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TWO NEW EXPONENTIAL FAMILIES OF LORENZ CURVES

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ABSTRACT

We present two new Lorenz curve families by using the basic model proposed by Sarabia, Castillo and Slottje (1999). We present estimations which show that the models in our new families are very efficient when applied to data on income distribution for a range of countries from Shorrocks (1983).

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1. Introduction

Research into finding satisfactory parametric Lorenz models has progressed in recent years. Several good performing Lorenz models have been developed. Sarabia, Castillo and Slottje (1999) (hereafter, SCS, 1999) propose a basic Lorenz model along with a generalized Pareto family of Lorenz curves. In a later contribution, Sarabia, Castillo and Slottje (2001) (hereafter SCS, 2001) develop an exponential family of Lorenz curves. Wang, Ng and Smyth (2007) (hereafter WNS, 2007) provide the condition for the SCS (1999) basic model to be a Lorenz curve and present a few families of Lorenz curves. The purpose of this letter is to provide two new Lorenz curve families by using the SCS (1999) basic model. We present estimations which show that the models in our new families are very efficient when applied to data on income distribution for a range of countries from Shorrocks (1983).

2. The ordered family of Lorenz curves

A function $L(p)$ defined on $[0,1]$ is a Lorenz curve if:

$L(0)=0$, $L(1)=1$, $L'(0^+)\geq 0$ and $L''(p)\geq 0$ for all $p\in[0,1]$.

The basic SCS (1999) Lorenz model is $\tilde{L}(p)=p^{\alpha}L(p)^{\eta}$ where $L(p)$ is a Lorenz curve. The main model in the SCS (1999) generalized Pareto family is:

$$S_3(p)=p^{\alpha}\left[1-(1-p)^{\beta}\right]^{\eta} \quad (1)$$

where $S_0(p)=1-(1-p)^{\beta}$ is called the generating curve of the family. The following result from WNS (2007), which is a generalization of the results of SCS (1999), can be used to create more efficient Lorenz models:

Theorem 1. Assume $L(p)$ is a Lorenz curve. $\tilde{L}(p)=p^{\alpha}L(p)^{\eta}$ is a Lorenz curve for any $\alpha\geq 0$ and $\eta\geq 1$. Furthermore, if $L'''(p)\geq 0$ for all $p\in[0,1]$, then $\tilde{L}(p)$ is a Lorenz curve if $\alpha\geq 0$, $\eta\geq 1/2$ and $\alpha+\eta\geq 1$.

The main model in the SCS (2001) exponential family of Lorenz curves is:

$$S(p) = p^{\alpha} L_{\lambda}(p)^{\eta} \tag{2}$$

where the generating model is

$$L_{\lambda}(p) = \frac{e^{\lambda p} - 1}{e^{\lambda} - 1}, \quad \lambda > 0$$

which is a Lorenz curve proposed by Chotikapanick (1993).

3. Two new families of Lorenz curves

We first describe some properties of $L_{\lambda}(p)$.

Lemma 1. $L_{\lambda}(p)$ possesses the following properties：

(a) For any $\lambda \neq 0$, $L_{\lambda}(p) \geq 0$ and $L'_{\lambda}(p) \geq 0$ on $[0,1]$.

(b) For any $p \in [0,1]$, $L''_{\lambda}(p) \geq 0$ if $\lambda > 0$ and $L''_{\lambda}(p) \leq 0$ if $\lambda < 0$. Therefore, $L_{\lambda}(p)$ is convex if $\lambda > 0$, but is concave if $\lambda < 0$.

(c) $L''_{\lambda}(p) = \lambda L'_{\lambda}(p)$ and, consequently, $L'''_{\lambda}(p) = \lambda L''_{\lambda}(p) = \lambda^2 L'_{\lambda}(p)$.

(d) $\lim_{\lambda \to 0} L_{\lambda}(p) = p$.

Proof. $L_{\lambda}(p) \geq 0$ (or $L'_{\lambda}(p) \geq 0$) on $[0,1]$ because its numerator and denominator will always have the same sign when $\lambda > 0$ and $\lambda < 0$. The definition of $L_{\lambda}(p)$ implies that (c) is true and (b) can be implied by (c) and (a). Meanwhile, (d) can be verified using the L'Hospital's rule. ***QED.***

To obtain another family, change the generating model of $S(p)$ from $L_{\lambda}(p)$ to

$$T_0(p) = 1 - L_{\lambda}(1-p)^{\beta} = 1 - \left(\frac{e^{\lambda(1-p)} - 1}{e^{\lambda} - 1} \right)^{\beta} \tag{3}$$

We consequently obtain our first family of Lorenz curves which contains $T_0(p)$ and

$$T_1(p) = p^{\alpha}\left[1 - L_{\lambda}(1-p)^{\beta}\right] \quad (4)$$

$$T_2(p) = \left[1 - L_{\lambda}(1-p)^{\beta}\right]^{\eta} \quad (5)$$

$$T_3(p) = p^{\alpha}\left[1 - L_{\lambda}(1-p)^{\beta}\right]^{\eta} \quad (6)$$

$T_3(p)$ is the main model in the family with the others being special cases. Note that $T_i(0) = 0$ and $T_i(1) = 1$ for $i = 0,1,2,3$ if $\alpha \ge 0$, $\beta > 0$, $\lambda > 0$ and $\eta > 0$.

While $T_3(p)$ differs from $S(p)$, it is closely related to $S_3(p)$. The generating curve $S_0(p) = 1-(1-p)^{\beta}$ of $S_3(p)$ can only be a convex curve on $[0,1]$ for any $\beta \in (0,1]$. Since $\lim_{\lambda \to 0} T_0(p) = 1-(1-p)^{\beta}$ from Lemma 1, $T_0(p)$ includes $S_0(p)$ as a special case. However, $T_0(p)$ can be concave as well as convex on $[0,1]$ from Lemma 1 and thus can be more flexible than $S_0(p)$. Thus, $T_3(p) = p^{\alpha}T_0(p)^{\eta}$ is a generalization of $S_3(p) = p^{\alpha}S_0(p)^{\eta}$ and is therefore more flexible. We expect that the performance of $T_3(p)$ will be at least as good as $S_3(p)$, which itself is a very good model as pointed out by SCS (1999). For the same reason, $T_3(p)$ must be more flexible than $S(p) = p^{\alpha}L_{\lambda}(p)^{\eta}$, since $L_{\lambda}(p)$ must represent a convex curve.

Lemma 2. Assume $\lambda < 0$. $T_0(p)$ is a Lorenz curve with $T_0'''(p) \ge 0$ if $\beta \in (0,1]$.

Proof. Note that $T_0(p)$ satisfies

$$T_0'(p) = \beta L_{\lambda}(1-p)^{\beta-1}L_{\lambda}'(1-p),$$

$$T_0''(p) = \beta L_{\lambda}(1-p)^{\beta-2}\left[(1-\beta)L_{\lambda}'(1-p)^2 - L_{\lambda}(1-p)L_{\lambda}''(1-p)\right] \quad (7)$$

Lemma 1 implies $L_{\lambda}(1-p) \ge 0$, $L_{\lambda}'(1-p) \ge 0$ and $L_{\lambda}''(1-p) \le 0$ if $\lambda < 0$.

Therefore, the two equations in (7) imply that $T_0(p)$ is a Lorenz curve if $\beta \in (0,1]$.

Moreover, note that

$$\frac{T_0'''(p)}{\beta L_\lambda'(1-p)L_\lambda(1-p)^{\beta-3}} = (1-\beta)(2-\beta)L_\lambda'(1-p)^2 + L_\lambda(1-p)^2\lambda^2 - 3\lambda(1-\beta)L_\lambda(1-p)L_\lambda'(1-p). \quad (8)$$

Therefore, $T_0'''(p) \geq 0$ if $\beta \in (0,1]$. ***QED.***

Lemma 3. Assume $\lambda \in (0, \ln \beta^{-1}]$. $T_0(p)$ is a Lorenz curve if $\beta \in (0,1)$. $T_0(p)$ is a Lorenz curve with $T_0'''(p) \geq 0$ if $\beta \in (0,1/2]$. $\ln \beta^{-1}$ is the natural log of β^{-1}.

Proof. Because $L_\lambda''(1-p) = \lambda L_\lambda'(1-p)$ from Lemma 1, the term between the braces on the right-hand side of (7) is

$$L_\lambda'(1-p)\left[(1-\beta)L_\lambda'(1-p) - \lambda L_\lambda(1-p)\right] = \frac{\lambda L_\lambda'(1-p)}{e^\lambda - 1}\left(1-\beta e^{\lambda(1-p)}\right).$$

But $1-\beta e^{\lambda(1-p)}$ is non-negative on $[0,1]$ if $\lambda \in (0, \ln \beta^{-1}]$ and $\beta \in (0,1)$. Therefore, we have $T_0''(p) \geq 0$. Thus the first statement of the lemma is true.

Furthermore, denoting the right-hand side of (8) as $f(p)$, we have

$$\frac{(e^\lambda - 1)}{\lambda^2 L_\lambda'(1-p)} f'(p) = (1-\beta)(2\beta-1)e^{\lambda(1-p)} + (1-3\beta)(e^{\lambda(1-p)} - 1)$$
$$= (1-\beta)(2\beta-1) - 2\beta^2(e^{\lambda(1-p)} - 1).$$

The right-hand side of this equation is non-positive if $\lambda > 0$ and $\beta \in (0,1/2]$. This implies that $f(p)$ is a decreasing function on $[0,1]$. Therefore $f(p) \geq 0$ and, consequently, $T_0'''(p) \geq 0$ on $[0,1]$ because $f(1) = (1-\beta)(2-\beta)L_\lambda'(0)^2 > 0$. Therefore the second statement of the lemma is also true. ***QED.***

Theorem 2. Assume $\alpha \geq 0$, $\eta \geq 1/2$ and $\alpha + \eta \geq 1$. $T_3(p)$ is a Lorenz curve if:

$$\beta \in (0,1], \quad \lambda < 0 \tag{9}$$

or

$$\beta \in (0,1/2], \quad \lambda \in (0, \ln \beta^{-1}] \tag{10}$$

Proof. Lemma 2, the second statement of Lemma 3 and Theorem 1 together imply that $T_3(p)$ is a Lorenz curve if (9) or (10) holds. ***QED***.

Note that $T_3(p)$ is also a Lorenz curve if $\alpha \geq 0$, $\lambda \in (-\infty, 0) \cup (0, \ln \beta^{-1}]$, $\beta \in (0,1]$ and $\eta \geq 1$. But the model with this parameterization appears inferior to the model with the parameter ranges given by Theorem 2. Using the concept of the hybrid model proposed by Ogwang and Rao (2000), we suggest another Lorenz family, which is a generalization of both $T_3(p)$ and $S(p)$:

$$V(p) = p^{\alpha}\left\{\delta\left(1 - L_{\lambda}(1-p)^{\beta}\right) + (1-\delta)L_{\lambda_1}(p)\right\}^{\eta}.$$

Drawing on the result of Lemma 2, Lemma 3 and Theorem 1, we have:

Corollary. Assume $\alpha \geq 0$, $\eta \geq 1/2$, $\alpha + \eta \geq 1$, $\delta \in [0,1]$ and $\lambda_1 > 0$. Then $V(p)$ is a Lorenz curve if β and λ satisfy either (9) or (10).

There is one major difference between the cases where $\lambda < 0$ and $\lambda > 0$ in Theorem 2. λ can be any negative number in (9). However, if λ is positive, then its admissible range must depend on β as described in (10). The parameter ranges for $T_0(p)$, $T_1(p)$ and $T_2(p)$ to be Lorenz curves, though, are much simpler since Lemma 2, the first part of Lemma 3 and Theorem 1 imply:

Theorem 3: Assume $\beta \in (0,1]$ and $\lambda \in (-\infty, 0) \cup (0, \ln \beta^{-1}]$. Then

(1) $T_0(p)$ is a Lorenz curve.

(2) $T_1(p)$ is a Lorenz curve for any $\alpha \geq 0$.

(3) $T_2(p)$ is a Lorenz curve for any $\eta \geq 1$.

One drawback of $V(p)$ is that it does not have a Gini index formula with closed form. While the Gini index formula for $T_3(p)$ can be obtained in a similar manner to that described in SCS (2001) if $\lambda < 0$, the formula becomes tedious to calculate and, furthermore, a convergence problem emerges if $\lambda > 0$. Thus we use numerical integral to obtain Gini indices in our estimation tests in the next section.

4. Some empirical results

We give results for models $T_3(p)$ and $V(p)$, imposing $\eta \geq 1/2$. Following SCS (1999, 2001), we use MSE, MAE and MAXABS as error measures. The Levenberg-Marquardt algorithm (Dennis & Schnabel, 1983) is used to solve the nonlinear least square problem of minimizing the sum of residual squares to obtain parameter estimates for the models. Similar parameter transformations to those described in WNS (2007) are adopted to enforce the parameter constraints.

Following SCS (1999, 2001) we use the Shorrocks (1983) income distribution data for a range of countries in the tests. The results are reported in Tables 1-3. Tables 2-3 contain the parameters of the two models, from which it can be seen that all the fitted curves satisfy the condition of the Lorenz curve. Table 1 shows that the fitted results for $T_3(p)$ and $V(p)$ are very good. The largest MSE values for both models are for the Norwegian data, being 8.75×10^{-6} for $T_3(p)$ and 5.49×10^{-6} for $V(p)$ respectively. Since the MAXABS values are all quite small for $V(p)$, we can conclude that each fitted curve is a good global approximation to the original data. We also tested the models by using US income distribution data from Basmann et al. (1993), where the sample size is larger, and also obtained satisfactory results. We do not report these results to conserve space, but they are available on request.

5. Conclusion

We have introduced two new families of Lorenz curves building on the models proposed in SCS (1999, 2001). The estimation tests show that the new families perform well in practice. The results presented here provide further evidence that the basic form suggested by SCS (1999) is very important and that the basic result of theorem 1 is useful in the creation of parametric Lorenz models.

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Table 1. Error measures for $T_3(p)$ and $V(p)$ using Shorrocks' (1983) income data

	Model $T_3(p)$				Model $V(p)$			
	MSE($\times 10^{-6}$)	MAE	MAXABS	Gini	MSE($\times 10^{-6}$)	MAE	MAXABS	Gini
Brazil	2.7074	0.0015	0.0030	0.6375	0.0463	0.0002	0.0005	0.6381
Columbia	0.5256	0.0006	0.0013	0.5579	0.0532	0.0002	0.0004	0.5589
Denmark	8.1865	0.0024	0.0058	0.3654	1.2141	0.0009	0.0018	0.3658
Finland	3.5935	0.0017	0.0032	0.4709	1.0652	0.0008	0.0021	0.4711
India	1.2945	0.0010	0.0023	0.4600	0.1676	0.0003	0.0009	0.4604
Indonesia	3.1508	0.0014	0.0032	0.4486	0.1849	0.0004	0.0009	0.4490
Japan	0.0924	0.0003	0.0005	0.3104	0.0639	0.0002	0.0004	0.3105
Kenya	5.2351	0.0020	0.0036	0.6236	0.0374	0.0002	0.0004	0.6244
Malaysia	0.9506	0.0008	0.0022	0.5113	0.3310	0.0004	0.0016	0.5124
Netherlands	1.3393	0.0008	0.0029	0.4479	0.6706	0.0006	0.0021	0.4486
N Zealand	3.8703	0.0013	0.0053	0.3691	2.2961	0.0011	0.0040	0.3695
Norway	8.7450	0.0023	0.0066	0.3611	5.4869	0.0017	0.0063	0.3605
Panama	1.5025	0.0009	0.0032	0.4467	0.6712	0.0005	0.0024	0.4474
Sri Lanka	0.6430	0.0007	0.0016	0.4087	0.1206	0.0003	0.0005	0.4090
Sweden	1.8838	0.0011	0.0030	0.3862	0.3623	0.0004	0.0016	0.3865
Tanzania	2.8785	0.0015	0.0025	0.5388	0.0426	0.0002	0.0005	0.5391
Tunisia	4.5335	0.0017	0.0042	0.5029	0.0185	0.0001	0.0003	0.5022
UK	0.7523	0.0006	0.0022	0.3630	0.4288	0.0004	0.0018	0.3633
Uruguay	5.3492	0.0021	0.0038	0.4963	0.0277	0.0001	0.0003	0.4968

Note: $\eta \geq 1/2$ is imposed for both $T_3(p)$ and $V(p)$.

Table 2. Parameter estimates for $T_3(p)$ using Shorrocks' (1983) income data

	α	β	λ	η
Brazil	0.9038607	0.1956482	-2.3391257	0.5000000
Columbia	1.0984900	0.2871671	-1.4966379	0.5025607
Denmark	0.0000000	0.8178719	-0.5454835	1.4227869
Finland	0.6222372	0.6247161	-0.0000051	1.2260239
India	0.9781033	0.2270743	0.7741400	0.5000000
Indonesia	0.8640722	0.1668873	1.7904335	0.5000110
Japan	0.1106829	0.7341653	-0.4107163	1.1252793
Kenya	0.6570113	0.1811202	-2.4278433	0.5000003
Malaysia	0.6253334	0.4713232	-0.0968370	1.0224873
Netherlands	0.6030711	0.5543365	-0.0000162	1.0319286
N Zealand	0.4678067	0.7183750	-0.0000003	1.1612433
Norway	1.2545605	0.4999966	0.6880046	0.5000116
Panama	0.6706023	0.5325137	-0.0000019	0.9563373
Sri Lanka	0.0000000	0.7227005	-0.5051129	1.3953376
Sweden	0.0000002	0.7452864	-0.2143127	1.4965412
Tanzania	0.6135106	0.2586841	-2.1928819	0.5000000
Tunisia	0.0009171	0.6882983	-2.4256180	0.9990832
UK	0.7887904	0.4999999	0.2792676	0.7002776
Uruguay	0.0000006	0.7653365	-1.5011102	1.3426007

Note: $\eta \geq 1/2$ is imposed so that β and λ satisfy (9) or (10).

Table 3. Parameter estimates for $V(p)$ using Shorrocks (1983) income data

	α	β	λ	λ_1	δ	η
Brazil	0.7588379	0.1358719	-2.3861370	5.0800772	0.8437318	0.5000000
Columbia	1.1250881	0.1945068	1.6372881	5.6692376	0.8000765	0.6373440
Denmark	0.5515430	1.0000000	-0.5265505	9.6912929	0.8354492	1.0103942
Finland	1.5025968	0.5853923	-7.4462709	0.0000014	0.4808688	0.5018076
India	0.9500295	0.1595836	1.8351873	3.3527612	0.8350063	0.5000002
Indonesia	0.7831276	0.1064557	2.2400265	2.0972553	0.7846015	0.5000403
Japan	0.6341660	0.6478339	-1.8545616	0.0000529	0.7272617	0.6280006
Kenya	0.0228880	0.1372374	0.2477313	2.9639857	0.6390415	0.9771120
Malaysia	1.1819108	0.2523612	1.3768940	6.4911370	0.7804873	0.5826837
Netherlands	1.2296029	0.3260155	1.1208101	6.8622157	0.7992656	0.5053963
N Zealand	1.1913785	0.6448899	-6.1424497	0.0000018	0.3433921	0.5356541
Norway	1.2305477	0.6644481	-9.9970779	0.0000031	0.2377121	0.5504111
Panama	1.2124715	0.3302206	1.1079939	7.1735019	0.8173589	0.5160027
Sri Lanka	0.9693821	0.6215003	-3.6374015	0.0000009	0.6720248	0.5546455
Sweden	1.0839604	0.6626720	-5.0729627	0.0000004	0.4836789	0.5462910
Tanzania	0.5045214	0.1952060	-2.4014569	4.1872337	0.8232045	0.5000000
Tunisia	0.0000000	0.7443686	-11.9728817	2.3710046	0.2420133	1.0397704
UK	1.0253910	0.3840919	0.9568732	5.9633874	0.8907825	0.5005821
Uruguay	0.0000000	0.8063398	-6.3830538	0.6346932	0.2316490	1.6407687

Note: $\eta \geq 1/2$ is imposed so that β and λ satisfy (9) or (10).