Exact and Superlative Price and Quantity Indicators

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Abstract

The traditional economic approach to index number theory is based on a ratio concept. The Konüs true cost of living index is a ratio of cost functions evaluated at the same utility level but with the prices of the current period in the cost function that appears in the numerator and the prices of the base period in the denominator cost function. The Allen quantity index is also a ratio of cost functions where the utility levels vary but the price vector is held constant in the numerator and denominator. There is a corresponding theory for differences of cost functions that was initiated by Hicks and the present paper develops this approach. Diewert defined superlative price and quantity indexes as observable indexes which were exact for a ratio of unit cost functions or for a ratio of linearly homogeneous utility functions. The present paper looks for counterparts to his results in the difference context, for both flexible homothetic and flexible nonhomothetic preferences. The Bennet indicators of price and quantity change turn out to be superlative for the nonhomothetic case. The underlying preferences are of the translation homothetic form discussed by Balk, Chambers, Dickenson, Färe and Grosskopf.

Key Words

Price and quantity aggregates, index number theory, equivalent and compensating variations, exact and superlative indexes, flexible functional forms, indicator functions, cost functions, money metric utility functions, duality theory, Bennet indicators, Konüs true cost of living index, Allen quantity index, welfare economics, decompositions of price and quantity change, translation homothetic preferences.

JEL Classification Numbers

C43, D11, D12, D60.

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1. Introduction

Traditional index number theory adopts a theoretical framework based on a ratio concept. In this approach, the ratio of the value aggregate between two periods is decomposed into the product of a price index and a quantity index. The price index, a function of the price and quantity data pertaining to the two periods under consideration, is interpreted as the ratio of the current price of the aggregate to the aggregate price in the base period. The quantity index, another function of the price and quantity data pertaining to the two periods, is interpreted as the ratio of the current period quantity aggregate to the base period quantity aggregate. In the economic approach to index number theory, it is assumed that the consumer has preferences over the individual quantities in the aggregate that can be represented by a utility function which has a dual cost function. This cost function is used to define consumer's family of Konüs (1939) price indexes or true cost of living indexes and the consumer's family of Allen (1949) quantity indexes.

If the consumer's preferences are homothetic (so that they can be represented by a linearly homogeneous utility function), then the family of Konüs price indexes collapses to a ratio of unit cost functions and the family of Allen quantity indexes collapses to a ratio of utility functions, where these functions are evaluated at the data of say period 1 in the numerator and the data of period 0 in the denominator. If preferences are homothetic, then Konüs and Byushgens (1926), Afriat (1972) and Pollak (1983) showed that certain numerical index number formula were exactly equal to the underlying theoretical economic indexes, provided that the consumer's utility function or dual unit cost function had certain functional forms. Diewert (1976) took this theory of exact indexes one step further and looked for indexes that were exact for flexible functional forms, for either the linearly homogeneous utility function or for the dual unit cost function and he called such indexes that were exact for flexible functional forms superlative. However, empirically, it has been shown that consumer preferences are generally not homothetic and hence the relevance of Diewert's concept of a superlative index is somewhat doubtful, at least in the consumer context. But Diewert (1976; 122) did implicitly develop a stronger concept for a superlative index in the context of general nonhomothetic preferences and we will formalize his idea in the present paper in section 2 below where we will define strongly superlative indexes. Section 2 will also review the standard definitions for exact and superlative indexes in the case of homothetic preferences.

In section 3, we switch from the traditional economic approach to index number theory, which is based on ratios, to an economic approach pioneered by Hicks (1942) (1943) (1945-46) which is based on differences. In the traditional approach to index number theory, a value ratio is decomposed into the product of a price index times a quantity index whereas in the difference approach, a value difference is decomposed into the sum of a price indicator (which is a measure of aggregate price change) plus a quantity indicator (which is a measure of aggregate quantity change). The difference analogue to a theoretical Konüs price index is a Hicksian price variation and the difference analogue to an Allen quantity index is a Hicksian quantity variation such as the equivalent or compensating variation. For normal index number theory, the theoretical Konüs and

Allen indexes are defined using ratios of cost functions but in the difference approach to index number theory, the theoretical price and quantity variation functions are defined in terms of differences of cost functions. In the difference approach, the counterparts to price and quantity index number formulae are price and quantity indicator functions.[2] Both index number formulae and indicator functions are known functions of the price and quantity data pertaining to the two periods under consideration. In section 3, we provide a definition for an exact price or quantity indicator function.

In sections 4 and 5, we develop further the difference approach to index number theory. In section 4, we will define a given price or quantity indicator function to be superlative if it is exactly equal to a corresponding theoretical price or quantity variation under the assumption that the consumer has homothetic preferences that are represented by a flexible linearly homogeneous utility function or which are dual to a flexible unit cost function. We draw on the theory of superlative price and quantity indexes to exhibit many superlative indicator functions. The theory that we develop in section 4 for the case of homothetic preferences turns out to be a variant of the theory of superlative indicators developed earlier by Diewert (2005).

In section 5, we will define a given price or quantity indicator function to be strongly superlative if it is exactly equal to a corresponding theoretical price or quantity variation, under the assumption that the consumer has (general) preferences which are dual to a flexible cost function that is subject to money metric utility scaling. The term money metric utility scaling is due to Samuelson (1974) and it is simply a convenient way of cardinalizing a utility function. It proves to be much more difficult to find strongly superlative price or quantity indicator functions but in section 5, we show that the Bennet (1920) indicator functions are strongly superlative. Our results require that the consumer's preferences be represented by a certain translation homothetic cost function that is a variant of the normalized quadratic cost function introduced by Diewert and Wales (1987) (1988a) (1988b). The flexibility of this functional form is shown in Appendix A. Our work draws on the earlier work on translation homothetic preferences (or linear parallel preferences) by Blackorby, Boyce and Russell (1978), Dickinson (1980), Chambers and Färe (1998), Chambers (2001; 111) and Balk, Färe and Grosskopf (2004).

The practical usefulness of the difference approach to the measurement of price and quantity change is illustrated at the end of section 5 where we show that under certain conditions including the assumption that each household faces the same prices in each period, it is possible to exactly measure the arithmetic average of the economy's sum of the individual household equivalent and compensating variations using only aggregate data since this aggregate measure of welfare change is exactly equal to the Bennet quantity indicator using aggregate quantity data. In other words, the difference approach to the measurement of aggregate price and quantity change has better aggregation properties than the traditional ratio approach.

[2] This indicator terminology was introduced by Diewert (1992a) (2005).

In section 6, we provide economic interpretations for each term in the sum of terms that make up the Bennet price and quantity indicators. The decomposition results developed here are analogues to similar results obtained by Diewert and Morrison (1986) and Kohli (1990) in the traditional approach to index number theory.

In section 7, we illustrate the use of the difference approach to measure aggregate Japanese consumption and we contrast the traditional ratio approach to the measurement of real consumption to our difference approach.

Section 8 concludes.

2. Exact and Superlative Price and Quantity Indexes

In preparation for the difference approach to aggregate price and quantity measurement, in this section, we review the standard ratio approach to the measurement of price and quantity change. Thus we will define exact price and quantity indexes and present two definitions for a superlative price index. In the following sections, we will attempt to adapt these standard index number theory concepts to the difference context.

The starting point for the economic approach to index number theory is the consumer's cost or expenditure function C. Thus suppose that the consumer has preferences that are defined by the *utility function* f(q) over all nonnegative N dimensional quantity vectors q $\equiv [q_1,...,q_N] \geq 0_N$.[3] In addition, suppose that f is a nonnegative, increasing,[4] continuous and quasiconcave function over the nonnegative orthant $\Omega \equiv \{q : q \geq 0_N\}$. Now suppose that the consumer faces the positive vector of commodity prices $p >> 0_N$ and suppose that the consumer wishes to attain the utility level u belonging to the range of f as cheaply as possible. Then the consumer will solve the following *cost minimization problem* and the consumer's *cost function*, C(u,p), will be the minimum cost of achieving the target utility level u:

(1) $C(u,p) \equiv \min_q \{p\cdot q : f(q) \geq u \,;\, q \geq 0_N\}$.

It can be shown[5] that C(u,p) will have the following properties: (i) C(u,p) is jointly continuous in u,p for $p >> 0_N$ and $u \in U$ where U is the range of f and is a nonnegative function over this domain of definition set; (ii) C(u,p) is increasing in u for each fixed p and (iii) C(u,p) is nondecreasing, linearly homogeneous and concave function of p for each $u \in U$.[6] Conversely, if a cost function is given and satisfies the above properties, then the utility function f that is dual to C can be recovered as follows.[7] For $u \in U$ and $q >> 0_N$, define the function F(u,q) as follows:

[3] Notation: $q \geq 0_N$ means each component of q is nonnegative; $q >> 0_N$ means each component of q is positive and $q > 0_N$ means $q \geq 0_N$ but $q \neq 0_N$ where 0_N denotes an N dimensional vector of zeros. Also $p\cdot q$ denotes the inner product of the vectors p and q; i.e., $p\cdot q = p^Tq \equiv \sum_{n=1}^N p_nq_n$.
[4] Thus if $q^2 >> q^1 \geq 0_N$, then $f(q^2) > f(q^1)$.
[5] See Diewert (1993; 124).
[6] Call these conditions on the cost function Conditions I.
[7] See Diewert (1974; 119) (1993; 129) for the details and for references to various duality theorems.

(2) $F(u,q) \equiv \max_p \{C(u,p) : p \cdot q \le 1 ; p \ge 0_N\}$.

Now solve the equation:

(3) $F(u,q) = 1$

for u^* and this solution u^* will equal $f(q)$.

The utility function $f(q)$ and the dual cost function $C(u,p)$ are used in order to define the consumer's *family of Konüs* (1939) *true cost of living indexes*, $P_K(p^0.p^1,f(q))$, where p^0 and p^1 are the vectors of positive commodity prices that the consumer faces in periods 0 and 1 respectively and $u = f(q)$ is a positive reference level of utility:

(4) $P_K(p^0.p^1,f(q)) \equiv C(f(q),p^1)/C(f(q),p^0)$.

Thus for each reference quantity vector q that gives rise to a positive utility level, $u = f(q) > 0$, the consumer's aggregate price index for that reference level of utility is the ratio of $C(u,p^1)$ to $C(u,p^0)$.

The consumer's utility and cost functions can be used in order to define the consumer's *family of Allen* (1949) *quantity indexes*, $Q_A(q^0.q^1,p)$, where q^0 and q^1 are the observed consumption vectors that the consumer chose in periods 0 and 1 respectively and $p >> 0_N$ is a strictly positive vector of reference prices:

(5) $Q_A(q^0,q^1,p) \equiv C(f(q^1),p)/C(f(q^0),p)$.

The meaning of (5) is that if the consumer faces the reference price vector p, then his or her period t utility, $f(q^t)$, is set equal to the minimum cost of achieving this utility level using the reference prices p, $C(f(q^t),p)$, for t = 0,1 and the consumer's quantity index is set equal to the ratio $C(f(q^1),p)/C(f(q^0),p)$. Samuelson (1974) called this type of cardinalization of utility, *money metric utility*.[8] However, note that different choices of p will generate different cardinalizations of utility and different Allen quantity indexes.

It is useful to specialize the above definitions for price and quantity indexes for the case where the consumer's preferences are *homothetic*[9] or *neoclassical*. We say that a utility function is *neoclassical* if it satisfies the following properties over the positive orthant: (i) f is a positive function; i.e., $f(q) > 0$ if $q >> 0_N$; (ii) f is positively linearly homogeneous; i.e., $f(\lambda q) = \lambda f(q)$ for all $\lambda > 0$ and $q >> 0_N$ and (iii) f is concave; i.e., for $0 < \lambda < 1$, $q^0 >> 0_N$ and $q^1 >> 0_N$, we have $f(\lambda q^0 + (1-\lambda)q^1) \ge \lambda f(q^0) + (1-\lambda)f(q^1)$. It turns out that a

[8] The basic idea can be traced back to Hicks (1942).

[9] Preferences are homothetic if the consumer's utility function can be written as G[f(q)] where f is neoclassical and G is a continuous increasing function of one variable. Note that the homothetic preferences G[f(q)] can be represented by the neoclassical utility function f. Thus, at times in what follows, we will sometimes refer to neoclassical preferences as homothetic preferences. The concept of homotheticity is due to Shephard (1953).

concave function defined over the positive orthant is also continuous over this domain of definition. Furthermore, f defined over the positive orthant has a continuous extension to the nonnegative orthant[10] and this extended f will also satisfy properties (ii) and (iii) above. The extended f(q) will also be nondecreasing in its variables q over the nonnegative orthant.[11]

If the consumer's preferences are neoclassical, then it turns out that the corresponding cost function defined by (1) above has the following representation:

(6) $C(u,p) = c(p)u$

where $c(p) \equiv C(1,p)$ is the consumer's *unit cost function*. It also turns out that the unit cost function, c(p), is also a neoclassical function, i.e., it is a positive, nondecreasing, continuous, concave and linearly homogeneous function of p over the positive orthant. Finally, the consumer's utility function f can be recovered from a knowledge of the unit cost function as follows:[12] for $q >> 0_N$,

(7) $f(q) = 1/\max_p \{c(p) : p \cdot q = 1; p \geq 0_N\}$.

The assumption that the consumer has neoclassical (or homothetic) preferences greatly simplifies index number theory. Under the assumption of neoclassical preferences, for each reference q such that f(q) is positive, we have[13]

$$
\begin{aligned}
(8)\ P_K(p^0,p^1,f(q)) &\equiv C(f(q),p^1)/C(f(q),p^0) && \text{using definition (4)} \\
&= c(p^1)f(q)/c(p^0)f(q) && \text{using (6)} \\
&= c(p^1)/c(p^0).
\end{aligned}
$$

Thus under the assumption of neoclassical preferences, the Konüs price index is equal to the unit cost ratio, $c(p^1)/c(p^0)$, and is independent of the reference utility level.

Similarly, under the assumption of neoclassical preferences, for each positive reference price vector p, we have

$$
\begin{aligned}
(9)\ Q_A(q^0,q^1,p) &\equiv C(f(q^1),p)/C(f(q^0),p). && \text{using definition (5)} \\
&= c(p)f(q^1)/c(p)f(q^0) && \text{using (6)} \\
&= f(q^1)/f(q^0).
\end{aligned}
$$

Thus under the assumption of neoclassical preferences, the Allen quantity index is equal to the utility ratio, $f(q^1)/f(q^0)$, and is independent of the reference price vector p.

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[10] See Fenchel (1953; 78) or Rockafellar (1970; 85).
[11] See Diewert (1974; 111).
[12] This is a version of the Samuelson (1953) Shephard (1953) duality theorem; see also Diewert (1974; 110-112) and Samuelson and Swamy (1974).
[13] See Shephard (1953) (1970), Pollak (1983) and Samuelson and Swamy (1974). Shephard in particular realized the importance of the homotheticity assumption in conjunction with separability assumptions in justifying the existence of subindexes of the overall cost of living index.

Now suppose that the consumer has homothetic preferences (which we represent by a neoclassical utility function f(q) or the dual unit cost function c(p)) and he or she faces prices $p^t >> 0_N$ in period t and minimizes the cost of achieving the utility level u^t in period t for t = 0,1. Let q^t be the consumer's observed quantity vector for period t so that $u^t = f(q^t)$ for t = 0,1. Then the consumer's observed period t cost, $p^t \cdot q^t$ can be written as follows:

(10) $p^t \cdot q^t = C(f(q^t),p^t) = c(p^t)f(q^t)$; $\qquad$ t = 0,1.

Under these assumptions, the consumer's ratio of period 1 expenditures to period 0 expenditures satisfies the following equations:

(11) $p^1 \cdot q^1/p^0 \cdot q^0 = [c(p^1)f(q^1)]/[c(p^0)f(q^0)]$ $\qquad$ using (10)
$= [c(p^1)/c(p^0)][f(q^1)/f(q^0)]$
$= P_K(p^0,p^1,f(q))Q_A(q^0,q^1,p)$ $\qquad$ for arbitrary reference q and p using (8) and (9).

Thus under the assumption of homothetic preferences and cost minimizing behavior on the part of the consumer for the two periods under consideration, the consumer's observed expenditure ratio is equal to the product of the Konüs price index for arbitrary reference vector q and the Allen quantity index for arbitrary reference vector p.

Note that in general, without a knowledge of the consumer's preferences, the Konüs price index and the Allen quantity index are not directly observable; i.e., they are *theoretical* indexes as opposed to the "*practical*" bilateral price and quantity *formulae*, say $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$, that are known functions of the observed consumer data pertaining to the two periods being compared. We assume that the bilateral index number formulae P and Q satisfy the following *product test* for all strictly positive price and quantity vectors:[14]

(12) $p^1 \cdot q^1/p^0 \cdot q^0 = P(p^0,p^1,q^0,q^1)Q(p^0,p^1,q^0,q^1)$.

Diewert (1976; 117) defined a quantity index $Q(p^0,p^1,q^0,q^1)$ to be *exact* for a neoclassical utility function f if under the assumption that the consumer minimizes the cost of achieving the utility level $u^t = f(q^t)$ in period t for t = 0,1, we have

(13) $Q(p^0,p^1,q^0,q^1) = f(q^1)/f(q^0)$;

i.e., the quantity index $Q(p^0,p^1,q^0,q^1)$ is exactly equal to the utility ratio which in turn is equal to the theoretical Allen quantity index under the assumption of neoclassical preferences.[15] Under the same assumptions of cost minimizing behavior and assuming that the preferences of the consumer can be represented by the dual unit cost function

[14] This is Fisher's (1922) weak factor reversal test.
[15] Diewert (1976) gave many examples of exact index number formulae drawing on the earlier work of Konüs and Byushgens (1926), Pollak (1983) (originally written in 1971) and Afriat (1972).

c(p), then Diewert (1976; 134) defined a price index $P(p^0,p^1,q^0,q^1)$ to be *exact* for c(p) if we have

(14) $P(p^0,p^1,q^0,q^1) = c(p^1)/c(p^0)$;

i.e., the price index $P(p^0,p^1,q^0,q^1)$ is exactly equal to the ratio of unit costs which in turn is equal to the theoretical Konüs price index under the assumption of neoclassical preferences.

Suppose the index number pair $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$ satisfy the product test (12) and either P is exact for c(p) or Q is exact for f(q).[16] Then Diewert (1976) defined P and Q to be *superlative* indexes if either c or f could provide a second order approximation to an arbitrary twice continuously differentiable neoclassical unit cost function $c^*(p)$ or to an arbitrary twice continuously differentiable neoclassical utility function $f^*(q)$.[17] Thus the advantage of superlative price and quantity indexes is that they can generate reasonably accurate price and quantity aggregates without having to undertake any econometric estimation of preferences, which becomes difficult or impossible as the number of commodities in the aggregate increases.

Examples of superlative price index formulae[18] are the Fisher (1922) ideal price index P_F and the Törnqvist (1936) (1937) Theil (1967) index P_T defined as follows:

(15) $P_F(p^0,p^1,q^0,q^1) \equiv [p^1\cdot q^0/p^0\cdot q^0]^{1/2}\,[p^1\cdot q^1/p^0\cdot q^1]^{1/2}$;
(16) $\ln P_T(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^{N} (1/2)[s_n^0 + s_n^1] \ln [p_n^1/p_n^0]$

where the period t expenditure share on commodity n is defined as $s_n^t \equiv p_n^t q_n^t/p^t\cdot q^t$ for n = 1,...,N and t = 0,1.

This completes our summary of the existing theory for superlative indexes in the case of homothetic preferences. Unfortunately, if the consumer's preferences are homothetic, then all income elasticities of demand are equal to unity and Engel's Law and other econometric evidence strongly suggests that income elasticities are not homothetic and hence consumer preferences are *not* homothetic. Thus while the theory of exact and superlative indexes may be very useful when we wish to construct subaggregate prices and quantities, it seems that superlative indexes may not be appropriate when constructing overall aggregate consumer price and quantity indexes. Thus we need to determine whether we can find indexes which are exact for more general nonhomothetic preferences. Fortunately, this can be done.

Suppose the consumer has general preferences defined by the utility function f(q) and the general cost function C(u,p) is dual to f. As usual, let p^t and q^t be the observed price and quantity data pertaining to period t and define the period t level of utility $u^t \equiv f(q^t)$ for t =

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[16] Of course, if P is exact for c, P and Q satisfy (12) and f is dual to c, then Q is exact for f (and vice versa).
[17] Blackorby and Diewert (1979) showed that if c is a differentiable flexible functional form and has a differentiable dual f(q), then f is also flexible in the class of neoclassical utility functions and vice versa.
[18] See Diewert (1976) for the details.

0,1. We assume that the consumer is minimizing the cost of achieving the utility level u^t in period t so we have:

(17) $p^t \cdot q^t = C(f(q^t),p^t)$; $\qquad t = 0,1.$

Under the above assumptions, we say that the bilateral price index number formula, $P(p^0,p^1,q^0,q^1)$, is *exact* for the cost function C if there exists a u^* such that u^* is between u^0 and u^1 so that

(18) either $u^0 \leq u^* \leq u^1$ or $u^1 \leq u^* \leq u^0$ and
(19) $P(p^0,p^1,q^0,q^1) = C(u^*,p^1)/C(u^*,p^0) \equiv P_K(p^0,p^1,u^*)$.

Thus P is an exact index number formula if under the assumption of cost minimizing behavior, $P(p^0,p^1,q^0,q^1)$ is exactly equal to the Konüs theoretical price index $P_K(p^0,p^1,u^*)$ where u^* is an intermediate reference level of utility. The requirement that the reference level of utility be between the period 0 and 1 utility levels (or possibly equal to one of these levels) is a natural one: we do not want the reference utility level to be too far from the two levels actually experienced by the consumer during the two periods under consideration.

Initially, we define P to be a *strongly superlative index number formula* if it is exact according to the definition immediately above and in addition, the cost function C(u,p) that P is exact for can approximate an arbitrary cost function to the second order.

Diewert (1976; 122) showed that the Törnqvist Theil index P_T defined by (16) is exact for a general translog cost function where the reference level of utility u^* is equal to $[u^0 u^1]^{1/2}$, the square root of the product of the period 0 and 1 utility levels. Since the general translog cost function is a fully flexible functional form, this shows that P_T is a strongly superlative price index.

Since the scaling of utility is arbitrary up to an increasing transformation of an initial representation of the utility function, we will find it convenient to impose money metric utility scaling on the underlying utility function f and its dual cost function C. Thus let $p^* >> 0_N$ be an arbitrary positive price vector. We will assume that the consumer's utility is scaled so that the dual cost function C satisfies the following equation:[19]

(20) $C(u,p^*) = u$ for all $u \in U$.

Thus our final definition for a *strongly superlative index number formula* P is that it is exact according to the above definition (18) and (19) and in addition, the cost function C(u,p) that P is exact for can approximate an arbitrary cost function (that satisfies the money metric utility scaling property (20)) to the second order.

[19] If the cost function C(u,p) satisfies Conditions I and in addition, satisfies the money metric utility scaling conditions (20), then we will say that C satisfies Conditions II.

An analogous definition of exactness can be made for a quantity index. Thus we say that the bilateral quantity index number formula, $Q(p^0,p^1,q^0,q^1)$, is *exact* for the cost function C if there exists a reference price vector $p^* \equiv [p_1{}^*,...,p_N{}^*]$ such that p^* is between p^0 and p^1 so that

(21) either $p_n{}^0 \leq p_n{}^* \leq p_n{}^1$ or $p_n{}^1 \leq p_n{}^* \leq p_n{}^0$ for n = 1,...,N and
(22) $Q(p^0,p^1,q^0,q^1) = C(f(q^1),p^*)/C(f(q^0),p^*) \equiv Q_A(q^0,q^1,p^*)$.

Thus Q is an exact index number formula if under the assumption of cost minimizing behavior, $Q(p^0,p^1,q^0,q^1)$ is exactly equal to the Allen theoretical quantity index $Q_A(q^0,q^1,p^*)$ where p^* is a vector of intermediate reference prices. The requirement that the reference price vector p^* be between the period 0 and 1 price vectors that the consumer faced (or possibly equal to one of these two vectors) is again a natural one: we do not want the money metric cardinalizing vector of reference prices to be too far from the two price vectors actually faced by the consumer during the two periods under consideration.

Finally, we define Q to be a *strongly superlative index number formula* if it is exact according to the definition immediately above and in addition, the cost function C(u,p) that is dual to the utility function f can approximate an arbitrary cost function (that has the money metric utility scaling property (20)) to the second order.

The above material summarizes the theory of exact and superlative indexes which is based on decompositions of the *value ratio* into price and quantity components that *multiply* together. In the following section, we will review and extend the companion theory that is based on decompositions of the *value difference* into a *sum* of a price change component and a quantity change component.

3. Value Differences, Variations and Indicators of Price and Quantity Change

Assume that the consumer's cost function, C(u,p), satisfies Conditions I and the dual utility function is f(q) as usual. Throughout this section, we will assume that p^t and q^t are the observed price and quantity data pertaining to period t and we define the consumer's period t observed level of utility $u^t \equiv f(q^t)$ for t = 0,1. We assume that the consumer is minimizing the cost of achieving the utility level u^t in period t so that conditions (17) hold; i.e., we have $p^t \cdot q^t = C(f(q^t),p^t)$ for t = 0,1. Our task in the present section is to decompose the consumer's observed value change over the two periods under consideration, $p^1 \cdot q^1 - p^0 \cdot q^0$, into the sum of two terms, one of which is the part of the value change that is due to price change and the other part due to quantity change. This is the *difference approach* to explaining a change in a value aggregate as opposed to the usual *ratio approach* used in index number theory.[20]

[20] Hicks (1942) seems to have been the first to explore the similarities between the two approaches.

The difference counterpart to the Allen (1949) quantity index explained in the previous section is the following *Hicks Samuelson quantity variation* Q_S: for each strictly positive reference price vector $p >> 0_N$, define $Q_S(q^0,q^1,p)$ as follows:[21]

$$(23)\ Q_S(q^0,q^1,p) \equiv C(f(q^1),p) - C(f(q^0),p).$$

Just as the Allen quantity index $Q_A(q^0,q^1,p)$ defined by (5) was an entire family of indexes (one for each reference price vector p), so too is the family of quantity variations, Q_S. Two special cases of (23) are of particular importance, the *equivalent and compensating variations*, Q_E and Q_C, defined as follows:[22]

$$(24)\ Q_E(q^0,q^1,p^0) \equiv Q_S(q^0,q^1,p^0) = C(f(q^1),p^0) - C(f(q^0),p^0)\ ;$$
$$(25)\ Q_C(q^0,q^1,p^1) \equiv Q_S(q^0,q^1,p^1) = C(f(q^1),p^1) - C(f(q^0),p^1)\ .$$

Thus the equivalent variation uses the period 0 price vector p^0 as the reference price vector while the compensating variation uses the period 1 price vector p^1 as the reference price vector.

Generalizing Hicks (1939; 40-41) (1946; 331-332), we will define a family of *Hicksian price variation functions* $P_H(p^0,p^1,f(q))$ as follows: for each nonnegative reference quantity vector q, define $P_H(p^0,p^1,f(q))$ as follows:

$$(26)\ P_H(p^0,p^1,f(q)) \equiv C(f(q),p^1) - C(f(q),p^0).$$

Just as the Konüs price index, $P_K(p^0.p^1,f(q))$, defined by (4) was an entire family of indexes (one for each reference quantity vector or reference utility level $u \equiv f(q)$), so too is the family of Hicksian price variations. Two special cases of (26) are of particular importance, the *Laspeyres and Paasche price variation functions*, P_{HL} and P_{HP}, defined as follows:[23]

[21] Samuelson (1974) recognized that C(f(q),p) was a valid cardinalization of utility for any reference price vector p and thus (23) is a valid cardinal measure of the utility difference between periods 0 and 1. Hicks on the other hand only considered the special cases (24) and (25) defined below.

[22] Henderson (1941; 120) introduced these variations in the N = 2 case and Hicks (1942) introduced them in the general case, although his exposition is difficult to follow. The term compensating variation is due to Henderson (1941; 118) and the term equivalent variation is due to Hicks (1942; 128). Hicks (1939; 40-41) initially defined the compensating variation as a measure of price change: "As we have seen, the best way of looking at consumer's surplus is to regard it as a means of expressing, in terms of money income, the gain which accrues to the consumer as a result of a fall in price. Or better, it is the *compensating variation* in income, whose loss would just offset the fall in price and leave the consumer no better off than before." However, later, Hicks (1942; 127-128), following Henderson (1941; 120) defined (geometrically) the compensating variation as $C(u^1,p^1) - C(u^0,p^1)$ and the equivalent variation as $C(u^1,p^0) - C(u^0,p^0)$, which are measures of welfare (or quantity) change.

[23] In the index number literature, $C(u^0,p^1)/C(u^0,p^0)$ is known as the Laspeyres Konüs (1939; 17) *true cost of living index* or price index and $C(u^1,p^1)/C(u^1,p^0)$ is known as the Paasche Konüs theoretical price index; see Pollak (1983). It can be seen that (27) and (28) are the difference counterparts to these ratio type indexes.

(27) $P_{HL}(p^0,p^1,f(q^0)) \equiv P_H(p^0,p^1,f(q^0)) = C(f(q^0),p^1) - C(f(q^0),p^0)$;[24]
(28) $P_{HP}(p^0,p^1,f(q^1)) \equiv P_H(p^0,p^1,f(q^1)) = C(f(q^1),p^1) - C(f(q^1),p^0)$.[25]

Thus the Laspeyres price variation uses the period 0 quantity vector q^0 as the reference quantity vector while the Paasche price variation uses the period 1 quantity vector q^1 as the reference quantity vector.

Let $M^0 \equiv p^0 \cdot q^0$ be the consumer's nominal "income" or expenditure on the N commodities in period 0. Then $P_{HL}(p^0,p^1,f(q^0))$ is the amount of nominal income that must be added to the period 0 income M^0 in order to allow the consumer, facing period 1 prices p^1, to achieve the same utility level as was achieved in period 0, which is $u^0 \equiv f(q^0)$. Similarly, let $M^1 \equiv p^1 \cdot q^1$ be the consumer's nominal "income" in period 1. Then $P_{HP}(p^0,p^1,f(q^1))$ is the amount of nominal income that must be subtracted from the period 1 income M^1 in order to allow the consumer, facing period 0 prices p^0, to achieve the same utility level as was achieved in period 1, which is $u^1 \equiv f(q^1)$.

Note that the equivalent quantity variation defined by (24) matches up with the Paasche price variation defined by (28) in order to provide an exact decomposition of the value change going from period 0 to 1; i.e., using these definitions and assumptions (17), it can be seen that:

(29) $p^1 \cdot q^1 - p^0 \cdot q^0 = C(f(q^1),p^1) - C(f(q^0),p^0) = Q_E(q^0,q^1,p^0) + P_{HP}(p^0,p^1,f(q^1)).$

Similarly, the compensating quantity variation defined by (25) matches up with the Laspeyres price variation defined by (27) in order to provide another exact decomposition of the value change going from period 0 to 1:

(30) $p^1 \cdot q^1 - p^0 \cdot q^0 = C(f(q^1),p^1) - C(f(q^0),p^0) = Q_C(q^0,q^1,p^1) + P_{HL}(p^0,p^1,f(q^0)).$

A problem with the quantity variations defined by (24) and (25) and the price variations defined by (27) and (28) is that they asymmetrically single out a reference price or quantity vector that pertains to a single period. Since both measures are equally valid and if a single measure of price or quantity change is required, then for some purposes, it may be useful to take an arithmetic average of the equivalent and compensating variations defined by (24) and (25) (denote the resulting average quantity variation as $Q_A(q^0,q^1,p^0,p^1)$) and to take an arithmetic average of the price variations defined by (27) and (28) (denote the resulting average price variation as $P_{HA}(p^0,p^1,q^0,q^1)$). It can be seen that these average price and quantity variations will also provide an additive decomposition of the value change; i.e., we have:

(31) $p^1 \cdot q^1 - p^0 \cdot q^0 = C(f(q^1),p^1) - C(f(q^0),p^0) = Q_A(q^0,q^1,p^0,p^1) + P_{HA}(p^0,p^1,f(q^0),f(q^1)).$

---

[24] Hicks (1945-46; 68) called this measure the 'price compensating variation' and distinguished this measure from a quantity compensating variation, which he did not define in a very clear manner. Hicks also considered price and quantity variations in Hicks (1943).
[25] Hicks (1945-46; 69) called this measure the "price equivalent variation".

All of the price and quantity variations defined above cannot be evaluated in general using observed price and quantity data pertaining to the two periods under consideration. Thus we now turn our attention to the problem of finding *observable approximations* to the above theoretical variation functions.

Looking at definition (24) for the equivalent variation, it can be seen that the term $C(f(q^0),p^0)$ is equal to period 0 expenditure on the N commodities, $p^0 \cdot q^0$, and hence this term is observable. The remaining term, $C(f(q^1),p^0)$, is not observable but we can use Shephard's (1953; 11) Lemma in order to obtain the following first order approximation to this term:

$$
\begin{aligned}
(32)\ C(f(q^1),p^0) &\approx C(f(q^1),p^1) + \nabla_p C(f(q^1),p^1)\cdot[p^0 - p^1] && \\
&= C(f(q^1),p^1) + q^1\cdot[p^0 - p^1] && \text{using Shepard's Lemma} \\
&= p^1\cdot q^1 + p^0\cdot q^1 - p^1\cdot q^1 && \text{using (17) for } t = 1 \\
&= p^0\cdot q^1. &&
\end{aligned}
$$

Using (17) for t = 0, (32) and definition (24), we obtain the following first order approximation to the equivalent variation:

$$
\begin{aligned}
(33)\ Q_E(q^0,q^1,p^0) &\approx p^0\cdot q^1 - p^0\cdot q^0 \\
&= p^0\cdot[q^1 - q^0] \\
&\equiv V_L(p^0,p^1,q^0,q^1)
\end{aligned}
$$

where the observable *Laspeyres indicator of quantity change*, $V_L(p^0,p^1,q^0,q^1)$, is defined as $p^0\cdot[q^1 - q^0]$, the inner product of the base period prices p^0 with the quantity change vector, $q^1 - q^0$. In a similar fashion, it can be shown that a first order approximation to the term $C(f(q^0),p^1)$ is $p^1\cdot q^0$ and so a first order approximation to the compensating variation $Q_C(q^0,q^1,p^1)$ defined by (25) is:[26]

$$
\begin{aligned}
(34)\ Q_C(q^0,q^1,p^1) &\approx p^1\cdot q^1 - p^1\cdot q^0 \\
&= p^1\cdot[q^1 - q^0] \\
&\equiv V_P(p^0,p^1,q^0,q^1)
\end{aligned}
$$

where the observable *Paasche indicator of quantity change*, $V_P(p^0,p^1,q^0,q^1)$, is defined as $p^1\cdot[q^1 - q^0]$, the inner product of the current period prices p^1 with the quantity change vector, $q^1 - q^0$.

Note that V_L and V_P are the difference counterparts to the ordinary Laspeyres and Paasche quantity indexes, Q_L, and Q_P, defined as follows:

$$(35)\ Q_L(p^0,p^1,q^0,q^1) \equiv p^0\cdot q^1/p^0\cdot q^0\ ;\ \ Q_P(p^0,p^1,q^0,q^1) \equiv p^1\cdot q^1/p^1\cdot q^0.$$

[26] The first order approximations (33) and (34) were obtained by Hicks (1942; 127-134); see also Diewert (1992a; 568).

We now turn our attention to the problem of finding observable approximations for the Laspeyres and Paasche price variation functions defined by (27) and (28) above. An observable first order approximation to the term $C(f(q^0),p^1)$ in (27) is

$$
\begin{aligned}
(36)\ C(f(q^0),p^1) &\approx C(f(q^0),p^0) + \nabla_p C(f(q^0),p^0)\cdot[p^1 - p^0] \\
&= C(f(q^0),p^0) + q^0\cdot[p^1 - p^0] && \text{using Shepard's Lemma} \\
&= p^0\cdot q^0 + p^1\cdot q^0 - p^0\cdot q^0 && \text{using (17) for } t = 0 \\
&= p^1\cdot q^0.
\end{aligned}
$$

Using (17) for t = 0, (36) and definition (27), we obtain the following first order approximation to the Laspeyres price variation:

$$
\begin{aligned}
(37)\ P_{HL}(p^0,p^1,f(q^0)) &\approx p^1\cdot q^0 - p^0\cdot q^0 \\
&= q^0\cdot[p^1 - p^0] \\
&\equiv I_L(p^0,p^1,q^0,q^1)
\end{aligned}
$$

where the observable *Laspeyres indicator of price change*, $I_L(p^0,p^1,q^0,q^1)$, is defined as $q^0\cdot[p^1 - p^0]$, the inner product of the base period quantity vector q^0 with the price change vector, $p^1 - p^0$. In a similar fashion, it can be shown that a first order approximation to the term $C(f(q^1),p^0)$ is $p^0\cdot q^1$ and so a first order approximation to the Paasche price variation $P_{HP}(p^0,p^1,f(q^1))$ defined by (28) is:

$$
\begin{aligned}
(38)\ P_{HP}(p^0,p^1,f(q^1)) &\approx p^1\cdot q^1 - p^0\cdot q^1 \\
&= q^1\cdot[p^1 - p^0] \\
&\equiv I_P(p^0,p^1,q^0,q^1)
\end{aligned}
$$

where the observable *Paasche indicator of price change*, $I_P(p^0,p^1,q^0,q^1)$, is defined as $q^1\cdot[p^1 - p^0]$, the inner product of the current period quantity vector q^1 with the price change vector, $p^1 - p^0$.[27]

Note that I_L and I_P[28] are the difference counterparts to the ordinary Laspeyres and Paasche price indexes, P_L, and P_P, defined as follows:

$$(39)\ P_L(p^0,p^1,q^0,q^1) \equiv p^1\cdot q^0/p^0\cdot q^0\ ;\ \ P_P(p^0,p^1,q^0,q^1) \equiv p^1\cdot q^1/p^0\cdot q^1.$$

[27] The first order approximations (37) and (38) were obtained by Hicks (1945-46; 72-73) (1946; 331).

[28] Hicks (1942; 128) (1945-46; 71) called I_L and I_P the Laspeyres and Paasche variations but we will reserve the term "variation" for the (unobservable) theoretical measures of price and quantity change defined by (23) for changes in quantities and by (26) for changes in prices. We will follow Diewert (1992a; 556) (2005; 313) and use the term "indicator" to denote a given function of the price and quantity data pertaining to the two periods under consideration so that the term *indicator* becomes the difference theory counterpart to an *index number formula* in the ratio approach to the measurement of price and quantity change. Since P and Q are usually used to denote price and quantity indexes, a different notation is required to denote price and quantity indicators. Using I to denote a price indicator and V to denote a quantity (or volume) indicator follows the conventions used by Diewert (2005). Note that national income accountants use the term "volume index" to denote a quantity index.

In the usual approach to index number theory, it proves to be useful to take the *geometric* average of the Laspeyres and Paasche price indexes, leading to the Fisher price index P_F defined by (15), since the Fisher index has very good properties from the viewpoint of the test or axiomatic approach to index number theory; see Diewert (1992b) and Balk (1995). However, in the axiomatic approach[29] to price and quantity measurement in the difference context, it proves to be better to take the *arithmetic* average of the Paasche and Laspeyres indicators. This leads to the *Bennet* (1920) *indicators of price and quantity change* defined as follows:

(40) $I_B(p^0,p^1,q^0,q^1) \equiv (1/2)I_L(p^0,p^1,q^0,q^1) + (1/2)I_P(p^0,p^1,q^0,q^1) = (1/2)[q^0+q^1]\cdot[p^1-p^0]$;
(41) $V_B(p^0,p^1,q^0,q^1) \equiv (1/2)V_L(p^0,p^1,q^0,q^1) + (1/2)V_P(p^0,p^1,q^0,q^1) = (1/2)[p^0+p^1]\cdot[q^1-q^0]$.

Note that Hicks (1942; 134) (1945-46; 73) obtained the Bennet quantity indicator V_B as an approximation to the arithmetic average of the equivalent and compensating variations and he also identified V_B as a generalization to many markets of Marshall's consumer surplus concept.

It can be verified that the Laspeyres, Paasche and Bennet price and quantity indicators can be used in order to obtain the following exact decompositions of the value change in the aggregate over the two periods under consideration:

(42) $p^1\cdot q^1 - p^0\cdot q^0 = I_L(p^0,p^1,q^0,q^1) + V_P(p^0,p^1,q^0,q^1)$;
(43) $p^1\cdot q^1 - p^0\cdot q^0 = I_P(p^0,p^1,q^0,q^1) + V_L(p^0,p^1,q^0,q^1)$;
(44) $p^1\cdot q^1 - p^0\cdot q^0 = I_B(p^0,p^1,q^0,q^1) + V_B(p^0,p^1,q^0,q^1)$.

We conclude this section by defining indicator counterparts to our index number definitions of exactness in the case of nonhomothetic preferences. As usual, we assume that the consumer minimizes cost in periods 0 and 1 so that the consumer has the utility function $f(q)$ that satisfies the usual regularity Conditions I and has the dual cost function $C(u,p)$ so that equations (17) are satisfied. Recall that the price index number formula $P(p^0,p^1,q^0,q^1)$ was defined to be exact for the cost function C if conditions (18) and (19) were satisfied. The price indicator counterpart to this definition is as follows: $I(p^0,p^1,q^0,q^1)$ is *exact* for the cost function C if there exists a u^* such that u^* is between $u^0 \equiv f(q^0)$ and $u^1 \equiv f(q^1)$ so that

(45) either $u^0 \le u^* \le u^1$ or $u^1 \le u^* \le u^0$ and
(46) $I(p^0,p^1,q^0,q^1) = C(u^*,p^1) - C(u^*,p^0) = P_H(p^0,p^1,u^*)$.

Thus $I(p^0,p^1,q^0,q^1)$ is exact for the preferences that are dual to $C(u,p)$ if under the assumption of cost minimizing behavior on the part of the consumer, $I(p^0,p^1,q^0,q^1)$ is exactly equal to the theoretical Hicksian price variation function $P_H(p^0,p^1,u^*)$ defined by (26) for a reference utility level u^* that is between the period 0 and 1 utility levels attained by the consumer.

[29] See Diewert (2005) and Balk (2007) on the axiomatic approach to measures of price and quantity change using differences.

Recall that the quantity index number formula $Q(p^0,p^1,q^0,q^1)$ was defined to be exact for the cost function C if conditions (21) and (22) were satisfied. The quantity indicator counterpart to this definition is as follows: $V(p^0,p^1,q^0,q^1)$ is *exact* for the cost function C if there exists a reference price vector $p^* \equiv [p_1^*,...,p_N^*]$ such that p^* is between p^0 and p^1 so that

(47) either $p_n^0 \leq p_n^* \leq p_n^1$ or $p_n^1 \leq p_n^* \leq p_n^0$ for n = 1,...,N and
(48) $V(p^0,p^1,q^0,q^1) = C(f(q^1),p^*) - C(f(q^0),p^*) = Q_S(q^0,q^1,p^*)$.

Thus $V(p^0,p^1,q^0,q^1)$ is exact for the preferences that are dual to C(u,p) if under the assumption of cost minimizing behavior on the part of the consumer, $V(p^0,p^1,q^0,q^1)$ is exactly equal to the theoretical Hicks Samuelson quantity variation function $Q_S(q^0,q^1,p^*)$ defined by (23) for a reference price vector p^* that is between the period 0 and 1 price vectors faced by the consumer..

In the following section, we will assume that the consumer has homothetic preferences and we will attempt to find price and quantity indicators that are exact and superlative in this case. In section 5, we will drop the assumption of homothetic preferences and we will attempt to find superlative indicators in this more general context.

4. Superlative Price and Quantity Indicators in the Homothetic Preferences Case

We now suppose that the consumer's utility function f(q) is neoclassical and the dual unit cost function is c(p). Under these conditions, using (6), we have

(49) $C(f(q),p) = c(p)f(q)$.

Thus the family of Hicks Samuelson quantity variations Q_S defined by (23) and the family of Hicksian price variations P_H defined by (26) have the following structures under the assumption of neoclassical preferences:

(50) $Q_S(q^0,q^1,p) \equiv C(f(q^1),p) - C(f(q^0),p) = [f(q^1) - f(q^0)]c(p)$;
(51) $P_H(p^0,p^1,f(q)) \equiv C(f(q),p^1) - C(f(q),p^0) = [c(p^1) - c(p^0)]f(q)$.

It turns out that if we choose the vector of reference prices p in (50) to be equal to p^0 or p^1, then we can find exact quantity indicator functions $V(p^0,p^1,q^0,q^1)$ and if we choose the reference quantity vector q in (51) to be equal to q^0 or q^1, then we can find exact price indicator functions $I(p^0,p^1,q^0,q^1)$, by drawing on exact index number theory in the case of homothetic preferences. Thus let $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$ be an exact pair of price and quantity indexes; i.e., they satisfy (12), (13) and (14) in section 2. Now let the reference price vector p in (50) above equal the period 0 price vector, p^0. Then $Q_S(q^0,q^1,p^0)$ becomes the equivalent variation $Q_E(q^0,q^1,p^0)$ and thus (50) becomes the following equation:

(52) $Q_E(q^0,q^1,p^0) = [f(q^1) - f(q^0)]c(p^0)$

$$= [\{f(q^1)/f(q^0)\} - 1]c(p^0)f(q^0)$$
$$= [Q(p^0,p^1,q^0,q^1) - 1]p^0 \cdot q^0 \qquad \text{using (13) and (10) for } t = 0$$
$$\equiv V_E(p^0,p^1,q^0,q^1).$$

Thus the observable function of the data, $V_E(p^0,p^1,q^0,q^1)$, defined to be equal to $[Q(p^0,p^1,q^0,q^1) - 1]p^0 \cdot q^0$, is *exactly equal to the equivalent variation*, $Q_E(q^0,q^1,p^0)$, and hence is an exact quantity indicator function. If in addition, Q is exact for a *flexible* neoclassical utility function f, then we say that the corresponding $V_E(p^0,p^1,q^0,q^1)$ is a *superlative quantity indicator*.

Now let the reference price vector p in (50) above equal the period 1 price vector, p^1. Then $Q_S(q^0,q^1,p^1)$ becomes the compensating variation $Q_C(q^0,q^1,p^1)$ and thus (50) becomes the following equation:

$$(53)\ Q_C(q^0,q^1,p^1) = [f(q^1) - f(q^0)]c(p^1)$$
$$= [1 - \{f(q^0)/f(q^1)\}]c(p^1)f(q^1)$$
$$= [1 - Q(p^0,p^1,q^0,q^1)^{-1}]p^1 \cdot q^1 \qquad \text{using (13) and (10) for } t = 1$$
$$\equiv V_C(p^0,p^1,q^0,q^1).$$

Thus the observable function of the data, $V_C(p^0,p^1,q^0,q^1)$, is *exactly equal to the compensation variation*, $Q_C(q^0,q^1,p^0)$, and hence is an exact quantity indicator function. If in addition, Q is exact for a *flexible* neoclassical utility function f, then we say that the corresponding $V_C(p^0,p^1,q^0,q^1)$ is a *superlative quantity indicator*.

Thus each superlative quantity index function, $Q(p^0,p^1,q^0,q^1)$, generates two superlative quantity indicator functions, $V_E(p^0,p^1,q^0,q^1)$ defined in (52) which is exact for the theoretical equivalent variation, and $V_C(p^0,p^1,q^0,q^1)$ defined in (53) which is exact for the theoretical compensating variation. Since there are an infinite number of superlative quantity indexes[30], there are an infinite number of superlative quantity indicators.

The above analysis can be repeated with some modifications in order to find superlative price indicator functions. Thus again let $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$ be an exact pair of price and quantity indexes. Now let the reference quantity vector q in (51) above equal the period 0 quantity vector, q^0. Then $P_H(p^0,p^1,f(q^0))$ becomes the Laspeyres price variation $P_{HL}(p^0,p^1,f(q^0))$ defined by (27) and thus (51) becomes the following equation:

$$(54)\ P_{HL}(p^0,p^1,f(q^0)) = [c(p^1) - c(p^0)]f(q^0)$$
$$= [\{c(p^1)/c(p^0)\} - 1]c(p^0)f(q^0)$$
$$= [P(p^0,p^1,q^0,q^1) - 1]p^0 \cdot q^0 \qquad \text{using (14) and (10) for } t = 0$$
$$\equiv I_{HL}(p^0,p^1,q^0,q^1).$$

Thus the observable function of the data, $I_{HL}(p^0,p^1,q^0,q^1)$, defined to be equal to $[P(p^0,p^1,q^0,q^1) - 1]p^0 \cdot q^0$, is *exactly equal to the Laspeyres price variation*, $P_{HL}(p^0,p^1,f(q^0))$, and hence is an exact price indicator function. If in addition, P is exact

[30] See Diewert (1976).

for a *flexible* unit cost function c, then we say that $I_{HL}(p^0,p^1,q^0,q^1)$ is a *superlative price indicator*.

Now let the reference quantity vector q in (51) above equal the period 1 quantity vector, q^1. Then $P_H(p^0,p^1,f(q^1))$ becomes the Paasche price variation $P_{HP}(p^0,p^1,f(q^1))$ and thus (51) becomes the following equation:

$$
\begin{aligned}
(55)\ P_{HP}(p^0,p^1,f(q^1)) &= [c(p^1) - c(p^0)]f(q^1) \\
&= [1 - \{c(p^0)/c(p^1)\}]c(p^1)f(q^1) \\
&= [1 - P(p^0,p^1,q^0,q^1)^{-1}]p^1\cdot q^1 && \text{using (14) and (10) for } t = 1 \\
&\equiv I_{HP}(p^0,p^1,q^0,q^1).
\end{aligned}
$$

Thus the observable function of the data, $I_{HP}(p^0,p^1,q^0,q^1)$, defined to be equal to $[1 - P(p^0,p^1,q^0,q^1)^{-1}]p^1\cdot q^1$, is *exactly equal to the Paasche price variation*, $P_{HP}(p^0,p^1,f(q^1))$, and hence is an exact price indicator function. If in addition, P is exact for a *flexible* unit cost function c, then we say that $I_{HP}(p^0,p^1,q^0,q^1)$ is a *superlative price indicator*. Again, since there are many superlative price index functions $P(p^0,p^1,q^0,q^1)$, there will be many superlative price indicator functions.[31]

There is one more detail to be settled in this analysis of superlative price and quantity indicator functions that are generated by traditional index number formulae: we want the sum of the price indicator and quantity indicator to be exactly equal to the value difference. Thus suppose that we are given bilateral index number formulae P and Q that satisfy the product test (12) and we use these indexes to define the quantity indicators $V_E(p^0,p^1,q^0,q^1)$ by (52) and $V_C(p^0,p^1,q^0,q^1)$ by (53) and the price indicators $I_{HL}(p^0,p^1,q^0,q^1)$ by (54) and $I_{HP}(p^0,p^1,q^0,q^1)$ by (55). Then using (12), it can be shown that numerically, the following equations will hold:

$$(56)\ p^1\cdot q^1 - p^0\cdot q^0 = I_{HP}(p^0,p^1,q^0,q^1) + V_E(p^0,p^1,q^0,q^1)\ ;$$
$$(57)\ p^1\cdot q^1 - p^0\cdot q^0 = I_{HL}(p^0,p^1,q^0,q^1) + V_C(p^0,p^1,q^0,q^1)\ .$$

Thus the equivalent variation indicator $V_E(p^0,p^1,q^0,q^1)$ generated by Q needs to be matched up with the Paasche price variation indicator $I_{HP}(p^0,p^1,q^0,q^1)$ generated by P and the compensating variation indicator $V_C(p^0,p^1,q^0,q^1)$ generated by Q needs to be matched up with the Laspeyres price variation indicator $I_{HL}(p^0,p^1,q^0,q^1)$ generated by P in order for the value difference to equal the sum of a price and quantity indicator.

[31] Diewert (2005; 333-337) also defined superlative price and quantity indicators in the case where consumer preferences were homothetic. Diewert's (2005; 336) *superlative economic indicator of price change* was defined as $I_E(p^0,p^1,q^0,q^1) = (1/2)I_{HL}(p^0,p^1,q^0,q^1) + (1/2)I_{HP}(p^0,p^1,q^0,q^1)$ where I_{HL} and I_{HP} are defined by the third equation in (54) and (55) respectively where the index number formula $P(p^0,p^1,q^0,q^1)$ is superlative. Thus our present definition of a superlative price or quantity indicator is a variation of Diewert's earlier definition. It should be noted that Fox (2006) generalized Diewert's (2005) bilateral approach to multilateral comparisons.

This completes our discussion of superlative indicators when the consumer's preferences are homothetic. In the following section, we address the much more difficult task of finding superlative indicators in the nonhomothetic case.

5. Strongly Superlative Price and Quantity Indicators

The holy grail of applied welfare economics is to obtain a quantity variation indicator that is exact for fully flexible preferences. To our knowledge, no one yet has succeeded in this quest.[32] In this section, we will show that the Bennet quantity indicator is exact for fully flexible preferences, subject to the money metric cardinalization of utility defined by (20), except that normalized prices that are adjusted for general inflation between the two periods must be used in place of the original prices facing the consumer. Since our focus is on quantity variations, this scaling of prices does not seem to be too serious a drawback to our suggested indicator of quantity change.

We now distinguish the original (unscaled) price vector $P^t \equiv [P_1^t,...,P_N^t] >> 0_N$ that the consumer faces in period t for t = 0,1 from the scaled or normalized price vector p^t which is proportional to P^t and will be defined shortly. As in previous sections, the consumer's observed quantity vector in period t is q^t for t = 0,1. Let the consumer's utility function f(q) satisfy Conditions II (which are the usual nonhomothetic assumptions plus the assumption of money metric utility scaling (20) for some strictly positive reference prince vector $P^* >> 0_N$) and let the corresponding dual cost function be C(u,P). We assume that the consumer's cost function has the following *translation homothetic normalized quadratic* functional form,[33] which is a special case of *translation homothetic preferences*:[34]

$$(58)\ C(u,P) \equiv b\cdot P + (1/2)\,(\alpha\cdot P)^{-1}P\cdot BP + c\cdot Pu$$

[32] Recent attempts by Weitzman (1988) and Diewert (1992a) ended up making homotheticity assumptions or in the case of Diewert's (1992a) Theorems 2 and 4, unrealistic assumptions relating the parameters of preferences to utility levels were made. Chambers and Färe (1998) and Chambers (2001) also came close but their preference classes fell short of being fully flexible; Chambers (2001; 111) explained the problem with his class of preferences. Diewert (1976; 123-124) had a fully flexible result but his result was exact for a Malmquist (1953) quantity index which is not an exact result for a quantity variation and moreover, the Malmquist index does not have the convenient aggregation properties that a Hicks-Samuelson quantity variation possesses.

[33] Diewert and Wales (1987) (1988a) (1988b) introduced the normalized quadratic cost function which can be defined as $C(u,P) \equiv b\cdot P + [(1/2)\,(\alpha\cdot P)^{-1}P\cdot BP + c\cdot P]u$ where b, c and B satisfy (59)-(62) and they showed that this functional form was flexible in the class of cost functions that satisfy the money metric utility scaling restrictions (20) for any predetermined parameter vector $\alpha > 0_N$. The advantage of this functional form is that it contains a flexible unit cost function as a special case (just set $b = 0_N$). However, since preferences are generally nonhomothetic, this advantage is not necessarily a huge one.

[34] Chambers and Färe (1998; 640) and Chambers (2001; 111) introduced the term "translation homothetic preferences" and studied these preferences in some detail and noted their importance for the measurement of welfare change; see also Balk, Färe and Grosskopf (2004). Blackorby, Boyce and Russell (1978; 348) introduced this class of preferences and Dickinson (1980; 1713) referred to this class of preferences as linear parallel preferences. Dickinson (1980; 1715-1717) exhibited several examples of this class of preferences that were flexible.

where $\alpha > 0_N$, b and c are N dimensional parameter vectors and B is parameter matrix. These parameter vectors and matrix satisfy the following restrictions, where $P^* >> 0_N$ is the reference vector which appears in (20), the definition for C to satisfy money metric utility scaling at the reference prices P^*:

(59) $B = B^T$ so that B is symmetric and B is negative semidefinite;
(60) $BP^* = 0_N$;
(61) $b \cdot P^* = 0$ and
(62) $c \cdot P^* = 1$.

Using the techniques in Diewert and Wales (1987), it can be shown that (59) implies that the C defined by (58) is globally concave. In the Appendix, we show that this functional form is flexible in the class of preferences satisfying the money metric utility scaling restrictions in (20) for any predetermined parameter vector $\alpha > 0_N$; i.e., given any $\alpha > 0_N$, we can find vectors b and c and a matrix of parameters B such that the restrictions (59)-(62) are satisfied and the resulting C defined by (58) is flexible at the arbitrary point (u^*,P^*). However, in general, this flexible functional form may not satisfy Conditions II for all $u > 0$ and all $P >> 0_N$. In the Appendix, we will define the region of prices and utility levels where the functional form satisfies the required regularity conditions for a cost function.

Assuming that the consumer's preferences can be represented by the cost function defined by (58)-(62) for the two periods under consideration, then assuming cost minimizing behavior on the part of the consumer, the following equations will hold:

$$(63)\ P^t \cdot q^t = C(f(q^t),P^t) = b \cdot P^t + (1/2)\,(\alpha \cdot P^t)^{-1} P^t \cdot BP^t + c \cdot P^t\, f(q^t)\ ; \qquad t = 0,1.$$

Using Shephard's Lemma, the consumer's observed period t demand vector q^t is equal to the following expression:

$$(64)\ q^t = \nabla_P C(f(q^t),P^t) = b + (\alpha \cdot P^t)^{-1}\, BP^t - (1/2)(\alpha \cdot P^t)^{-2}\, P^t \cdot BP^t\, \alpha + c\, f(q^t)\ ; \qquad t = 0,1.$$

If there is a great deal of general inflation between periods 0 and 1, then the compensating variation will be much larger than the equivalent variation simply due to this general inflation and taking an average of these two variations will be difficult to interpret due to the change in the scale of prices. In order to eliminate the effects of general inflation between the two periods being compared, it will be useful to *scale* the prices in each period by a fixed basket price index of the form $\alpha \cdot P$ where $\alpha \equiv [\alpha_1,...,\alpha_N] > 0_N$ is a nonnegative, nonzero vector of price weights.[35] Thus, having chosen the price

[35] A reasonable "standard" choice for the weighting vector α is $\alpha \equiv q^0/P^0 \cdot q^0$. For this choice of α, the vector of period t normalized prices, $p^t \equiv P^t/P^t \cdot \alpha$, can be interpreted as a period t vector of "real" prices using a fixed base Laspeyres price index to do the deflation of nominal prices. Diewert (2005; 340-341) commented on the general inflation problem as follows: "The above quotation alerts us to a potential problem with our treatment of value changes; namely, if there is a great change in the general purchasing power of money between the two periods being compared, then our indicators of volume change may be "excessively" heavily weighted by the prices of the period that has the highest general price level. Put

weighting vector α, the period t *real prices* that the consumer faces p^t are defined as follows:

(65) $p^0 \equiv P^0/\alpha \cdot P^0$; $p^1 \equiv P^1/\alpha \cdot P^1$.

Note that these real price vectors will satisfy the following restrictions:

(66) $\alpha \cdot p^t = \alpha \cdot P^t/\alpha \cdot P^t = 1$; $t = 0,1.$

Divide both sides of equation t in (63) by $\alpha \cdot P^t$ and using definitions (65), the resulting equations become:

(67) $p^t \cdot q^t = C(f(q^t),p^t) = b \cdot p^t + (1/2)p^t \cdot Bp^t + c \cdot p^t f(q^t)$; $t = 0,1.$

Similarly, substituting equations (65) and (66) into equations (64) leads to the following equations relating the consumer's period t quantity vectors q^t to the real price vectors p^t:

(68) $q^t = \nabla_P C(f(q^t),p^t) = b + Bp^t - (1/2)p^t \cdot Bp^t \alpha + cf(q^t)$; $t = 0,1.$

With the above preliminaries out of the way, we are ready to state our first Proposition which relates the Bennet quantity indicator defined earlier by (41), $V_B(p^0,p^1,q^0,q^1) \equiv (1/2)[p^0+p^1] \cdot [q^1-q^0]$, to the theoretical equivalent and compensating variations defined by (24) and (25), $Q_E(q^0,q^1,p^0) \equiv C(f(q^1),p^0) - C(f(q^0),p^0)$ and $Q_C(q^0,q^1,p^1) \equiv C(f(q^1),p^1) - C(f(q^0),p^1)$, where we are using the scaled real price vectors p^t defined by (65) as reference price vectors in place of the original nominal price vectors P^t.

Proposition 1: Let the consumer's observed period t data be (P^t,q^t) and suppose that the consumer minimizes the cost of achieving the period t utility level for each period t = 0,1. Let $\alpha > 0_N$ be a given vector of price weights that are used in order to construct the period t real price vectors, $p^t \equiv P^t/\alpha \cdot P^t$ for t = 0,1. Suppose a consumer has preferences f(q) which are dual to the translation homothetic normalized quadratic cost function C(u,P) defined by (58)-(62) and define $u^t \equiv f(q^t)$ for t = 0,1. Then the Bennet quantity indicator defined by (41) using the real prices defined by (65) is exactly equal to the arithmetic average of the equivalent and compensating variations defined by (24) and (25) using the

another way, the units that quantities are measured in do not require any comparisons with other quantities but the dollar price of a quantity is the valuation of a unit of a commodity *relative to a numeraire commodity, money*. Thus the indicators of price change that we have discussed in this paper encompass both *general changes* in the purchasing power of money as well as *changes in inflation adjusted* prices. Thus if there is high inflation between periods 0 and 1 and quantities have increased, then the use of symmetric in prices and quantities indicators (like the Bennet and Montgomery indicators) will shift some of the inflationary increase in values over to the indicator of volume change." Diewert (2005; 341) suggested deflating the prices of the second period by a general index of inflation going from period 0 to 1 whereas our solution is more specific in that we choose a Laspeyres type index to do the deflation. Diewert (1992a; 566) discussed other normalizations that have been used historically by various authors in order to construct suitable real prices for use in the measurement of welfare change by volume or quantity indicators.

real price vectors as reference prices rather than the original nominal price vectors; i.e., we have

(69) $V_B(p^0,p^1,q^0,q^1) = (1/2)Q_E(q^0,q^1,p^0) + (1/2)Q_C(q^0,q^1,p^1)$.

Proof:

$$
\begin{aligned}
(70)\ 2V_B(p^0,p^1,q^0,q^1) &= [p^0 + p^1]\cdot[q^1 - q^0] && \text{using definition (41)}\\
&= p^0\cdot q^1 - C(f(q^0),p^0) + C(f(q^1),p^1) - p^1\cdot q^0 && \text{using (67)}\\
&= p^0\cdot[b + Bp^1 - (1/2)p^1\cdot Bp^1\alpha + cf(q^1)] - C(f(q^0),p^0)\\
&\quad + C(f(q^1),p^1) - p^1\cdot[b + Bp^0 - (1/2)p^0\cdot Bp^0\alpha + cf(q^0)] && \text{using (68)}\\
&= p^0\cdot b + p^0\cdot Bp^1 - (1/2)p^1\cdot Bp^1 + p^0\cdot cf(q^1) - C(f(q^0),p^0)\\
&\quad + C(f(q^1),p^1) - [p^1\cdot b + p^1\cdot Bp^0 - (1/2)p^0\cdot Bp^0 + p^1\cdot cf(q^0)] && \text{using (66)}\\
&= [p^0\cdot b + (1/2)p^0\cdot Bp^0 + p^0\cdot cf(q^1)] - C(f(q^0),p^0)\\
&\quad + C(f(q^1),p^1) - [p^1\cdot b + (1/2)p^1\cdot Bp^1 + p^1\cdot cf(q^0)] && \text{using (59)}\\
&= C(f(q^1),p^0) - C(f(q^0),p^0) + C(f(q^1),p^1) - C(f(q^0),p^1) && \text{using (67)}\\
&= Q_E(q^0,q^1,p^0) + Q_C(q^0,q^1,p^1) && \text{using definitions (24) and (25)}
\end{aligned}
$$

which is equivalent to (69). Q.E.D.

Corollary 1: Under the conditions of the above Proposition, the following equality holds:

$$
\begin{aligned}
(71)\ V_B(p^0,p^1,q^0,q^1) &= C(f(q^1),(1/2)[p^0+p^1]) - C(f(q^0),(1/2)[p^0+p^1])\\
&\equiv Q_S(q^0,q^1,(1/2)[p^0+p^1]).
\end{aligned}
$$

Proof: From (70), we have the following equality:

$$
\begin{aligned}
(72)\ 2V_B(p^0,p^1,q^0,q^1) &= C(f(q^1),p^0) - C(f(q^0),p^0) + C(f(q^1),p^1) - C(f(q^0),p^1)\\
&= p^0\cdot c[f(q^1) - f(q^0)] + p^1\cdot c[f(q^1) - f(q^0)] && \text{using definition (67)}\\
&= [p^0 + p^1]\cdot c[f(q^1) - f(q^0)]\\
&= C(f(q^1),p^0+p^1) - C(f(q^0),p^0+p^1)
\end{aligned}
$$

where the last equality follows adding and subtracting terms and using definition (67) for C. Using the linear homogeneity property of $C(u,p)$ in p, it can be seen that (72) implies (71). Q.E.D.

The equality (71) shows that *the Bennet quantity indicator*, $V_B(p^0,p^1,q^0,q^1)$, *is a strongly superlative indicator*, since it is exact for the theoretical quantity variation, $Q_S(q^0,q^1,(1/2)p^0 + (1/2)p^1)$, using reference prices that are between p^0 and p^1, namely the arithmetic average reference prices $(1/2)p^0 + (1/2)p^1$.

There is a counterpart to Proposition 1 for the Bennet price indicator. Proposition 2 relates the Bennet price indicator defined earlier by (40), $I_B(p^0,p^1,q^0,q^1) \equiv (1/2)[q^0+q^1]\cdot[p^1-p^0]$, to the theoretical Laspeyres and Paasche price variation functions defined by (27) and (28), $P_{HL}(p^0,p^1,f(q^0)) \equiv C(f(q^0),p^1) - C(f(q^0),p^0)$ and $P_{HP}(p^0,p^1,f(q^1)) \equiv$

$C(f(q^1),p^1) - C(f(q^1),p^0)$, where again we use the scaled real price vectors p^t defined by (65) as reference price vectors in place of the original nominal price vectors P^t.

Proposition 2: Under the hypotheses listed in Proposition 1, the Bennet price indicator defined by (40) using the real prices defined by (65) is exactly equal to the arithmetic average of the Laspeyres and Paasche price variations defined by (27) and (28) using the real price vectors as reference prices rather than the original nominal price vectors; i.e., we have

(73) $I_B(p^0,p^1,q^0,q^1) = (1/2)\, P_{HL}(p^0,p^1,f(q^0)) + (1/2)P_{HP}(p^0,p^1,f(q^1))$.

Proof:[36]

$$
\begin{aligned}
(74)\ 2I_B(p^0,p^1,q^0,q^1) &= [q^0 + q^1]\cdot[p^1 - p^0] && \text{using definition (40)}\\
&= p^1\cdot q^0 - C(f(q^0),p^0) + C(f(q^1),p^1) - p^0\cdot q^1 && \text{using (67)}\\
&= p^1\cdot[b + Bp^0 - (1/2)p^0\cdot Bp^0\alpha + cf(q^0)] - C(f(q^0),p^0)\\
&\quad + C(f(q^1),p^1) - p^0\cdot[b + Bp^1 - (1/2)p^1\cdot Bp^1\alpha + cf(q^1)] && \text{using (68)}\\
&= p^1\cdot b + p^1\cdot Bp^0 - (1/2)p^0\cdot Bp^0 + p^1\cdot cf(q^0) - C(f(q^0),p^0)\\
&\quad + C(f(q^1),p^1) - [p^0\cdot b + p^0\cdot Bp^1 - (1/2)p^1\cdot Bp^1 + p^0\cdot cf(q^1)] && \text{using (66)}\\
&= [p^1\cdot b + (1/2)p^1\cdot Bp^1 + p^1\cdot cf(q^0)] - C(f(q^0),p^0)\\
&\quad + C(f(q^1),p^1) - [p^0\cdot b + (1/2)p^0\cdot Bp^0 + p^0\cdot cf(q^1)] && \text{using (59)}\\
&= C(f(q^0),p^1) - C(f(q^0),p^0) + C(f(q^1),p^1) - C(f(q^1),p^0) && \text{using (67)}\\
&= P_{HL}(p^0,p^1,f(q^0)) + P_{HP}(p^0,p^1,f(q^1)) && \text{using definitions (27) and (28)}
\end{aligned}
$$

which is equivalent to (73). Q.E.D.

Corollary 2: Under the conditions of the above Proposition, the following equality holds:

$$
\begin{aligned}
(75)\ I_B(p^0,p^1,q^0,q^1) &= C((1/2)f(q^0)+(1/2)f(q^1),p^1) - C((1/2)f(q^0)+(1/2)f(q^1),p^0)\\
&\equiv P_H(p^0,p^1,(1/2)f(q^0)+(1/2)f(q^1)) .
\end{aligned}
$$

Proof: From (74), we have the following equality:

$$
\begin{aligned}
(76)\ 2I_B(p^0,p^1,q^0,q^1) &= C(f(q^0),p^1) - C(f(q^0),p^0) + C(f(q^1),p^1) - C(f(q^1),p^0)\\
&= [p^1\cdot b + (1/2)p^1\cdot Bp^1 + p^1\cdot cf(q^0)] - [p^0\cdot b + (1/2)p^0\cdot Bp^0 + p^0\cdot cf(q^0)]\\
&\quad + [p^1\cdot b + (1/2)p^1\cdot Bp^1 + p^1\cdot cf(q^1)] - [p^0\cdot b + (1/2)p^0\cdot Bp^0 + p^0\cdot cf(q^1)] && \text{using (58)}\\
&= 2[p^1\cdot b + (1/2)p^1\cdot Bp^1 + p^1\cdot c(1/2)\{f(q^0)+f(q^1)\}]\\
&\quad - 2[p^1\cdot b + (1/2)p^1\cdot Bp^1 + p^1\cdot c(1/2)\{f(q^0)+f(q^1)\}] && \text{rearranging terms}\\
&= 2C((1/2)f(q^0)+(1/2)f(q^1),p^1) - 2C((1/2)f(q^0)+(1/2)f(q^1),p^0) && \text{using definition (58)}\\
&= 2\, P_H(p^0,p^1,(1/2)f(q^0)+(1/2)f(q^1)) && \text{using definition (26)}
\end{aligned}
$$

which is equivalent to (75). Q.E.D.

[36] Our technique of proof is closely related to the techniques used by Balk, Färe and Grosskopf (2004; 160-161) but our functional form assumptions are different and they do not establish a flexibility result for the class of functional forms that they use in their proofs.

The equality (75) shows that *the Bennet price indicator*, $I_B(p^0,p^1,q^0,q^1)$, *is a strongly superlative indicator*, since it is exact for the theoretical Hicksian price variation, $P_H(p^0,p^1,(1/2)u^0+(1/2)u^1)$, using the arithmetic average of the period 0 and 1 utility levels, u^0 and u^1, as the reference utility level.

Bennet (1920) showed that the sum of the Bennet price and quantity indicators, $I_B(p^0,p^1,q^0,q^1)$ plus $V_B(p^0,p^1,q^0,q^1)$, is numerically equal to the value difference, $p^1 \cdot q^1 - p^0 \cdot q^0$; recall (44) above. The above two Propositions show that the Bennet indicators have strong economic interpretations if we use real prices instead of nominal prices when calculating these indicators; i.e., they are both strongly superlative indexes.[37]

Another advantage of the Bennet quantity indicator is that it has a nice aggregation over households property. Thus let $\alpha > 0_N$ and suppose that there are H households in the economy and household h has normalized quadratic translation homothetic preferences $f^h(q)$ that are dual to the following cost function C^h for h = 1,...,H:

$$(77)\ C^h(u^h,P) \equiv b^h \cdot P + (1/2)\,(\alpha \cdot P)^{-1} P \cdot B^h P + c^h \cdot P u^h$$

where b^h, c^h and B^h satisfy the restrictions (59)-(62) for h = 1,...,H. Let q^{ht} be household h's observed consumption vector for period t and let household h face the price vector P^{ht} in period t for h = 1,...,H and t = 0,1. Define the vector of real prices that household h faces in period t, p^{ht}, as follows:

$$(78)\ p^{ht} \equiv P^{ht}/\alpha \cdot P^{ht}\ ; \qquad t = 0,1\ ;\ h = 1,\ldots,H.$$

Now make the hypotheses in Proposition 1 for each household and we find that the sum over households of the Bennet quantity indicators $V_B(p^{h0},p^{h1},q^{h0},q^{h1})$ for each household h is equal to the average of the sum of the household h equivalent and compensating variations, $Q_E{}^h(q^{h0},q^{h1},p^{h0})$ and $Q_C{}^h(q^{h0},q^{h1},p^{h1})$; i.e., using Proposition 1, we have:

$$\begin{aligned}(79)\ \textstyle\sum_{h=1}^H V_B(p^{h0},p^{h1},q^{h0},q^{h1}) &\equiv \textstyle\sum_{h=1}^H (1/2)[p^{h0} + p^{h1}] \cdot [q^{h1} - q^{h0}] \\ &= (1/2)\textstyle\sum_{h=1}^H Q_E{}^h(q^{h0},q^{h1},p^{h0}) + (1/2)\textstyle\sum_{h=1}^H Q_C{}^h(q^{h0},q^{h1},p^{h0}) \\ &= \textstyle\sum_{h=1}^H Q_S{}^h(q^{h0},q^{h1},(1/2)[p^{h0}+p^{h1}]) \qquad \text{using Corollary 1}\end{aligned}$$

where for h = 1,...,H, $Q_S{}^h(q^{h0},q^{h1},(1/2)[p^{h0}+p^{h1}])$ is the Hicks Samuelson theoretical quantity variation for household h using the vector of average real prices facing household h for the two periods under consideration, $(1/2)p^{h0} + (1/2)p^{h1}$, as the reference price vector. Thus if individual household price and quantity data are available, the sum of these theoretical quantity variations can be calculated as the sum of the observable Bennet quantity indicators.

[37] Diewert (2005) and Balk (2007) indicated that the Bennet indicators had excellent axiomatic properties as well. Thus the Bennet indicators seem to be the difference counterparts to the Fisher indexes in normal ratio index number theory, since the Fisher indexes also have strong economic and axiomatic properties.

If in addition, each household faces the same vector of prices p^0 in period 0 and p^1 in period 1, then (79) simplifies as follows:

$$(80)\ V_B(p^0,p^1,q^0,q^1) \equiv (1/2)[p^0 + p^1]\cdot[q^1 - q^0]$$
$$= (1/2)\sum_{h=1}^{H} Q_E^{\,h}(q^{h0},q^{h1},p^0) + (1/2)\sum_{h=1}^{H} Q_C^{\,h}(q^{h0},q^{h1},p^0)$$
$$= \sum_{h=1}^{H} Q_S^{\,h}(q^{h0},q^{h1},(1/2)[p^0+p^1])$$

where the aggregate period t quantity vectors q^t are defined as the sum of the individual household quantity vectors:

$$(81)\ q^0 \equiv \sum_{h=1}^{H} q^{h0}\ ;\ q^1 \equiv \sum_{h=1}^{H} q^{h1}\ .$$

Thus under the assumptions of Proposition 1 and the assumption that each household faces the same prices in each period, the aggregate Bennet indicator of quantity change, $V_B(p^0,p^1,q^0,q^1)$ defined by the first line in (80), is *exactly* equal to the arithmetic average of the sum of the individual household equivalent variations, $\sum_{h=1}^{H} Q_E^{\,h}(q^{h0},q^{h1},p^0)$, plus the sum of the individual compensating variations, $\sum_{h=1}^{H} Q_C^{\,h}(q^{h0},q^{h1},p^0)$. Under these hypotheses, the aggregate Bennet indicator of quantity change is also *exactly* equal to the sum over households of the Hicks Samuelson theoretical quantity variations using the vector of average real prices facing household h for the two periods under consideration, $\sum_{h=1}^{H} Q_S^{\,h}(q^{h0},q^{h1},(1/2)[p^0+p^1])$.[38]

6. The Decomposition Properties of the Bennet Indicators

In the production context, Diewert and Morrison (1986), Morrison and Diewert (1990) and Kohli (1990) (1991) developed a methodology that enables one to obtain exact decompositions of various Törnqvist indexes into explanatory factors for each price or quantity change using the assumption of a translog technology.[39] It would be useful if we could provide a similar decomposition result for the Bennet indicators but we are not able to accomplish this task. However, Diewert and Morrison (1986; 674-676) developed an *average of first order approximations methodology* which gave very similar results to their translog methodology[40] and so we will use this second approach below in order to provide economic interpretations for each separate term in the Bennet indicators.

In this section, we will not make any specific parametric assumptions; we will assume only that the consumer's cost function $C(u,P)$ satisfies conditions I and in addition, $C(u,P)$ and the dual $f(q)$ are once differentiable in a neighbourhood around the observed period t real price and quantity vectors, $p^t \equiv P^t/\alpha\cdot q^t$ and q^t, and around the observed period t utility levels, $u^t \equiv f(q^t)$, for $t = 0,1$. Hence the following equations will be satisfied by the data under the assumption that the consumer minimizes costs in each period:

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[38] This result is analogous to Chamber's (2001; 114) exact result for an aggregate normalized Bennet quantity indicator in the context of Chamber's benefit function framework.
[39] Diewert (2002) also developed some decomposition results for the Fisher indexes but these results lack the simplicity of the Törnqvist decomposition results.
[40] See Morrison and Diewert (1990) and Diewert and Lawrence (2006).

(82) $p^t \cdot q^t = C(f(q^t),p^t)$; $\quad t = 0,1;$
(83) $\quad q^t = \nabla_P C(f(q^t),P^t) = \nabla_p C(f(q^t),p^t)$; $\quad t = 0,1.$

The first set of equalities in (83) follows from Shephard's Lemma and the second set follows from the proportionality of the real prices p^t to the corresponding nominal prices P^t and the linear homogeneity of the cost function $C(u,P)$ in the components of P so that the partial derivative functions $\partial C(u,P)/\partial P_n$ are homogeneous of degree 0 in their price variables.

Define the *nth partial Bennet price and quantity indicators*, $I_n(p_n^0,p_n^1,q_n^0,q_n^1)$ and $V_n(p_n^0,p_n^1,q_n^0,q_n^1)$, as follows:

(84) $I_{Bn}(p_n^0,p_n^1,q_n^0,q_n^1) \equiv (1/2)[q_n^0 + q_n^1][p_n^1 - p_n^0]$; $\quad n = 1,...,N;$
(85) $V_{Bn}(p_n^0,p_n^1,q_n^0,q_n^1) \equiv (1/2)[p_n^0 + p_n^1][q_n^1 - q_n^0]$; $\quad n = 1,...,N.$

Note that the above partial indicators using real prices sum up to the overall Bennet indicators using real prices; i.e., we have:

(86) $I_B(p^0,p^1,q^0,q^1) = \sum_{n=1}^N I_{Bn}(p_n^0,p_n^1,q_n^0,q_n^1)$;
(87) $V_B(p^0,p^1,q^0,q^1) = \sum_{n=1}^N V_{Bn}(p_n^0,p_n^1,q_n^0,q_n^1)$.

We will relate the above observable partial indicators to *theoretical partial indicators*: for each n, define the *Laspeyres and Paasche partial price variations*, α_{Ln} and α_{Pn}, as follows:[41]

(88) $\alpha_{Ln} \equiv C(f(q^0),p_1^0,...,p_{n-1}^0, p_n^1, p_{n+1}^0,...,p_N^0) - C(f(q^0),p^0)$; $\quad n = 1,...,N$;
(89) $\alpha_{Pn} \equiv C(f(q^1),p^1) - C(f(q^1),p_1^1,...,p_{n-1}^1, p_n^0, p_{n+1}^1,...,p_N^1)$; $\quad n = 1,...,N$.

Thus the nth partial price Laspeyres variation, α_{Ln}, is the difference in real expenditure that would result if the standard of living of the consumer were held constant at the period 0 utility level, $u^0 \equiv f(q^0)$, and all real prices are also held constant at their period 0 levels except that we allow the nth real price to increase from the period 0 level, p_n^0, to the period 1 level, p_n^1. The nth partial price Paasche variation, α_{Pn}, has a similar interpretation except that the reference utility level is held constant at the period 1 level, $u^1 \equiv f(q^1)$, and all real prices are held constant at their period 1 levels and as before, we allow the nth real price to increase from the period 0 level, p_n^0, to the period 1 level, p_n^1.

It is possible to adapt the first order approximation methods used to derive the approximations (36) and (38) in the present context. Thus first order approximations to the unobservable terms in (88) and (89) can be obtained as follows: for n = 1,...,N, we have:

(90) $C(f(q^0),p_1^0,...,p_{n-1}^0, p_n^1, p_{n+1}^0,...,p_N^0) \approx C(f(q^0),p^0) + [\partial C(f(q^0),p^0)/\partial p_n][p_n^1 - p_n^0]$

[41] These variations are difference counterparts to the partial indexes defined in Diewert and Morrison (1986) and Kohli (1990).

$= C(f(q^0),p^0) + q_n^0[p_n^1 - p_n^0]$ using Shepard's Lemma (83) ;

(91) $C(f(q^1),p_1^1,\ldots,p_{n-1}^1, p_n^0, p_{n+1}^1,\ldots,p_N^1) \approx C(f(q^1),p^1) + [\partial C(f(q^1),p^1)/\partial p_n][p_n^0 - p_n^1]$
$= C(f(q^1),p^1) + q_n^1[p_n^0 - p_n^1]$ using Shepard's Lemma (83).

Substituting (90) and (91) into (88) and (89) leads to the following *observable first order approximations*, a_{Ln} and a_{Pn} to the Laspeyres and Paasche partial price variations, α_{Ln} and α_{Pn}:[42]

(92) $\alpha_{Ln} \approx q_n^0[p_n^1 - p_n^0] \equiv a_{Ln}$; $n = 1,\ldots,N$;
(93) $\alpha_{Pn} \approx q_n^1[p_n^1 - p_n^0] \equiv a_{Pn}$; $n = 1,\ldots,N$.

Thus using definitions (84) for the Bennet partial price indicators, $I_{Bn}(p_n^0,p_n^1,q_n^0,q_n^1)$, it can be seen that they are exactly equal to the arithmetic average of the Laspeyres and Paasche partial price indicators, $(1/2)a_{Ln} + (1/2)a_{Pn}$, which in turn approximate the average of theoretical Laspeyres and Paasche partial price variations, $(1/2)\alpha_{Ln} + (1/2)\alpha_{Pn}$, to the first order; i.e., we have:

Proposition 3:

(94) $I_{Bn}(p_n^0,p_n^1,q_n^0,q_n^1) = (1/2)a_{Ln} + (1/2)a_{Pn} \approx (1/2)\alpha_{Ln} + (1/2)\alpha_{Pn}$; $n = 1,\ldots,N$.

The above results are nonparametric; i.e., the approximations given by (92)-(94) are first order Taylor series approximations that are valid no matter what (once differentiable) preferences the consumer holds. However, if we assume that the consumer has preferences that can be represented by the translation homothetic normalized quadratic cost function C(u,P) defined by (58)-(62), then we can obtain an exact expression for the gap between the Bennet partial indicator on the left hand side of (94) and the average of the theoretical partial variations on the right hand side of (94); i.e., we can obtain the following expression for the *bias* BBP_n in the nth Bennet partial price indicator; i.e., we have:

(95) $I_{Bn}(p_n^0,p_n^1,q_n^0,q_n^1) = (1/2)\alpha_{Ln} + (1/2)\alpha_{Pn} + BBP_n$; $n = 1,\ldots,N$;
(96) $BBP_n \equiv -(1/2)\alpha_n[p^0 \cdot Bp^0 + p^1 \cdot Bp^1][p_n^1 - p_n^0]$

where α_n is the nth component in the weighting vector α that is used to form real prices. Since $\sum_{n=1}^N \alpha_n p_n^t = 1$ for t = 0,1, it can be seen that:

(97) $\sum_{n=1}^N BBP_n = 0$

[42] Using simple feasibility arguments for the cost minimization problems defined by the left hand sides of (90) and (91), it can be shown that $a_{Ln} \geq \alpha_{Ln}$ and $a_{Pn} \leq \alpha_{Pn}$ so that the Laspeyres partial price indicators a_{Ln} generally biased upwards for the true partial Laspeyres price indexes α_{Ln} and the Paasche partial price indicators a_{Pn} generally biased downwards for the true partial Paasche price indexes α_{Pn}; i.e., these partial price indicators will generally have some substitution bias, which will tend to cancel out when we take their averages.

so that the sum of the bias terms in the Bennet partial indicators $I_{Bn}(p_n^0,p_n^1,q_n^0,q_n^1)$ sums to zero.[43] Let P^* be the money metric utility scaling vector which appears in (60)-(62) and define its real counterpart by $p^* \equiv P^*/P^*\cdot\alpha$. If p^0 is proportional to p^*, then $p^0\cdot Bp^0$ is equal to 0 and if p^1 is proportional to p^*, then $p^1\cdot Bp^1$ is equal to 0 and under these conditions, it can be seen that all of the bias terms BBP_n will be equal to 0 as well. Hence if p^0 and p^1 are close to each other, then we can choose the reference price vector p^* to be close to p^0 and p^1 and the bias terms will all be close to 0.

Finding economic interpretations for the Bennet partial quantity indicators, $V_{Bn}(p_n^0,p_n^1,q_n^0,q_n^1)$, is more difficult. For each n, we first define the theoretical *Laspeyres and Paasche partial quantity variations*, β_{Ln} and β_{Pn}, as follows:

(98) $\beta_{Ln} \equiv C(f(q_1^0,...,q_{n-1}^0, q_n^1, q_{n+1}^0,...,q_N^0),p^0) - C(f(q^0),p^0)$; $\quad n = 1,...,N$;
(99) $\beta_{Pn} \equiv C(f(q^1),p^1) - C(f(q_1^1,...,q_{n-1}^1,q_n^0,q_{n+1}^1,...,q_N^1),p^1)$; $\quad n = 1,...,N$.

Thus the nth partial quantity Laspeyres variation, β_{Ln}, is the difference in real expenditure that would result if the real prices of the consumer are held constant at their period 0 levels p^0 and all quantities are also held constant at their period 0 except that we allow the nth quantity to increase from the period 0 level, q_n^0, to the period 1 level, q_n^1. The nth partial quantity Paasche variation, β_{Pn}, has a similar interpretation except that the reference prices are held constant at their period 1 levels p^1 and all quantities are also held constant at their period 1 levels except that we allow the nth quantity to increase from the period 0 level, q_n^0, to the period 1 level, q_n^1.

In order to obtain observable first order approximations to the theoretical quantity variations defined by (98) and (98), it is first necessary to develop some preliminary material. Define the function $h^t(q)$ for q's in a neighborhood of q^t as follows:

(100) $h^t(q) \equiv C(f(q),p^t)$; $\quad t = 0,1.$

Under our assumptions, $h^t(q)$ is once differentiable at q^t and we can calculate the vector of first order partial derivatives as follows:

(101) $\nabla_q h^t(q^t) = [\partial C(f(q^t),p^t)/\partial u]\, \nabla_q f(q^t)$; $\quad t = 0,1.$

Under our assumptions, q^t solves the cost minimization problem defined by $C(f(q^t),p^t)$ for $t = 0,1$ and since $f(q)$ is differentiable at q^t, there exists a nonnegative Lagrange multiplier λ^t such that the following first order necessary conditions for the period t cost minimization problem are satisfied:[44]

(102) $p^t = \lambda^t\, \nabla_q f(q^t)$; $\quad t = 0,1.$

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[43] This must be the case in order for Proposition 2 to hold.
[44] Strictly speaking, we require $q^t >> 0_N$ to ensure that conditions (102) are satisfied and later we will also require that marginal cost be positive so that $\partial C(f(q^t),p^t)/\partial u > 0$ for $t = 0,1$.

But Samuelson (1947) showed that the period t Lagrange multiplier λ^t which appears in (102) is also equal to the period t marginal cost around the equilibrium point so that we have:

$$(103)\ \lambda^t = \partial C(f(q^t),p^t)/\partial u\ ; \qquad t = 0,1.$$

Substituting (102) and (103) into (101) gives us the following simple expression for the derivatives of the function $h^t(q)$ defined by (100):

$$(104)\ [\partial C(f(q^t),p^t)/\partial u]\nabla_q f(q^t) \equiv \nabla_q h^t(q^t) = p^t\ ; \qquad t = 0,1.$$

Equations (104) seem to have been first derived by Balk (1989; 166) so we can call these relationships *Balk's Lemma*. With the above preliminary material out of the way, we can now proceed to the task of finding first order approximations to the theoretical partial quantity variations β_{Ln} and β_{Pn} defined by (98) and (99). Thus a first order approximation to the unobservable term $C(f(q_1^0,...,q_{n-1}^0, q_n^1, q_{n+1}^0,...,q_N^0),p^0)$ in (98) is:

$$\begin{aligned}(105)\ C(f(q_1^0,...,q_{n-1}^0, q_n^1, q_{n+1}^0,...,q_N^0),p^0) & & n = 1,...,N\\ \approx C(f(q^0),p^0) + [\partial h^0(q^0)/\partial q_n][q_n^1 - q_n^0] & & \text{using (100) for } t = 0\\ = C(f(q^0),p^0) + p_n^0[q_n^1 - q_n^0] & & \text{using (104) for } t = 0.\end{aligned}$$

Similarly, a first order approximation to the unobservable term $C(f(q_1^1,...,q_{n-1}^1,q_n^0,q_{n+1}^1,...,q_N^1),p^1)$ in (99) is:

$$\begin{aligned}(106)\ C(f(q_1^1,...,q_{n-1}^1,q_n^0,q_{n+1}^1,...,q_N^1),p^1) & & n = 1,...,N\\ \approx C(f(q^1),p^1) + [\partial h^1(q^1)/\partial q_n][q_n^0 - q_n^1] & & \text{using (100) for } t = 1\\ = C(f(q^1),p^1) + p_n^1[q_n^0 - q_n^1] & & \text{using (104) for } t = 1.\end{aligned}$$

Substituting (105) and (106) into (98) and (99) leads to the following *observable first order approximations*, b_{Ln} and b_{Pn} to the Laspeyres and Paasche partial quantity variations, β_{Ln} and β_{Pn}:[45]

$$(107)\ \beta_{Ln} \approx p_n^0[q_n^1 - q_n^0] \equiv b_{Ln}\ ; \qquad n = 1,...,N\ ;$$
$$(108)\ \beta_{Pn} \approx p_n^1[q_n^1 - q_n^0] \equiv b_{Pn}\ ; \qquad n = 1,...,N\ .$$

Thus using definitions (85) for the Bennet partial quantity indicators, $V_{Bn}(p_n^0,p_n^1,q_n^0,q_n^1)$, it can be seen that they are exactly equal to the arithmetic average of the Laspeyres and Paasche partial quantity indicators, $(1/2)b_{Ln} + (1/2)b_{Pn}$, which in turn approximate the

[45] Using simple feasibility arguments for the cost minimization problems defined by the left hand sides of (102) and (103), it can be shown that $b_{Ln} \geq \beta_{Ln}$ and $b_{Pn} \leq \beta_{Pn}$ so that the Laspeyres partial quantity indicators b_{Ln} generally biased upwards for the true partial Laspeyres quantity indexes β_{Ln} and the Paasche partial quantity indicators b_{Pn} generally biased downwards for the true partial Paasche quantity indexes β_{Pn}; i.e., these partial quantity indicators will generally have some substitution bias, which will tend to cancel out when we take their averages.

average of the theoretical Laspeyres and Paasche partial price variations, $(1/2)\beta_{Ln}$ + $(1/2)\beta_{Pn}$, to the first order; i.e., we have:

Proposition 4:

(109) $V_{Bn}(p_n^0,p_n^1,q_n^0,q_n^1) = (1/2)b_{Ln} + (1/2)b_{Pn} \approx (1/2)\beta_{Ln} + (1/2)\beta_{Pn}$; $n = 1,...,N$.

This completes our theoretical discussion of the properties of the Bennet indicators. In the following section, we illustrate the use of these indicators for a Japanese data set.

7. The Bennet Indicators using Japanese Data

In this section, we apply our methodology to Japanese consumption data. These data were constructed from the Japanese national accounts for 12 classes of expenditure for the period 1980-2006. The prices for each commodity class were normalized to equal one in 1980; see Tables B-1 and B-2 in Appendix B for a listing of the data. We chose food and non-alcoholic beverages to be our numeraire commodity and the resulting real prices are listed in Table B-3.[46] Aggregate expenditures evaluated in terms of real prices are 127753 billion yen in 1980 and 232679 in 2006. Therefore, household expenditures evaluated in real prices increased by 104927 billion yen over the last 27 years. We calculate Bennet indicators of quantity changes and real price changes to decompose the expenditure difference for every year. Table 1 lists value the real expenditure differences and the Bennet indicators for the period 1981-2006. Table 2 lists their annual averages. It tells us that the effects of real price changes are much smaller than the effects of quantity changes. However, the impact of real price changes has been significant for the last decade.

Table 1: Real Expenditure Differences and Bennet Indicators, 1981-2006

[46] This choice of deflator means that we used the weighting vector $\alpha = (1,0,\ldots 0)^T$.

Year	Difference	Bennet Quantity Indicator	Bennet Price Indicator
1981	2657.4	1966.8	690.7
1982	8980.9	6252.6	2728.3
1983	3191.4	4054.7	-863.3
1984	2352.7	3456.0	-1103.2
1985	6652.0	6002.3	649.7
1986	6622.4	4996.6	1625.7
1987	9040.9	6538.3	2502.6
1988	8122.2	7847.4	274.8
1989	8089.1	8398.1	-309.0
1990	5992.3	8497.3	-2505.0
1991	1489.5	5275.9	-3786.4
1992	6674.9	4586.8	2088.1
1993	2324.7	2438.6	-113.9
1994	5214.8	5253.6	-38.8
1995	6675.9	3224.2	3451.7
1996	4329.8	5314.7	-984.9
1997	1297.4	1683.8	-386.3
1998	-4934.2	-2125.0	-2809.2
1999	1424.7	1188.5	236.3
2000	4120.6	1851.2	2269.3
2001	3656.7	3969.5	-312.8
2002	1932.1	2188.8	-256.7
2003	4.3	1433.0	-1428.7
2004	480.4	3708.4	-3228.0
2005	4579.9	3944.9	635.1
2006	3953.4	6167.5	-2214.1

Table 2: Annual Averages of Real Expenditure Differences and Bennet Indicators

Year	Real Expenditure Difference	Bennet Quantity Indicator	Bennet Price Indicator
1980-2006	4035.6	4158.3	-122.6
1981-1990	6170.1	5801.0	369.1
1991-2000	2861.8	2869.2	-7.4
2001-2006	2434.5	3568.7	-1134.2

Our focus is on real consumption that measures the overall utility or volume of aggregate consumption. Real consumption can be computed throughout either the traditional ratio approach to quantity indexes or by the difference approach as outlined in this paper. However, if we use the ratio approach, the choice of specific index number formula could matter for the value of real consumption. Therefore, we use the difference approach as well as alternative index number formulae in order to evaluate the performance of the difference approach relative to that of the ratio approach.

Real consumption coincides with the corresponding nominal value at the reference year. Setting 1980 to be the reference year, we calculate different versions of real consumption for all years using the ratio approach and the difference approach. Fixed base and chained quantity indexes were computed using the Laspeyres, Paasche, Fisher and Törnqvist-Theil formulae.[47] The results are listed in Table 3 below. The last column of Table 3 lists the corresponding Bennet estimate of total consumption. The first entry in this column is simply the 1980 measure of Japanese total consumption expenditures divided by the price of food; i.e., the first entry in the second column of the Table. The next entry in the Bennet column just adds the Bennet measure of quantity change or volume change V_B defined by (41) above where the real price vectors and quantity vectors pertaining to the years 1980 and 1981 are used in the formula. The 1982 entry in the Bennet column is just the 1981 entry plus the Bennet measure of quantity change going from 1981 to 1982 and so on.

Looking at Table 3, it can be seen that all of the index number estimates of real Japanese consumption are very close to each other with the exception of the fixed base Laspeyres and Paasche estimates. This lack of correspondence is normal since these indexes are known to differ from their superlative counterparts when a fixed base is used. The superlative chained indexes are particularly close to each other. But how do these chained superlative indexes compare to the corresponding Bennet estimates of real consumption listed in the last column of Table 3? It can be seen that the Bennet measures are always equal to or greater than their chained superlative counterparts but the differences are not very large: on average, the Bennet estimate exceeds its chained Fisher counterpart by 0.74% per year, with a maximum deviation of 1.1%.

Table 3: Comparison of Japanese Real Consumption, 1980-2006

[47] Two versions of the Törnqvist-Theil were computed for both fixed base and chained indexes: one that constructed the price index first using the usual formula (and then the quantity index was defined by dividing real expenditures by this direct price index) and the other was constructed by directly comparing a share weighted average of log quantity changes and exponentiating. These two quantity indexes are superlative as is the Fisher index; see Diewert (1976) for the formula details.

Year	Total Expenditures deflated by the Price of Food	Ratio Approach										Difference Approach
		Fixed Base Index					Chained Index					
		Laspeyres Quantity Index	Paasche Quantity Index	Fisher Quantity Index	Tornqvist Quantity Index	Implicit Tornqvist Quantity Index	Laspeyres Quantity Index	Paasche Quantity Index	Fisher Quantity Index	Tornqvist Quantity Index	Implicit Tornqvist Quantity Index	Bennet Quantity Indicator
1980	127752.7	127752.7	127752.7	127752.7	127752.7	127752.6	127752.7	127752.7	127752.7	127752.7	127752.7	127752.7
1981	130410.1	129723.2	129705.2	129714.2	129714.5	129713.9	129723.2	129705.2	129714.2	129714.5	129714.1	129719.5
1982	139391.1	135942.2	135820.7	135881.4	135885.0	135881.6	135909.6	135830.6	135870.1	135871.1	135870.4	135972.1
1983	142582.5	139930.0	139723.9	139826.9	139829.8	139825.8	139890.4	139778.2	139834.3	139835.3	139834.5	140026.9
1984	144935.2	143397.3	143032.0	143214.5	143216.7	143211.8	143312.0	143160.8	143236.4	143237.4	143236.6	143482.8
1985	151587.2	149444.7	148913.2	149178.7	149179.0	149173.5	149246.3	149063.8	149155.0	149156.0	149155.2	149485.1
1986	158209.6	154316.2	153786.8	154051.3	154058.6	154048.9	154151.3	153940.0	154045.6	154047.0	154045.8	154481.7
1987	167250.5	160639.7	160067.6	160353.4	160365.5	160354.8	160474.4	160251.2	160362.8	160364.3	160363.1	161020.0
1988	175372.7	168160.1	167493.2	167826.3	167841.9	167828.9	168006.2	167755.6	167880.8	167882.7	167881.1	168867.4
1989	183461.8	176183.0	175390.8	175786.5	175810.7	175789.5	176072.5	175781.1	175926.7	175928.8	175927.1	177265.5
1990	189454.1	184481.0	183247.0	183863.0	183935.2	183864.1	184306.9	183951.3	184129.0	184132.2	184129.5	185762.8
1991	190943.7	189745.3	188051.9	188896.7	188939.3	188886.0	189504.2	189110.7	189307.3	189310.6	189307.8	191038.7
1992	197618.6	194323.3	192481.4	193400.1	193406.1	193384.4	194039.0	193622.0	193830.4	193833.8	193830.9	195625.6
1993	199943.2	196770.9	194760.7	195763.2	195728.0	195727.1	196436.0	196010.0	196222.9	196226.7	196223.3	198064.1
1994	205158.0	202159.1	199719.7	200935.7	200849.2	200882.8	201593.5	201165.2	201379.3	201383.0	201379.6	203317.7
1995	211834.0	205423.2	202857.5	204136.3	204059.7	204090.2	204745.7	204290.4	204517.9	204521.8	204518.2	206541.9
1996	216163.8	210890.5	207390.9	209133.4	209019.5	209049.0	209921.7	209399.4	209660.4	209665.1	209660.4	211856.6
1997	217461.2	212945.5	208639.4	210781.4	210627.0	210645.4	211585.7	211004.4	211294.8	211298.6	211294.5	213540.4
1998	212527.0	210785.1	206006.4	208382.1	208221.2	208186.7	209497.1	208936.5	209216.6	209219.0	209216.1	211415.4
1999	213951.7	212406.8	206864.1	209617.1	209381.3	209366.7	210687.9	210084.2	210385.9	210386.6	210385.3	212603.8
2000	218072.3	214655.4	208394.7	211501.9	211186.4	211205.9	212530.9	211863.0	212196.7	212196.6	212195.9	214455.0
2001	221729.0	219124.7	211575.8	215317.2	214843.1	214905.3	216449.2	215674.6	216061.6	216061.0	216060.5	218424.6
2002	223661.1	221564.5	213242.1	217363.5	216757.6	216910.3	218589.8	217802.2	218195.6	218194.5	218194.6	220613.4
2003	223665.4	223647.4	213901.5	218720.2	218000.7	218186.1	220041.5	219154.9	219597.8	219596.3	219596.3	222046.3
2004	224145.8	228274.4	216395.4	222255.5	221340.7	221579.9	223775.6	222753.6	223264.0	223261.8	223261.8	225754.8
2005	228725.7	233194.6	219024.9	225998.7	224784.3	225100.2	227751.1	226625.5	227187.6	227184.5	227184.7	229699.6
2006	232679.2	241317.8	223461.4	232218.0	230683.0	230972.0	234019.0	232665.4	233341.2	233336.2	233336.5	235867.2

What are we to conclude from the above results? For the Japanese data, it seems that a standard superlative index number approach to measuring aggregate real consumption will be fairly close to the results generated by the theoretically preferable Bennet approach, which has better aggregation over consumer properties and is consistent with nonhomothetic preferences. However, there seem to be small but significant differences between the Bennet estimates and those generated by chained superlative indexes.

8. Conclusion

This paper has established satisfactory difference theory counterparts to the standard results on exact and superlative indexes in the ratio approach to the aggregation over commodities problem. The counterpart to a superlative index number formula is a superlative indicator formula. We found that the Bennet indicators of price and quantity change were (strongly) superlative and thus we recommend their use in practical applications of cost benefit analysis when ex post variations must be calculated.

In section 7 above, we found that, somewhat surprisingly, the results using the Bennet indicator of quantity change are rather close to the quantity aggregates generated by a superlative quantity index. This is somewhat reassuring in that the ratio and difference approaches to economic aggregation seem to give more or less the same answer, at least for our Japanese data set.

Finally, we mention one strong advantage of the difference approach over the ratio approach: the ratio approach fails if the quantity aggregate has a value equal to zero in the base period whereas the difference approach is unaffected by this complication. This observation is important if labour supply enters the consumer's utility function (negatively rather than positively) since in this case, zero or negative value aggregates can readily occur. Although we did not formally model this situation, we are confident that our techniques can be generalized to cover this situation.

Appendix A: On the Flexibility of the Translation Homothetic Normalized Quadratic Cost Function.

Let $P^* >> 0_N$ be an arbitrary predetermined reference price vector and let $\alpha > 0_N$ be a predetermined weighting vector. Define the translation homothetic normalized quadratic cost function, C(u,P) by (58) where the two parameter vectors b and c and the parameter matrix B satisfy the restrictions (59)-(62). The restrictions (61) and (62) imply that the b and c vectors each have only N − 1 independent parameters, b_n and c_n respectively, while the restrictions (59) and (60) imply that the N by N matrix $B \equiv [b_{nm}]$ has only N(N−1)/2 independent parameters b_{mn}. Thus this functional form has 2N − 2 + N(N−1)/2 independent parameters in all.[48] In this Appendix, we will show that this C(u,P) is flexible in the class of cost functions satisfying Conditions II over a region of utility levels u and price vectors P.

Let $C^*(u,P)$ be an arbitrary cost function satisfying Conditions II and suppose that it is twice continuously differentiable at $u^* > 0$ and $P^* >> 0_N$. We assume that it satisfies *money metric utility scaling* at the reference prices P^* so that

(A1) $C^*(u,P^*) = u$ for all $u \geq 0$.

In order for C(u,P) defined by (58)-(62) to be flexible at (u^*,P^*), the following equations need to be satisfied for some choice of b, c and B:

(A2) $C^*(u^*,P^*) = C(u^*,P^*) = u^*$ using (58)-(62);
(A3) $\nabla_P C^*(u^*,P^*) = \nabla_P C(u^*,P^*) = b + cu^*$ using (58)-(62);
(A4) $\partial C^*(u^*,P^*)/\partial u = \partial C(u^*,P^*)/\partial u = 1$ using (58) and (62)
(A5) $\nabla^2_{PP} C^*(u^*,P^*) = \nabla^2_{PP} C(u^*,P^*) = (\alpha \cdot P^*)^{-1} B$ using (58)-(60);
(A6) $\partial^2 C^*(u^*,P^*)/\partial u^2 = \partial^2 C(u^*,P^*)/\partial u^2 = 0$ using (58);
(A7) $\nabla^2_{Pu} C^*(u^*,P^*) = \nabla^2_{Pu} C(u^*,P^*) = c$ using (58).
(A8) $\nabla^2_{uP} C^*(u^*,P^*) = \nabla^2_{uP} C(u^*,P^*) = c^T$

The number of equations in (A2)-(A8) is $1+N+1+N^2+1+N+N$. However, Young's Theorem on the equality of second order partial derivatives implies that there are only N(N+1)/2 independent equations in (A5) instead of N^2 and the N equations in (A8) are implied by the N equations in (A7). This leaves 3 + 2N + N(N+1)/2 equations to be satisfied. However, both $C^*(u,P)$ and C(u,P) are positively linearly homogeneous in the prices P. Hence Euler's Theorem on homogeneous functions implies the following 3 sets of further restrictions on the derivatives of C^* and C:

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[48] This turns out to be the minimal number of parameters required for a functional form to be flexible in the class of cost functions satisfying Conditions II; hence this flexible functional form is also *parsimonious*.

$$\text{(A9)}\quad C^*(u^*,P^*) = P^*\cdot\nabla_P C^*(u^*,P^*)\ ; \qquad C(u^*,P^*) = P^*\cdot\nabla_P C(u^*,P^*)\ ;$$
$$\text{(A10)}\quad \nabla^2_{PP}C^*(u^*,P^*)P^* = 0_N\ ; \qquad \nabla^2_{PP}C(u^*,P^*)P^* = 0_N\ ;$$
$$\text{(A11)}\quad P^*\cdot\nabla^2_{Pu}C^*(u^*,P^*) = \partial C^*(u^*,P^*)/\partial u\ ; \qquad P^*\cdot\nabla^2_{Pu}C(u^*,P^*) = \partial C(u^*,P^*)/\partial u\ .$$

Thus there are 1+N+1 further equations which can be dropped which leaves $1 + 2N + N(N-1)/2$ equations to be satisfied.

Finally, both C^* satisfies (A1) and C satisfies (20) in the main text; i.e., both C and C^* satisfy money metric utility scaling at the reference prices P^*. Differentiation of (A1) and (20) gives us the following additional 3 restrictions on the levels and derivatives of C and C^*:

$$\text{(A12)}\quad u^* = C^*(u^*,P^*) = P^*\cdot\nabla_P C^*(u^*,P^*)\ ; \qquad u^* = C(u^*,P^*) = P^*\cdot\nabla_P C(u^*,P^*) \qquad \text{using (A9);}$$
$$\text{(A13)}\quad \partial C^*(u^*,P^*)/\partial u = 1\ ; \qquad \partial C(u^*,P^*)/\partial u = 1;$$
$$\text{(A14)}\quad \partial^2 C^*(u^*,P^*)/\partial u^2 = 0\ ; \qquad \partial^2 C(u^*,P^*)/\partial u^2 = 0.$$

Thus we will require that C have at least $2N - 2 + N(N-1)/2$ free parameters so that this number of independent equations can be satisfied. Using the above material, it can be seen that we will satisfy all of the equations (A2)-(A8) if we can find b, c and B which satisfy equations (A3), (A5) and (A7) where the chosen b,c and B must satisfy the restrictions (59)-(62). This can readily be done. Use equations (A5) in order to define B as follows:

$$\text{(A15)}\quad B \equiv \alpha\cdot P^*\ \nabla^2_{PP}C^*(u^*,P^*).$$

Since $\alpha > 0_N$ and $P^* \gg 0_N$, $\alpha\cdot P^*$ is greater than 0. Since $C^*(u,P)$ is concave in P, $\nabla^2_{PP}C^*(u^*,P^*)$ is a negative semidefinite symmetric matrix and hence so is B. Since $C^*(u,P)$ is linearly homogeneous in P, (A10) holds and so $\nabla^2_{PP}C^*(u^*,P^*)P^* = 0_N$. Thus the B defined by (A15) satisfies the restrictions (59) and (60). Now use equations (A7) in order to define c:

$$\text{(A16)}\quad c \equiv \nabla^2_{Pu}C^*(u^*,P^*).$$

(A11) and (A13) will imply that the c defined by (A16) satisfies the restrictions (62) in the main text. Now define u^* using (A2):

$$\text{(A17)}\quad u^* \equiv C^*(u^*,P^*).$$

Finally, define b using (A3) and definitions (A16) and (A17):

$$\text{(A18)}\quad b \equiv \nabla_P C^*(u^*,P^*) - \nabla^2_{Pu}C^*(u^*,P^*)C^*(u^*,P^*).$$

We need to verify that the b defined by (A18) satisfies the restriction (61) in the main text. Using definition (A18), we have:

(A19) $P^{*}\cdot b = P^{*}\cdot[\nabla_P C^{*}(u^{*},P^{*}) - \nabla^2_{Pu}C^{*}(u^{*},P^{*})C^{*}(u^{*},P^{*})]$
$= C^{*}(u^{*},P^{*}) - [\partial C^{*}(u^{*},P^{*})/\partial u]C^{*}(u^{*},P^{*})]$ using (A9) and (A11)
$= C^{*}(u^{*},P^{*}) - 1C^{*}(u^{*},P^{*})$ using (A13)
$= 0.$

Thus the parameter vector b defined by (A18) does indeed satisfy the restriction (61) and this completes our proof of the flexibility of the translation homothetic normalized quadratic functional form.

Suppose C(u,P) is defined by (58) where B, b and c satisfy the restrictions (59)-(62). The region of prices and utility levels where C satisfies the appropriate regularity conditions for a cost function are the set of P and u which satisfy the following inequalities:

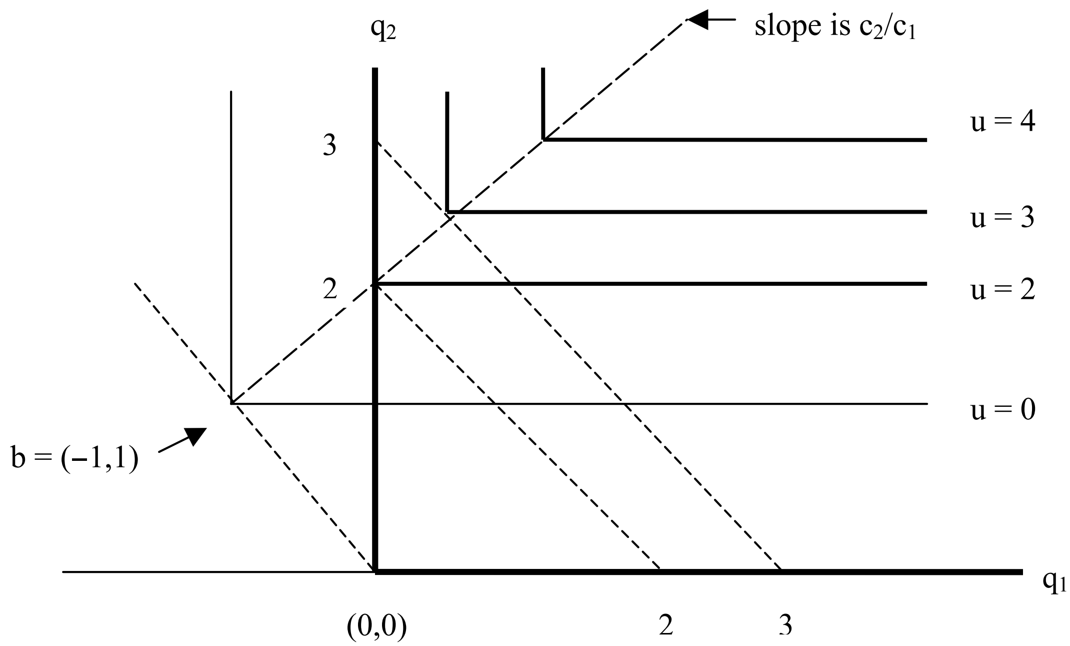
(A20) $u \geq 0$; $P \geq 0_N$;
(A21) $\nabla_P C(u,P) = b + (\alpha\cdot P)^{-1}BP - (1/2)\,(\alpha\cdot P)^{-2}\,P\cdot BP\alpha + cu \geq 0_N$;
(A22) $\partial\, C(u,P)/\partial u = c\cdot P > 0.$

We will illustrate what the preferences dual to the normalized quadratic translation homothetic cost function look like for the case of two commodities. For all of the examples defined below, we choose the price weighting vector α which is used to form real prices as $\alpha^T = [\alpha_1,\alpha_2] \equiv [1,0]$ and the reference price vector for money metric utility scaling P^* to be $P^{*T} = [P_1^{*},P_2^{*}] \equiv [1,1]$. For *Example 1*, define the parameters in (58) as follows:

(A23) $b^T = [b_1,b_2] \equiv [-1,1]$; $c^T = [c_1,c_2] \equiv [1/2,1/2]$; $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} \equiv \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

The preferences corresponding to this functional form are graphed in Figure 1.

Figure 1: Leontief Translation Homothetic Preferences with No Inferior Goods



It can be seen that the regular region of utility levels and price vectors for this cost function is $u \geq 2$ and $P \geq 0_N$. The dual direct utility function is defined only over the set of quantity vectors such that $q_1 \geq 0$ and $q_2 \geq 2$. Note that all of the indifference curves are simply parallel shifts of a base Leontief or L shaped indifference curve that goes through the point $b = [-1,1]$. The consumer's income expansion path or Engel curve is the dashed line that passes through (0,2) and has slope $c_2/c_1 = 1$. Note also that $b \cdot P^* = (-1)(1) + (1)(1) = 0$. Any point $q \equiv [q_1,q_2]$ that lies along the dashed line through b and the origin will have the property that $q \cdot P^*$ will equal 0 and so as we vary b along this dashed line and vary the c vector, we will be able to approximate an arbitrary Engel curve locally by using this functional form.[49] Finally, to illustrate money metric utility scaling, shift the dashed line through the point b and the origin (0,0) in a parallel fashion until it is just tangent to an indifference curve; we have drawn two of these parallel budget lines that are tangent to the u = 2 and u = 3 indifference curves. The distance of these lines from the origin serves to cardinalize utility.

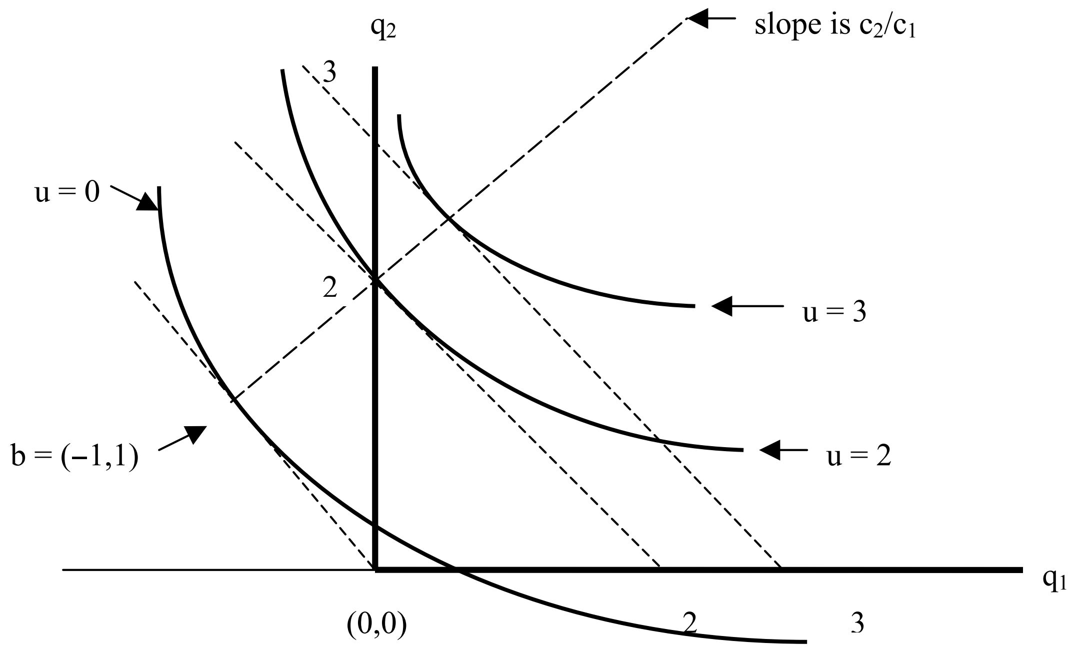
Now consider *Example 2* and define the parameters in (58) as follows:

(A24) $b^T = [b_1,b_2] \equiv [-1,1]$; $c^T = [c_1,c_2] \equiv [1/2,1/2]$; $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} \equiv \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$.

The preferences that correspond to this functional form are graphed in Figure 2.

Figure 2: Translation Homothetic Preferences with No Inferior Goods

[49] This is a special case of the class of cost functions studied by Lau and Tamura (1972).



The dashed line through the point b and the origin is the set of q_1 and q_2 that satisfy the equation

(A25) $P_1^* q_1 + P_2^* q_2 = P_1^* b_1 + P_2^* b_2 = 0$.

Note that the base indifference curve (which corresponds to the zero utility level u = 0) is tangent to this budget line defined by (A26) and the higher utility indifference curves are simply parallel shifts of this base indifference curve. It can be shown that the points $(0, 2-3^{1/2})$ and (1/2, 0) are on the base indifference curve. It can also be shown that as P_2/P_1 tends to zero, the upper limiting point on the base indifference curve tends to the point (−3/2, 2)[50] and as P_2/P_1 tends to plus infinity, the lower limiting point on the base indifference curve tends to (∞, −∞). As was the case with the Figure 1 preferences, The consumer's income expansion path or Engel curve is the dashed line that passes through (0,2) and has slope $c_2/c_1 = 1$. Again, to illustrate money metric utility scaling, shift the dashed line through the point b and the origin (0,0) in a parallel fashion until it is just tangent to an indifference curve; we have drawn two of these parallel budget lines that are tangent to the u = 2 and u = 3 indifference curves. The distance of these lines from the origin serves to cardinalize utility. Thus for example, all of the points on the u = 2 indifference curve are assigned the utility level 2 while all of the points on the u = 3 indifference curve are assigned the utility level 3.

[50] This indifference curve can be extended to cover higher levels of q_2 in the obvious way.

The regular region of prices and utility levels is more difficult to describe in a succinct fashion. Basically, given u[51] and $P >> 0_N$, use Shephard's Lemma to generate $q \equiv \nabla_P C(u,P)$ and check whether this q is nonnegative. If so, then the given u and P belong to the regular region. In our example, the vector c had positive components so the regularity condition $\partial C(u,P)/\partial u = c \cdot P > 0$ will automatically be satisfied.

Note that the vector c is equal to $\nabla^2_{uP} C(u,P)$ which in turn is equal to the vector of derivatives of the consumer's Hicksian demand functions, $q(u,P) \equiv \nabla_P C(u,P)$, with respect to an increase in utility u and thus indicates how the household's consumption changes as utility increases. If $\partial q_n(u,P)/\partial u = \partial^2 C(u,P)/\partial P_n \partial u$ is negative, then we say that commodity n is an *inferior* commodity at u, P. Although it is easy to show that not all commodities can be inferior at a particular point, there is nothing to prevent one or more commodities from being inferior. Hence if the consumer's preferences are represented by a translation homothetic normalized quadratic cost function in a neighbourhood of a point where the consumer has one or more inferior commodities, then the parameter vector c will have one or more negative components. This means that there will exist positive price vectors P such that $c \cdot P$ is negative and hence C(u,P) defined by (58) cannot be regular at these price vectors. Fortunately, it is possible to show that the cost function defined by (58) can still be locally regular and provide a valid representation of a consumer's nonhomothetic preferences in a neighborhood of a point (u,P) where C(u,P) satisfies the required regularity conditions for a cost function locally.

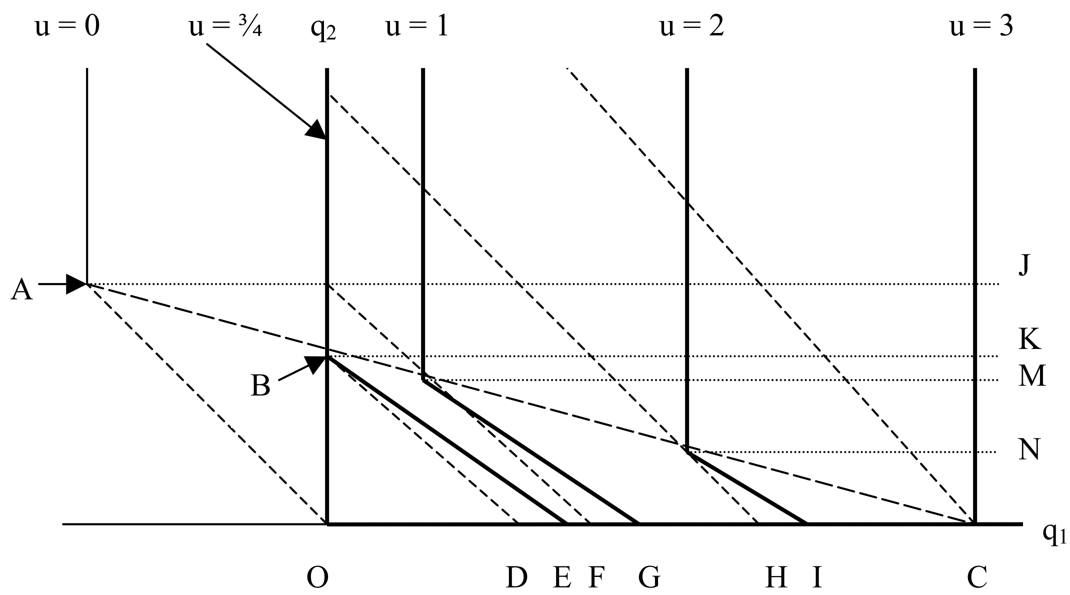
We conclude this section by considering an example where the second commodity is inferior. For simplicity, we consider again the case of Leontief preferences. Thus for *Example 3*, define the parameters in (58) as follows:

(A26) $b^T = [b_1,b_2] \equiv [-1,1]$; $c^T = [c_1,c_2] \equiv [4/3,-1/3]$; $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} \equiv \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

The preferences corresponding to this functional form are graphed in Figure 3.

Figure 3: Leontief Translation Homothetic Preferences with an Inferior Good

[51] The utility level u does not have to be nonnegative in this example. If we allow negative utility levels, then the dual preferences f(q) are well defined for all $q \geq 0_N$; i.e., if we restrict ourselves to $u \geq 0$, then we will not be able to define the consumer's preferences in the little triangle which is below the u = 0 indifference curve intersected with the nonnegative quadrant.



The u = 0 indifference curve is the L shaped curve that has its corner at the point A, which is the point $(q_1,q_2) = (-1,1) = (b_1,b_2) = b$. As usual, the point b lies on the line $P^*\cdot q = 0$ and the dashed line segment AO is part of this line. The dashed line segments ending in D, F, H and C are all parallel to the line segment AO. The consumer's Engel curve at the reference prices $P^{*T} = [P_1^*,P_2^*] = [1,1]$ is the intersection of the line segment AC with the nonnegative orthant which is the line segment BC. The first regular point on this line segment is the point B which corresponds to the point $(q_1,q_2) = (0,3/4)$ and the corresponding money metric utility level is u = 3/4. L shaped indifference curves that are translations of the base indifference curve corresponding to u = 0 have been drawn for the utility levels u = 3/4, u = 1, u = 2 and u = 3. However, it can be seen that there is a problem with these indifference curves: they cross each other! Geometrically, it is easy to solve this problem: for the u = 3/4 indifference curve, replace the lower line segment BK by the line segment BE, where the point E on this line segment must be strictly between the points D and C. Similarly replace the bottom part of the u = 1 indifference curve that passes through M by the line segment that ends at the point G where this line segment is parallel to the line segment BE, replace the bottom part of the u = 2 indifference curve that passes through N by the line segment that ends at the point I where this line segment is also parallel to the line segment BE, and so on. The resulting system of indifference curves no longer cross and are well behaved. Algebraically, we need to restrict the prices P_1 and P_2 so that the restrictions (A22), $c\cdot P > 0$, are satisfied. Under assumptions (A26), (A22) becomes the following restriction on prices:

(A27) $P_2 < 4\,P_1$.

In order to satisfy the restrictions (A21), we require the following restrictions on u:

(A28) $\frac{3}{4} \le u \le 3$.

The inequalities (A27) and (A28) define the regular region for the Example 3 preferences.

For more information on how local information on a cost function can be used to form local approximations to utility functions, see Blackorby and Diewert (1979).

Viewing this last example, it can be seen that when there are inferior goods, the regular region may not be very large; i.e., our suggested functional form will not be able to provide an adequate global approximation to arbitrary preferences. However, with all of the examples, it can be seen that if relative prices do not change too much and the utility levels in the two periods being compared are fairly close, translation homothetic normalized quadratic preferences will be able to provide a good local approximation to arbitrary preferences. On the other hand, if relative prices differ markedly and/or utility levels differ considerably, then the approximation may not be very close. But having results that are exact for second order approximations to arbitrary preferences is better than having results that are exact for only first order approximations!

Appendix B: The Japanese Consumption Data

We use the household consumption data in the Japanese national accounts. Household consumption is classified into 12 categories; (1) Food and non-alcoholic beverages; (2) Alcoholic beverages and tobacco; (3) Clothing and footwear; (4) Housing, electricity, gas and water supply; (5) Furnishings, household equipment and household services; (6) Health; (7) Transport; (8) Communication; (9) Recreation and culture; (10) Education; (11) Restaurants and hotels; (12) Miscellaneous goods and services.

Expenditure series for each category are provided at current prices and constant prices in the national accounts. For the years 1980-2003, current and constant yen series for the 12 consumption goods are found in the Annual Report on National Accounts of 2005; see the Economic and Social Research Institute (2005); Part 1 Flows; 5. Supporting Tables; (13) Composition of Final Consumption Expenditure of Households classified by Purpose; Calendar Year, in billions of yen. The constant yen series are at the market prices of 1995. For the years 1996-2006, current and constant yen series for the 12 consumption goods are found on Annual Report on National Accounts of 2008; see the Economic and Social Research Institute (2007). The constant yen series are at the market prices of 2000. We link the current and constant yen series of these two reports to construct current and constant prices for the 12 goods for the period 1980-2006. For the years 1996-2006, we use current and constant yen series taken from the Annual Report on the National Accounts of 2008. We interpolate these series backward by using the growth rate of current and constant yen series taken from Annual Report on the National Accounts of 2005. The price of each good is implicitly derived from the current and constant yen series. We normalize all the prices so that prices in 1980 are one and adjust the corresponding quantity series in order to preserve values. We regard the constant yen series as the quantity series. We list the resulting price and quantity series for 1980-2006 in Table B-1 and Table B-2 respectively.

Table B-1: Prices of Consumption Goods

Year	Food and non-alcoholic beverages	Alcoholic beverages and tobacco	Clothing and footwear	Housing ,electricity ,gas and water supply	Furnishings ,household equipment and household services	Health	Transport	Communication	Recreation and culture	Education	Restaurants and hotels	Miscellaneous goods and services
1980	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1981	1.04162	1.07216	1.03888	1.05957	1.02152	1.02092	1.04887	1.04322	1.06233	1.07679	1.05108	1.03687
1982	1.04697	1.08538	1.06201	1.10067	1.02653	1.04762	1.12421	1.03614	1.07852	1.14704	1.08435	1.05841
1983	1.07685	1.13687	1.08312	1.13579	1.02254	1.05256	1.12714	1.03037	1.10095	1.20658	1.11704	1.07842
1984	1.11502	1.21343	1.10829	1.16822	1.03326	1.06930	1.14733	1.01435	1.13302	1.26061	1.14997	1.10152
1985	1.12865	1.22243	1.13071	1.19664	1.03306	1.10449	1.16704	1.00670	1.17007	1.31554	1.17421	1.09722
1986	1.12573	1.26110	1.15100	1.20559	1.04714	1.13071	1.15367	1.00081	1.18575	1.36529	1.20012	1.09963
1987	1.11440	1.27879	1.16626	1.22047	1.04753	1.15577	1.16040	1.00150	1.18393	1.41058	1.21627	1.10285
1988	1.11997	1.27435	1.17891	1.23856	1.04888	1.15943	1.16186	0.98687	1.18153	1.45914	1.23302	1.10879
1989	1.14457	1.27041	1.22581	1.26618	1.05149	1.17543	1.17658	0.99035	1.20139	1.51821	1.28285	1.12579
1990	1.18888	1.28750	1.28017	1.30016	1.05341	1.19413	1.18787	0.97857	1.23202	1.60228	1.32780	1.14023
1991	1.24622	1.30227	1.33633	1.33486	1.06092	1.19979	1.21371	0.95705	1.26447	1.66855	1.37310	1.15870
1992	1.25307	1.30740	1.37554	1.36658	1.06566	1.24310	1.22328	0.93804	1.29031	1.73603	1.40949	1.17392
1993	1.26774	1.30678	1.37512	1.39606	1.05435	1.26089	1.23661	0.91270	1.29919	1.80541	1.43427	1.18747
1994	1.27496	1.32120	1.36187	1.42254	1.02534	1.27442	1.23729	0.88941	1.28709	1.86929	1.44748	1.20909
1995	1.25026	1.32541	1.35712	1.44468	1.00444	1.28208	1.22791	0.92536	1.25411	1.92655	1.44297	1.21485
1996	1.25226	1.32096	1.36823	1.46079	0.97937	1.29761	1.21633	0.88694	1.19769	1.97597	1.44650	1.22327
1997	1.27192	1.33666	1.39597	1.50130	0.97486	1.31024	1.22771	0.86328	1.18973	2.01820	1.48891	1.23723
1998	1.29105	1.34005	1.41566	1.51774	0.95657	1.30974	1.21124	0.82059	1.17403	2.05449	1.49427	1.24117
1999	1.28641	1.38709	1.41436	1.50993	0.94419	1.29259	1.21288	0.79750	1.15311	2.08789	1.49443	1.25487
2000	1.26024	1.38261	1.40445	1.50566	0.90247	1.28611	1.22926	0.77160	1.11376	2.12104	1.48423	1.25409
2001	1.24679	1.37807	1.38329	1.49958	0.85999	1.28711	1.23033	0.73706	1.04903	2.13690	1.47162	1.27311
2002	1.23008	1.37027	1.35960	1.48518	0.81462	1.27004	1.21938	0.73040	1.00066	2.16401	1.46972	1.26947
2003	1.22405	1.39253	1.34381	1.47242	0.77259	1.26163	1.22346	0.73076	0.94145	2.18028	1.47433	1.26684
2004	1.23003	1.41678	1.34030	1.45903	0.72654	1.24966	1.23720	0.72653	0.88308	2.19088	1.48340	1.26366
2005	1.21146	1.41617	1.34573	1.44856	0.69341	1.24689	1.26614	0.69996	0.81847	2.21460	1.48515	1.26429
2006	1.21304	1.43922	1.35569	1.44506	0.66537	1.23537	1.28849	0.68186	0.75676	2.21964	1.49338	1.26790

Table B-2: Quantities of Consumption Goods

Year	Food and non-alcoholic beverages	Alcoholic beverages and tobacco	Clothing and footwear	Housing ,electricity ,gas and water supply	Furnishings ,household equipment and household services	Health	Transport	Communication	Recreation and culture	Education	Restaurants and hotels	Miscellaneous goods and services
1980	27979.8	5406.9	10906.6	22524	6773.2	5243.7	12836.1	1337.6	9851.1	2899	9279.5	12715.1
1981	28510.8	5253	10648.9	23231.6	6794	5989.5	12830.9	1519.3	9993.3	2900.1	9029.5	13022.1
1982	29430.6	5447.7	11260.5	23815.8	7622	6779.4	12701.3	1793.1	10970.9	2871	9412.9	13836.9
1983	29734.4	5486.4	11390.7	24493.4	8262	7002.9	13329.4	2013.9	11506.7	2895.2	9432.9	14382.1
1984	29726.9	5152.7	11562.3	25417.8	8863.7	7072.5	13368.6	2252.5	12186	2890.3	9751.7	15152.2
1985	30525.2	5111.2	11698.3	26347.2	9722.7	7448.2	13454.1	2418	12879.5	2865.2	10264.9	16710.1
1986	30637.9	5078.4	12083.5	26918.3	9555	7117.5	14553.8	2512.3	14328.7	2933.2	10415.1	18182.5
1987	30901.4	5153.8	12567	27922.6	9930.5	7009.9	15752.4	2534.7	15707.7	2987.1	10705	19467.8
1988	31245.2	5296.6	12889.7	29059	10414.3	6843	17832.4	2631.9	17404.3	3065	11013.4	20465.4
1989	31361	5580.8	13143	30659.8	10639.8	6614.8	20079.2	2690.6	19274.4	3167	11027.1	21945.4
1990	31661.6	5936.2	13194.3	32219.9	11520.2	6523.1	21872.1	2820.6	22051.8	3178.2	10896.8	22606.2
1991	31936.4	6075.1	13347.7	33795.6	12338.9	6547.8	22633.1	3095.2	22331.2	3231.9	11266.2	23146.1
1992	32724.6	6207.7	12904.5	35358.9	12584.2	6392.3	23059.4	3321.5	22187.6	3259	12007.8	24315.8
1993	32801.6	6247.8	12806.9	36709.1	12854.6	6338.2	23170.8	3956.9	22198.2	3263.7	12463.3	23959.9
1994	33203.4	6511.2	12823.5	37792	13528	6643.4	23599.9	4324	22050.8	3199.5	13140.9	25342.4
1995	34143.8	6710.4	13395.8	38563.1	14169.2	6587.9	23772.7	4586.7	22544.2	3134.6	13191.4	24623.6
1996	34131.8	6762.4	13509.2	39533.8	14231.2	6566.5	24697.8	5685.3	23595.5	3101.3	13681.5	25394.3
1997	33560.4	6729.2	12286.8	40206.5	14178.9	6911.9	24888.4	6564	25154	3054.5	13693	25717.9
1998	34117.3	7017.8	11293.5	40810.9	13574.4	7132.6	22911	7283.1	24820.6	2960.5	13794.4	25069.1
1999	34389.4	6973	9924.6	41648.9	13579.8	7798.1	23125.4	8342.6	25104	2901.3	13873.5	24746.1
2000	34508.9	6952.9	9027.5	42669.4	13236	8045.9	23367.4	8946.8	27752.1	2852	13658	23638.4
2001	34920.3	7002.2	8662.9	43505.4	13498.9	8424.1	23786.1	9856.6	29575.4	2836	14079.2	22977.7
2002	35032.3	7064.9	7988.3	44344.6	13301.9	8677	23854.9	10529.2	30515.1	2812.5	14169.5	23274.2
2003	34544.8	6660.2	7662.5	45170.2	13587.1	9231.2	23633	11030.7	32334.1	2773.9	14009.1	23010.8
2004	34388.1	6540	7383.2	46038.6	14163.5	9202.4	23726.1	11584.3	34894.6	2829	14029.6	23494.8
2005	33583.9	6277.2	7347.2	47045.9	14945.4	9563.4	23990.4	12139.4	36975.4	2801.8	14243.5	24281.1
2006	33424	6098.5	7582.7	48066.7	15915.5	9644.2	24167.1	12878.3	41188.1	2783.1	14401.1	25168.5

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