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# SCALING RELATIONSHIPS OF GAUSSIAN PROCESSES

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# SCALING RELATIONSHIPS OF GAUSSIAN PROCESSES

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**Abstract**

Asset returns conforming to a Gaussian random walk are characterised by the temporal independence of the moments of the distribution. Employing currency returns, this note demonstrates the conditions that are necessary for risk to be estimated in this manner.

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# SCALING RELATIONSHIPS OF GAUSSIAN PROCESSES

## 1. Introduction

Under the random walk model, the temporal dimension of the risk of an asset is irrelevant. The convention is for risk to be scaled at the square root of time, though due to the presence of conditional heteroskedasticity, some financial time series do not conform to the linear scaling of variances rule and require a more complicated function of variance (Diebold *et al*. 1998). A Gaussian process may also be described in terms of the value of an exponent originally determined by Hurst (1951). Combining the scaling properties of Gaussian series and the value of the Hurst exponent implies a number of empirical conditions that provide an insight into the distributional properties of time series. The objective of this note is to identify and explain, using an example from currency returns, these empirical conditions. We begin by developing the principles of fractal geometry.

## 2. Scaling and self-similarity

The concepts of standard Brownian motion (sBm) and fractional Brownian (fBm) motion may be defined in terms of the relative level of dependence between successive increments. One characteristic of these processes is their self-similar behaviour. Self-similarity and scale invariance can be used to describe the relationship between the parts and the whole of the trail (plot) of a function. Consider a function (S) made up of the points $X = [X_0, X_1,..., X_n]$, where the probability of incremental movement is unrestricted with respect to the direction of the movement. Changing the length of the function by a common factor $r<1$, such that $rX = [rX_0, rX_1,..., rX_n]$, will yield a new function rS, whose geometric length is less than that of the original function. For the appropriate value of r, self-similarity implies the original function S can be recovered by N times contiguous replications of the self-similar rescaled function rS

When the trail of the function is measured with respect to time (as in the case of the line-to-line function) Mandelbrot (1977) shows that the function will instead be self-affine. Consider the same function (S), measured now as a line-to-line function comprising the points $X(t) = [X(t_0),$

$X(t_1)$,..., $X(t_n)$], in time t. Changing the time scale of the function by the ratio r<1, the required change in scale of the amplitude (measured along the vertical axis) is shown to be $r^H$ for a self-affine function, where the H-exponent is the statistic proposed by Hurst (1951).

Given the function S is a Brownian line-to-line function, the distance from $X(t_0)$ to a point $X(t_0 + t)$ is shown by Mandelbrot (1977) to be a random multiple of $\sqrt{t}$. Setting $t_0 = 0$ it follows for $t>t_0$ that

(1) $$X(t_0+t)-X(t_0)\approx e\,|(t_0+t)-t_0|^{0.5}\approx e\,t^{0.5}$$

where e is a random variable with zero mean and unit variance. Properly rescaled in time by r, and in amplitude by $\sqrt{r}$, the increments of the self-affine rescaled function $(rS)/\sqrt{r}$ will be

(2) $$\frac{X(t_0+rt)-X(t_0)}{\sqrt{r}}$$

For the correct choice of scaling factor, the two functions S and $(rS)/\sqrt{r}$ are statistically indistinguishable, such that they have the same finite dimensional distribution functions for all $t_0$ and all r>0.

Self-affine Brownian line-to-line functions are described by Mandelbrot as exhibiting scale invariance with the scaling factor $\sqrt{r} = r^H$. Allowing for $0\leq H\leq 1$, $H\neq 0.5$ it follows that Equation (1) can be generalised by

(3) $$X(t_0+t)-X(t_0)\approx e\,|(t_0+t)-t_0|^{H}\approx e\,t^{H}$$

Equation (2) may be similarly generalised by

(4) $$\frac{X(t_0+rt)-X(t_0)}{r^H}$$

for a fractional line-to-line function.

Fractional Brownian line-to-line functions exhibit statistical self-affinity at all time scales. Independent therefore of the incremental length (or frequency of observation) of $X(t_0 + t) - X(t_0)$, the relative level of positive dependence (1>H>0.5) or negative dependence (0<H<0.5) should remain constant. Modelling financial asset returns as line-to-line functions, the self-affine relation described by Equation (3) and Equation (4) can be tested by an examination of the scaling relations between the risk of long-interval returns ($\sigma_k$) and the risk of short-interval returns ($\sigma_n$)

(5) $$[\,s^2(P_t - P_{t-k})\,]^{0.5} = (k/n)^H\,[\,s^2(P_t - P_{t-n})\,]^{0.5}$$

The constants k and n may take the values 1 (daily), 5 (weekly), 12 (fortnightly), 22 (monthly) and 252 (annual), where $\infty \geq n \geq 1$ and $\infty \geq k \geq n$. $P_t$ are $P_{t-k}$ are the natural logarithms of daily prices, and H is the scale exponent necessary to estimate the long-interval risk of the asset from the observed short-interval risk(s).

If the underlying return series exhibits conditional heteroskedasticity, linear rescaling by Equation (5) will tend to overestimate the long-interval conditional standard deviations. Drost and Nijman (1993) provide an alternate model to Equation (5) for series where the underlying form of the returns (GARCH) process is known a priori. In the case of Gaussian processes, two Hypotheses based upon Equation (5) are:

Hypothesis 1: The estimated risk of an asset ($\sigma_q$*) over a long time interval (k) is a linear function of the observed risk over a shorter time interval (n) scaled at the square root of time

$H_0$ : $\sigma_q^* = (k/n)^{0.5}\,\sigma(P_t - P_{t-n})$

$H_a$ : $\sigma_q^* \neq (k/n)^{0.5}\,\sigma(P_t - P_{t-n})$

Hypothesis 2: The estimated risk of an asset ($\sigma_q$*) over a long time interval (k) is a linear function of the observed risk over a shorter time interval (n) scaled at the Hurst exponent of the returns series

$H_0$ : $\sigma_q^* = (k/n)^{H}\,\sigma(P_t - P_{t-n})$

$H_a$ : $\sigma_q^* \neq (k/n)^{H}\,\sigma(P_t - P_{t-n})$

Implied standard deviations for each interval (k = 5, 12, 22 and 252) may be estimated from the standard deviation of the daily yields series (k = 1), and the results compared to the observed standard deviation of q interval yields. Implied daily yield ranges may be estimated from the q interval yield ranges using Equation (5) for H = 0.5 and the results compared to observed daily yield ranges for each series. For these tests, the acceptance of the null hypothesis that the scale factor (r) is r = 5, and that the appropriate scale exponent (H) is H = 0.5, will imply the series under observation conforms to a random Gaussian distribution.

For the alternate hypotheses, imputed values of the scale exponent (H) for the standard deviation and range may be estimated. The significance of the imputed scale exponent is that this represents the value of H for which the implied weekly standard deviations (and daily yield ranges) equals exactly their observed values. Observed values of H which are significantly different from H = 0.5, will imply the rejection of the null hypothesis. An example of these tests is provided in the following example.

### 2.1. *Example: Scaled currency returns*

The process of linear rescaling may be illustrated by the comparison of two time-series: a simulated Gaussian random walk; and a time-series of spot DMK/USD yields. Returns for each series are calculated for intervals of k = 1, 2, ..., 252 periods and the standard deviation of returns are then estimated. A summary of the log standard deviations over selected return intervals is provided in Table I. Observed and implied standard deviations for both series over all return intervals are also shown graphically in Figure 1. Implied standard deviations in the figure are estimated by rescaling the observed one-period standard deviations using the Gaussian exponent value H = 0.5. Comparison of the plot of observed versus implied standard deviations of the simulated Gaussian random walk, suggests that the implied standard deviations of k-interval DMK/USD returns consistently underestimate their corresponding observed values. An implication of the results shown in Figure 1 is that linear rescaling of short-interval risk using H = 0.5 may not be appropriate when the underlying series does not conform to Gaussian random walk.

(Insert TABLE I and FIGURE 1 about here)

Using data such as provided in Table I, the scale exponent (H) may be estimated either locally or globally. Local estimates of the scale exponent are estimated for individual pairs of standard deviations. Using observed daily and annual standard deviations to find the exponent value for which daily risk scales to its annual equivalent is one example of estimating the value of H locally. The estimation of local scale exponent values is analogous to solving Equation (5) for the exponent value H and may be completed using a model of the general form

$$(6) \qquad H = \frac{\log(s_k / s_n)}{\log(k/n)}$$

and assumes that the values of $\sigma_k$ and $\sigma_n$ are both known and observable. By substituting data in Table I into Equation (6), Table II provides local estimates of H for the scaling of short-interval risks into annual risk. Values in the table represent the rate at which the observed daily (n = 1) to monthly (n = 22) risk estimates scale to the observed annual risk for the simulated Gaussian series and spot DMK/USD yields. Local scale exponents for the Gaussian series are all approximately equal to H = 0.5. For the DMK/USD, the higher scale exponent values imply that scaling short-interval risk underestimates the real level of risk. One example of where linear rescaling may be used to estimate annual standard deviation is option pricing. All other things being equal, results from Table II suggest that premiums for DMK/USD currency options would be under-priced if annual risk was estimated conventionally by multiplying the short-interval risk by the square root of time (ie. H = 0.5).

(Insert TABLE II about here)

When the standard deviation of returns is known and observable over several consecutive return intervals, a global estimate of the scale exponent can be estimated using an OLS regression of the general form

$$(7) \qquad \log(s_k) = a + b(\log k)$$

where the beta coefficient is the scale exponent H. Global estimates of H for the simulated Gaussian series and standard deviation of DMK/USD returns are given in Table III .

(Insert TABLE III about here)

Consistent with local estimates provided in Table 2, the DMK/USD global scale exponent value is higher than the equivalent value for the simulated Gaussian series. It should be noted however that the problem of serial correlation between consecutive k-interval standard deviations requires that the statistical significance of the estimated scale exponent (beta) cannot be tested using standard parametric tests such as those shown in the table. As reported by Muller *et al*. (1990), the use of overlapping versus contiguous data series in the estimation of consecutive k-interval standard deviations has no real impact on the estimated scale exponent (beta coefficient) itself.

## 3. Conclusion

The objective of this note was to identify and explain, using an example of currency returns, the empirical conditions that are implied by combining the scaling properties of a Gaussian series with the value of the Hurst exponent. The example of currency returns illustrates the process by which linear rescaling may be used to estimate the risk of long-interval returns using observable short-interval returns. However we conclude that such an approach may not be appropriate when the return series under observation are not independent.

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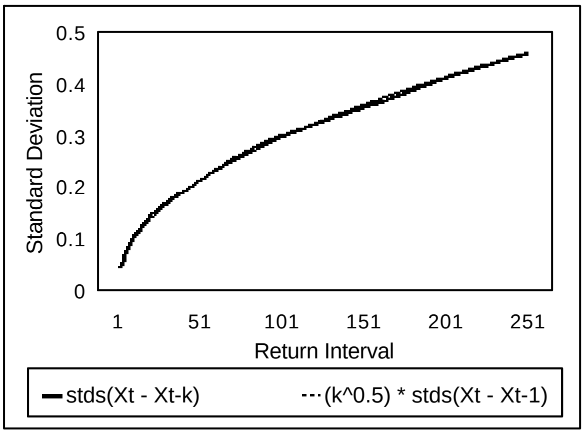
TABLE I

LOG STANDARD DEVIATIONS BY RETURN INTERVAL

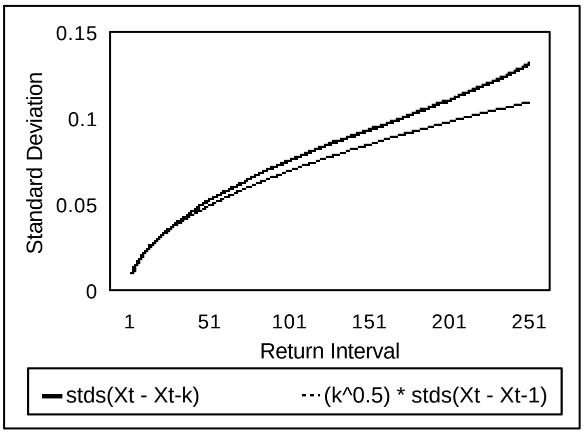
| Return interval | Log σ $_{sBm}$ | log σ $_{DMK/USD}$ |
|---|---|---|
| 1 | -1.4989 | -2.1590 |
| 2 | -1.3440 | -2.0046 |
| 3 | -1.2576 | -1.9196 |
| 4 | -1.1955 | -1.8609 |
| 5 | -1.1481 | -1.8128 |
| : | : | : |
| 22 | -0.8342 | -1.4774 |
| : | : | : |
| 252 | -0.3102 | -0.8766 |

FIGURE 1

OBSERVED AND IMPLIED STANDARD DEVIATIONS BY RETURN INTERVAL




Standard Brownian Motion (sBm)




DMK/USD

TABLE II

LOCAL H EXPONENT ESTIMATES FOR SCALING OBSERVED N-PERIOD RISK TO ANNUAL RISK

| Return interval (n) | 1 | 2 | 5 | 22 |
|---|---|---|---|---|
| SBm | 0.4950 | 0.4922 | 0.4921 | 0.4948 |
| DMK/USD | 0.5340 | 0.5499 | 0.5499 | 0.5674 |

TABLE III

REGRESSION SUMMARY STATISTICS

| | sBm | DMK/USD |
|---|---|---|
| R Square | 0.9992 | 0.9989 |
| Standard error | 0.0057 | 0.0074 |
| Observations | 252 | 252 |
| Degrees of Freedom | 251 | 251 |
| Intercept | -1.4650 | -2.1986 |
| Slope | **0.5013** | **0.5405** |
| (t-statistic) | 548.2941 | 482.5121 |
| p-value | 0.0000 | 0.0000 |