfolio. This holds true because in the Vasicek model, the prices at a future date of all zero coupon bonds involved are decreasing functions of the random spot rate at that date.

III. A COMONOTONIC UPPER BOUND APPROXIMATION

As opposed to the case of independent or comonotonic rv's X_i, it is in general not straightforward to determine the cdf of S. In the general case it may be helpful to find a dependency structure for the random vector $(X_1, X_2, \ldots, X_n)$ that leads to a 'less favorable' or 'more dangerous' sum for the marginal terms X_i and such that the cdf of this sum is easier to determine. Making decisions based on the 'less favorable' distribution will lead to prudent or conservative decisions.

In order to define what we mean by 'less favorable' we have to decide how to order risks. In this respect it is convenient to consider convex ordering: A r.v. X is smaller than a r.v. Y in convex order if $E[X] = E[Y]$ and $E[(X-d)_+] \leq E[(X-d)_+]$ for all real d. In this case, we write

$$X \leq_{cx} Y. \tag{13}$$

In von Neumann & Morgenstern's (von Neumann e.a. (1947)) 'Expected Utility Theory', as well as in Yaari's (Yaari (1987)) 'Dual Theory of Choice under Risk', convex order represents the common preferences of risk averse decision makers between risks with equal expectations. See for example (Wang e.a. (1998)).

When X and Y represent losses or future payments, $X \leq_{cx} Y$ means that every risk averse decision maker prefers paying X above paying Y. Hence, replacing (the distribution of) the real loss X by (the distribution of) the loss Y and making decisions based on (the distribution of) Y can be considered as a prudent strategy. On the other hand, when X and Y represent gains or incomes, $X \leq_{cx} Y$ means that every risk averse decision maker prefers gaining X to gaining Y. For more details on ordering (distributions of) r.v.'s, we refer to (Shaked e.a. (1997)). Actuarial applications of stochastic ordering concepts are described in detail in (Kaas e.a. (2001)) and (Denuit e.a. (2005)).

One can prove that for any random vector $(X_1, X_2, \ldots, X_n)$, the following ordering relation holds:

$$\sum_{i=1}^{n} X_i \leq_{cx} \sum_{i=1}^{n} F_{X_i}^{-1}(U). \tag{14}$$

This means that replacing (the distribution function of) S by (the distribution function of) S^c and making decisions based on the latter distribution function can be considered as a prudent strategy in the framework of expected utility theory as well as Yaari's dual theory of choice under risk. Moreover, quantiles and stop-loss premiums of S^c can easily be determined from (9) and (12). The comonotonic upper bound approximation F_{S^c} will be 'close' to the exact cdf F_{S^c} when the different U_i in (4) possess a strong positive dependency structure. An insightful geometric proof of (14) can be found in (Kaas e.a. (2002)). Earlier references to closely related results are (Meilijson e.a. (1979)), (Rüschendorf (1983)) and (Müller (1997)).

As $S \leq_{cx} S^c$ implies that $E[S]=E[S^c]$, it follows that the cdf.'s of S and S^c must cross at least once. Hence, apart from the case that $S \overset{d}{=} S^c$, we find that it is impossible that $F_{S^c}^{-1}(p)$ is an upper bound for $F_{S'}^{-1}(p)$ for all $0 < p < 1$. This implies that the quantile risk measure is not subadditive.

Several actuarial and financial problems that we mentioned in the previous section involve the evaluation of the net present value or the accumulated value of future cash flows, which can be expressed as a sum S as in (1) where the r.v.'s X_i are given by

$$X_i = a_i e^{Y_i}. \tag{15}$$

Here, the a_i are deterministic real numbers and $(Y_1, Y_2, \ldots, Y_n)$ is a random vector.

The accumulated value at time n of a series of future deterministic saving amounts a_i can be written in this form, where Y_i denotes the cumulative logreturn over the period $[i, n]$. Similarly, the present value of a series of future deterministic payments a_i can be written in this form where now e^{Y_i} denotes the random discount factor over the period $[0, i]$.

In both cases (compounding and discounting), the random vector $(X_1, X_2, \ldots, X_n)$ will not be comonotonic, although neighboring components X_i and X_j will be rather strongly dependent random variables. This is because there is a natural overlapping process when compounding (or discounting) over the different time periods. In case of discounting, the random variable S can be considered as the stochastic present value of an n-year term annuity. A continuous version (with payments continuously spread over time) is considered in (Defresne (1990)).

Let us now assume that the X_i are given by $X_i = a_i\ e^{Y_i}$ with $a_i > 0$. We also assume that any random variable Y_i is normally distributed. We find that

$$S^c = \sum_{i=1}^{n} a_i e^{\mathrm{E}[Y_i]+\sigma_{Y_i}\Phi^{-1}(U)}, \tag{16}$$

where Φ is the standard normal cdf. In this case the quantiles and the stop-loss premiums of S^c are given by

$$F_{S^c}^{-1}(p) = \sum_{i=1}^{n} a_i e^{\mathrm{E}[Y_i]+\sigma_{Y_i}\Phi^{-1}(p)}, \qquad 0 < p < 1, \tag{17}$$

and

$$E\left[S^c - d\right]_+ = \sum_{i=1}^{n} a_i e^{\mathrm{E}[Y_i]+\frac{1}{2}\sigma_{Y_i}^2}\Phi\left(\sigma_{Y_i} - \Phi^{-1}\left(F_{S^c}(d)\right)\right) - d\left(1 - F_{S^c}(d)\right), \qquad 0 < d < \infty, \tag{18}$$

respectively. The quality of this upper bound approximation is investigated in (Dhaene e.a. (2002)), (Huang e.a. (2004)) and (Vanduffel e.a. (2005)).

For a general random vector $(X_1, X_2, \ldots, X_n)$ and real d and $d_i\,(i = 1,2,\ldots n)$ such that $\sum_{i=1}^{n} d_i = d$ we have that

$$\left[\sum_{i=1}^{n} X_i - d\right]_+ \leq \sum_{i=1}^{n} \left[X_i - d_i\right]_+ \tag{19}$$

It can be proven that the minimum of the expectation of the right hand side in (19), taken over all d_i such that $\sum_{i=1}^{n} d_i = d$, is given by $E[S^c - d]_+$. Hence, in the case of strictly increasing cdf's F_{X_i}, we find from (12) that this minimum is obtained for the d_i^* as defined in (11). This result can be generalized to the case of general cdf's F_{X_i}.

When the X_i represent asset prices at some future date, say t, then the r.v. $\left[\sum_{i=1}^{n} X_i - d\right]_+$ can be interpreted as the pay-off of a European type basket call option at expiration date t, whereas each of the terms $[X_i - d_i]_+$ can be interpreted as the pay-off of a European call option on the i-th asset involved at the same expiration date. The inequality (19) provides an infinite number of ways to super-replicate the pay-off of the basket option in terms of the individual asset options involved. The super-hedging strategy consisting of buying the n European calls with respective exercise prices d_i^* corresponds to a cheapest super-replicating hedging strategy for the basket option under consideration. Similar results hold for Asian options. For more details, we refer to (Dhaene e.a. (2002)), (Simon e.a. (2000)), (Albrecher e.a. (2005)), (Hobson e.a. (2005)), (Vanmaele e.a. (2006)), (Teynaerts e.a. (2006)) and (Chen e.a. (2007)).

IV. COMONOTONIC LOWER BOUND APPROXIMATIONS

In the previous section, we introduced an approximation for the cdf F_S by keeping the marginal cdf's F_{X_i} unchanged while replacing the 'real' dependency structure by the comonotonic one. The crucial feature of comonotonicity is that only a one-dimensional randomness is involved. As a consequence, comonotonic sums have convenient additivity properties for quantiles and stop-loss premiums. In this section, we will look for less crude and hence better approximations for F_S without losing the convenient properties of the comonotonic upper bound approximation. The technique of taking conditional expectations will help us to achieve this goal.

For an appropriate random variable Λ, we consider the conditional expectations $E[S \mid \Lambda = \lambda]$ for all outcomes λ of Λ. Now, we propose to approximate the cdf of S by the cdf of S^l, which is defined by

$$S^l = E\left[S \mid \Lambda\right] = \sum_{i=1}^{n}\left[X_i \mid \Lambda\right] \qquad (20)$$

This approximation allows us to move from the multivariate randomness of the vector $(X_1, X_2, \ldots, X_n)$ to the univariate randomness of the conditioning random variable Λ. Notice that a continuous version of this technique applied to Asian option pricing is considered in (Rogers e.a. (1995)).

Let us now assume that all $E[X_i \mid \Lambda]$ are increasing in Λ. In this case, we find that S^l is a comonotonic sum. As a consequence, we have that

$$S^l \overset{d}{=} \sum_{i=1}^{n} F^{-1}_{E[X_i \mid \Lambda]}(U), \tag{21}$$

where the random variable U is uniformly distributed on the unit interval. Furthermore, the quantiles and the stop-loss premiums related with S^l can be expressed as a sum of corresponding quantities for the individual terms $E[X_i \mid \Lambda]$.

Concerning an appropriate choice for Λ, notice that when Λ is chosen equal to S, we find that $S^l = S$. Therefore, intuitively it is clear that the 'closer' Λ is to S, the better the approximation S^l will perform. However, for the Λ to be useful it must enable an explicit expression for the different $E[X_i \mid \Lambda]$.

The most prominent case which leads to closed form expressions for quantiles and stop-loss premiums of S^l is the one where $X_i = a_i\ e^{Y_i}$, with all $a_i > 0$ and $(Y_1, Y_2, \ldots, Y_n)$ a multivariate normally distributed random vector. In this section, we will further concentrate on this particular case.

We choose Λ to be a linear combination of the $Y_1, Y_2, \ldots, Y_n$:

$$\Lambda = \sum_{i=1}^{n} \gamma_i Y_i, \tag{22}$$

for appropriate choices of the coefficients γ_i. In the literature, several choices for these coefficients have been proposed. In (Kaas e.a. (2000)) it is proposed to determine Λ such that it can be interpreted as a first-order approximation for the original sum S. In (Vanduffel e.a. (2005)) the conditioning r.v. Λ is chosen such that a first-order approximation for the variance of S^l is is maximized. In (Vanduffel e.a. (2006)) it is argued that both choices for Λ in some sense provide an overall goodness of fit for the cdf of S, based on S^l, and one can further improve the choice for Λ when concentrating on a particular neighborhood of the distribution function such as the extreme lower or upper tails.

For the general Λ as considered in (22), we find that

$$S^l = \sum_{i=1}^{n} \alpha_i\, e^{E[Y_i] + \frac{1}{2}(1 - r_i^2)\sigma^2_{Y_i} + r_i \sigma_{Y_i} \frac{\Lambda - E[\Lambda]}{\sigma_\Lambda}}, \tag{23}$$

where the r_i are the correlations between the Y_i and Λ:

$$r_i = \frac{\sum_{j=1}^{i}\sum_{k=j}^{n}\gamma_k}{\sqrt{i\sum_{j=1}^{n}\left(\sum_{k=j}^{n}\gamma_k\right)^2}} \tag{24}$$

From (23), we see that S^l is a comonotonic sum when all correlation coefficients r_i are non-negative. Notice that the particular choices for the γ_i as proposed in (Kaas e.a. (2000)) and in (Vanduffel e.a. (2005)) lead to non-negative r_i. In the comonotonic case the quantiles of S^l are given by

$$F_{S^l}^{-1}(p) = \sum_{i=1}^{n} a_i e^{\mathrm{E}[Y_i]+\frac{1}{2}(1-r_i^2)\sigma_{Y_i}^2+r_i\sigma_{Y_i}\Phi^{-1}(p)}, \qquad 0<p<1, \tag{25}$$

whereas the stop-loss premiums are given by

$$\begin{aligned}\mathrm{E}\left[S^l-d\right]_+ = \\ \sum_{i=1}^{n} a_i e^{\mathrm{E}[Y_i]+\frac{1}{2}\sigma_{Y_i}^2}\Phi\left(r_i\sigma_{Y_i}-\Phi^{-1}\left(F_{S^l}(d)\right)\right) \\ -d\left(1-F_{S^l}(d)\right), \qquad 0<d<\infty.\end{aligned} \tag{26}$$

As mentioned above the expressions (23)-(26) hold when all cash flows a_i and correlations r_i are positive. These results can be generalized. In (Vanduffel e.a. (2005)) a particular pattern of cash flows with mixed signs of the a_i is considered, whereas in (Deelstra e.a. (2006)) the case that some of the r_i are negative is dealt with.

Using Jensen's inequality, one can prove that

$$S^l \leq_{cx} S, \tag{27}$$

which means that S^l is 'less dangerous' than S. At first sight, it seems counter-intuitive for a risk-averse decision maker to make his decisions based on the 'less dangerous' S^l. However, numerical comparisons reveal that, at least when $X_i = a_i\ e^{Y_i}$ and assuming the $(Y_1, Y_2, \ldots, Y_n)$ to be multivariate normally distributed, the risk measures of S^l can, statistically speaking, barely be distinguished from the risk measures of the random variable S, obtained by simulation, provided an appropriate choice is made for the conditioning r.v. Λ., see for example (Albrecher e.a. (2005)). This observation may outweigh the fact that the lower bound S^l is 'less dangerous' and the cdf of S^l may generically be considered to be an accurate approximation for the cdf of S.

V. DEPENDENCIES IN A NON-GAUSSIAN WORLD

In the previous two sections, we considered the problem of how to determine comonotonic lower and upper bounds for sums of r.v.'s. We illustrated the technique by deriving explicit expressions for sums of lognormal r.v.'s. The latter case can directly be applied for the discounting and compounding applications described above, provided the investment returns can be described by a lognormal process. It is well-known that daily returns are correlated and exhibit fat tails, which implies that they cannot be adequately modelled through normal random variables. However, several of the applications we encountered concern long time investments horizons (typically some decades) and hence, also the time unit will be expressed in months or years. As soon as the time unit is sufficiently long, assuming a Gaussian model for the $(Y_1, Y_2, \ldots, Y_n)$ seems to be appropriate in many cases, see for instance (Cesari, e.a. (2003)) and (Levy (2004)).

The theoretical developments concerning the comonotonic lower and upper bounds continue to hold for non-Gaussian random vectors. The comonotonic upper bound can readily be applied in the general case. For sums of logelliptical r.v.'s, we refer to (Valdez e.a. (2003)). The performance of the upper bound in case Lévy processes are involved is investigated in (Albrecher e.a. (2005)) and (Valdez e.a. (2003)).

The comonotonic lower bound results are more difficult to use for general distribution functions, mainly because closed form expressions for $E[X_i \mid \Lambda]$ are in general not available. In (Dhaene e.a. (2005)), the lower bound based on the conditioning technique is investigated for sums consisting of a combination of lognormal and normal r.v.'s. The case of sums of logelliptical r.v.'s is considered in (Valdez e.a. (2003)). They illustrate that in the general logelliptical case, no closed-form expressions for S^l are readily available.

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