New Methodological Developments for the International Comparison Program

Erwin Diewert,[1]
Discussion Paper 08-08,
Department of Economics,
The University of British Columbia,
Vancouver, Canada, V6T 1Z1.
email: diewert@econ.ubc.ca

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Abstract

The paper explains the new methodology that was used in the 2005 International Comparison Program (ICP) that compared the relative price levels and GDP levels across 146 countries. In this round of the ICP, the world was divided into 6 regions: OECD, CIS, Africa, South America, Asia Pacific and West Asia. What is new in this round compared to previous rounds of the ICP is that each region was allowed to develop its own product list and collect prices on this list for countries in the region. The regions were then linked using another separate product list and 18 countries across the 6 regions collected prices for products on this list and this information was used to link prices and quantities across the regions. An additional complication was that the final linking of prices and volumes across regions had to respect the regional price and volume measures that were (separately) constructed by the regions. The paper also studies the properties of the Iklé Dikhanov Balk multilateral system of index numbers which was used by Africa.

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Keywords

Index numbers, multilateral comparison methods, GEKS, EKS, Geary-Khamis, Balk, Dikhanov, Iklé, Country Product Dummy (CPD) method, basic headings, Structured Product Descriptions, Purchasing Price Parities (PPPs), representative products, spatial chaining, fixity.

1. Introduction

The final results for the 2005 International Comparison Program (ICP) have been released in February; for a tabulation of the results, see the World Bank (2008). The

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program compared the level of prices and the quantities or volumes of GDP (and its components) for 146 countries for the year 2005. International price statisticians developed Structured Product Descriptions (SPDs) for approximately 1000 products[2] and the individual countries collected price information on these products for the year 2005. The 1000 products were grouped into 155 Basic Heading (BH) categories. The price information collected in each country was then compared across countries, leading to a matrix of 155 basic heading prices by 146 countries. The precise way in which the individual product prices in each BH category were aggregated into a single country price for each BH heading is the topic which will be investigated in sections 2 and 3 below.

The 2005 ICP differed from previous ICP rounds.[3] In previous rounds, each country attempted to find prices in their country for a common product list. However, it is difficult to find products that are representative for *all* countries in the world and so the decision was made to break up the world into 6 regions and price statisticians developed *separate* product lists for each region. The 6 regions were: (1) Africa with 48 participating countries; (2) South America with 10 countries; (3) Asia Pacific with 23 countries; (4) The Commonwealth of Independent States (CIS) with 10 countries; (5) West Asia with 11 countries and (6) the OECD and other European countries covered by Eurostat plus Israel and Russia adding up to 46 countries in this region. This sums to 148 countries but Egypt appears in both the African and West Asia regions and Russia appears in both the OECD and CIS regions so there are 146 participating countries in all.

The fact that the product lists in each region were allowed to be different across regions means that without further information, prices and volumes could not be compared across regions. However, the World Bank, in cooperation with other national and international statistical agencies, developed an additional product list, which was priced out by 18 selected countries across the regions. These 18 countries were called *ring countries*. The prices that were collected by the ring countries using this final product list enabled price comparisons to be made across the 6 regions. We will indicate how this was done at the Basic Heading level in section 3 below and in section 5, we will indicate how comparisons at higher levels of aggregation between regions were made.

There was another methodological innovation made in this current ICP round in addition to having regional product lists: the price parities or Purchasing Power Parities (PPPs) and relative volumes for each country were determined using information on prices and GDP expenditure shares that pertained only to countries within the given region and these parities and relative volumes were preserved in the world comparison. Thus each region was independently allowed to determine its country PPPs and volume shares and the final linking of the regional results into a global world comparison left these regional relative parities undisturbed.[4]

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[2] Most of the products referred to are components of individual consumption: "There are about 830 SPDs that cover 100 Basic Headings for individual consumption. Each SPD contains price determining characteristics that will define unique products from any corner of the world." Dennis Trewin (2008; 8). For an overview of the organization and methodology used in the 2005 ICP, see Vogel (2008).

[3] For an overview of previous ICP rounds and an assessment of the current round, see Heston and Summers (2008).

[4] Egypt is an exception to this statement as will be explained below.

The final results from the 2005 International Comparison Program for the 146 participating countries are available on the World Bank website; see the World Bank (2008) for these results and explanations for various difficulties that were encountered. This publication explained the basic framework for the provision of the data as follows:

"The purchasing power parities and the derived indicators in this report are the product of a joint effort by national statistical offices, regional coordinators, and the ICP global office. PPPs cannot be computed in isolation by a single country. However, each country was responsible for submitting official estimates of 2005 gross domestic product and its components, population counts, and average exchange rates. The regional coordinators worked with the national statistical offices to review the national accounts data to ensure that they conformed to the standards of the 1993 System of National Accounts. Similar reviews were conducted for population and exchange rate data." The World Bank (2008; 2)

The World Bank noted that the data provided by China were not quite complete and that the Tables broke China into 4 separate regions:

"China submitted prices for 11 administrative areas and the urban and rural components. The World Bank and the Asian Development Bank extrapolated these 11 city prices to the national level. The China data do not include Hong Kong, Macao, and Taiwan, China." The World Bank (2008; 2).

The World Bank publication also explained how the ICP dealt with the fact that Egypt appeared in two regions (and priced out the product lists for both regions):

"Egypt participated in both the Africa and West Asia ICP programs by providing prices for the products included in each comparison. Therefore, it was possible to compute PPPs for Egypt separately for Africa and West Asia. Both regions included Egypt results in their regional reports. Egypt appears in the global report in both regions. The results for Egypt from each region were averaged by taking the geometric mean of the PPPs, allowing Egypt to be shown in each region with the same ranking in the world comparison." The World Bank (2008; 2).

Finally, the World Bank explained how the CIS regional results were obtained:

"Russia participated in the price collection for both the CIS and OECD comparisons. As with Egypt, PPPs for Russia were computed separately for the OECD and CIS comparisons. However, the CIS region did not participate in the Ring. Therefore, following past practices the CIS region was linked to Eurostat-OECD using Russia as a link. For comparison purposes, Russia is shown in both regions in the report." The World Bank (2008; 2).

Thus since Russia is the only country that belongs to both the OECD region and the CIS region, linking the two regions at both the Basic Heading level and higher levels of aggregation can be done though Russia. The same linking strategy could have been used to link the Africa and West Asia regions using Egypt as the linking country (or bridge country using ICP parlance) but a decision was made not to do this.[5]

[5] The problems in the case of Egypt are more complicated than in the case of Russia since there were more than one ring countries in Africa and in West Asia. Hill (2007c; 13) listed the 18 ring countries as Brazil, Cameroon, Chile, Egypt, Estonia, Hong Kong, Japan, Jordan, Kenya, Malaysia, Oman, Philippines, Senegal, Slovenia, South Africa, Sri Lanka, United Kingdom and Zambia. Thus Cameroon, Jordan, Kenya, Oman, Senegal, South Africa and Zambia join Egypt as ring countries that are present in either the African or West Asian regions.

The above material presents a quick overview of the ICP. Our specific task in the present paper is to present some of the methodological details of the methods that were used to:

- Link the Basic Heading PPPs across the regions (sections 2 and 3) and
- Link the price levels and volumes for each country within a region across the regions in a way that preserves the regional relative price and volume measures (sections 4 and 5).

Thus sections 2 and 3 deal with the problems associated with the aggregation of price information at the lowest level of aggregation where information on expenditures or quantities is not available. Sections 4 and 5 deal with aggregation problems at higher levels of aggregation where expenditure information by category and country is available. It should be noted that the material to be covered in sections 2-5 below overlaps substantially with the material in the *ICP 2003-2006 Handbook*; see Hill (2007a) (2007b) (2007c) (2007d) (2007e). Also the material in sections 2 and 3 overlaps with Hill (2008) and the material in sections 3 and 5 overlaps substantially with Diewert (2004b).

Section 6 lists some of the methodological problems that require additional research before the next round of the ICP program, which is scheduled to take place in 2011.

Section 7 provides some concluding remarks to the main text. An Appendix, which looks at the properties of the relatively new Iklé Dikhanov Balk multilateral system used by the African region, concludes the paper.

2. The Comparison of Prices Across Countries Within a Region at the BH Level

Three distinct methods for linking prices across countries within a region at the Basic Heading level were used by the regions in the 2005 ICP:

- The Country Product Dummy (CPD) method (used by the African, Asian Pacific and West Asian regions);
- The Extended Country Product Dummy (CPRD) method (used by South America) and
- The EKS* method used by the OECD/Eurostat and CIS regions.

The most widely used statistical approach to the multilateral aggregation of prices at the first stage of aggregation is the *Country Product Dummy* (CPD) method, proposed by Robert Summers (1973). This method for making international comparisons of prices can be viewed as a very simple type of hedonic regression model where the only characteristic of the commodity is the commodity itself. The CPD method can also be viewed as an example of the stochastic approach[6] to index numbers. Since an extension

[6] See Selvanathan and Rao (1994) for examples of the stochastic approach to index number theory. A main advantage of the CPD method for comparing prices across countries over traditional index number methods is that we can obtain *standard errors* for the country price levels. This advantage of the stochastic approach to index number theory was stressed by Summers (1973).

of this method was used to link prices across regions, we will outline the algebra behind this approach.

Suppose that we are attempting to make an international comparison of prices between C countries over a reasonably homogeneous group of say N items.[7] In this section, we also assume that no expenditure weights are available for the price comparisons. Let p_{cn} denote the average price of item n[8] in country c for c = 1,…,C; n = 1,…,N. Each item n must be measured in the same quantity units across countries but the prices are collected in local currency units. The basic statistical model that is assumed is the following one:

(1) $p_{cn} = a_c b_n e_{cn}$; $\qquad$ c = 1,…,C; n = 1,…,N.

where the a_c and b_n are unknown parameters to be estimated and the e_{cn} are independently distributed error terms with means 1 and constant variances. The parameter a_c is to be interpreted as the *average level of prices* (over all items in this group of items) in country c relative to other countries and the parameter b_n is to be interpreted as the *average* (over all countries) *multiplicative premium* that item n is worth relative to an average item in this grouping of items. Thus the a_c are the basic heading country price levels that we want to determine while the b_n are item or individual product effects. The basic hypothesis is that the price of item n in country c is equal to a country price level a_c times an item commodity adjustment factor b_n times a random error that fluctuates around 1. Taking logarithms of both sides of (1) leads to the following model:

(2) $y_{cn} = \alpha_c + \beta_n + \varepsilon_{cn}$; $\qquad$ c = 1,…,C; n = 1,…,N

where $y_{cn} \equiv \ln p_{cn}$, $\alpha_c \equiv \ln a_c$, $\beta_n \equiv \ln b_n$ and $\varepsilon_{cn} \equiv \ln e_{cn}$.

The model defined by (2) is obviously a linear regression model where the independent variables are dummy variables. The least squares estimators for the α_c and β_n can be obtained by solving the following minimization problem:[9]

(3) $\min_{\alpha\text{'s}, \beta\text{'s}} \{\sum_{c=1}^{C} \sum_{n=1}^{N} [y_{cn} - \alpha_c - \beta_n]^2\}$.

However, it can be seen that the solution for the minimization problem (3) cannot be unique: if α_c^* for c = 1,…,C and β_n^* for n = 1,…,N solve (3), then so does $\alpha_c^* + \gamma$ for c = 1,…,C and $\beta_n^* - \gamma$ for n = 1,…,N, for any arbitrary number γ. Thus it will be necessary to impose an additional restriction or normalization on the parameters α_c and β_n in order to

[7] Using the language of the International Comparison of Prices (ICP) project, we are making a comparison of prices at the basic heading level. In ICP 2005 project, there are 155 basic headings. Thus each region using this method would have to run 155 regressions of the type described here.
[8] In most cases, this item n price in country c was an unweighted arithmetic mean of prices collected over outlets and regions in the country during the reference year.
[9] Weighted (by expenditure shares) versions of the CPD model were considered by Prasada Rao (1990), (1995) (2001) (2002) (2004), Heston, Summers and Aten (2001), Sergueev (2002) (2003), Diewert (2004b) (2005), Hill (2007a; 23-24) and Hill and Timmer (2006).

obtain a unique solution to the least squares minimization problem (3). The simplest normalization is:

(4) $\alpha_1 = 0$ or $a_1 = 1$;

The normalization (4) means that country 1 is chosen as the numeraire country and the parameter a_c for c = 2,…,C is the PPP (Purchasing Power Parity) of country c relative to country 1 for the class of commodity prices that are being compared across the C countries.[10]

Cuthbert and Cuthbert (1988; 57) introduced an interesting generalization of the Country Product Dummy method that can be used if information on *representativity* of the prices is collected by the countries in the comparison project along with the prices themselves. Hill (2007a) (2008) explains this method in some detail and he called the method the *extended CPD Method* or *CPRD Method* and he justified the method as follows:

"The reason for distinguishing between representative and unrepresentative products is that the relative prices of representative products in a country may be expected to be low compared with relative prices of the same products in countries in which they are not representative. Conversely, of course, the relative prices of unrepresentative products will tend to be high. This will tend to happen as result of normal substitution effects. Products will tend to be purchased in relatively large (small) quantities precisely because their relative prices are low (high). This conclusion is not merely a theoretical deduction, as there is ample empirical evidence of the substitution effect at work in both inter-temporal and inter-national comparisons." Peter Hill (2007a; 3).

"The expected price depends on the interaction of three factors: the country, the product and its representativity. Given that the coefficient of a representative product is fixed at unity, the coefficient of an unrepresentative product may be expected to be greater than unity. The price of product is expected to be higher relatively to the reference product 1 in a country in which it is unrepresentative than in a country in which it is representative. The improvement over the traditional CPD method comes from the partial relaxation of the unrealistic assumption that the pattern of relative prices is the same in all countries. ...
The addition of the new variable, representativity, does not simply add another parameter to be estimated. It adds another dimension to the analysis. As there are three types of explanatory variables in the regression -- country, product and representativity -- the extended regression will described as the CPRD method to distinguish it from the traditional CPD method." Peter Hill (2007a; 26).

The basic idea is that representative products in a country should tend to be lower in price (and hence they should be more popular) compared to unrepresentative products; thus representativity becomes a price determining characteristic of the commodity.

The CPDR method generalizes the model (2) above as follows. Define $y_{cnu} = \ln p_{cnu}$ where p_{cnu} is the logarithm of the average product n price collected in country c and u is an index that denotes whether the collected price is unrepresentative (in which case u = 1) or representative (in which case u = 2). The basic (unweighted) statistical model that is assumed is the following one:

(5) $y_{cnu} = \alpha_c + \beta_n + \delta_u + \varepsilon_{cnu}$; $c = 1,\ldots,C;\ n = 1,\ldots,N;\ u = 1,2$

[10] See Rao (2004) and Hill (2007a) for further analysis of this model. We note that Hill uses a different but equivalent normalization.

where the α_c are the log country PPP's, the β_n are the log product price effects and the δ_u are the two log representativity effects and the ε_{cnu} are independently distributed random variables with mean zero and constant variances. In order to identify the parameters, the following normalizations can be used:

(6) $\alpha_1 = 0$; $\delta_1 = 0$.

Thus the present model is much the same as the basic CPD model except that we have 3 classifications instead of 2. For additional discussion of this model, the reader is referred to Cuthbert and Cuthbert (1988), Cuthbert (2000), Diewert (2004b) and Hill (2007a) (2008).

We agree with Hill in endorsing the method in theory. However, in practice, it seems it was at times difficult for national price statisticians to agree on a workable definition of representativity that was uniform across countries and regions. Thus in the end, it appears that only the South American region used CPRD method to construct its 155 by 10 matrix of PPP's by Basic Heading and country. The other regions used the basic CPD method or the EKS* method (which also used the representativity concept).

As mentioned at the beginning of this section, the EKS* method was used to aggregate prices at the lowest level of aggregation in the OECD and CIS regions. This method is explained by Hill as follows:

"Eurostat abandoned EKS 1 in 1982 and replaced it by the method described in the present section, which will be called the asterisk method or EKS*. A detailed exposition of EKS* and its properties is given by Sergey Sergeev (2003). The EKS* method is so called because it makes use of the distinction between representative and unrepresentative products, the representative products being identified in the product lists by an *. The EKS* method recognizes, and exploits, the fact that, as already explained, the prices of representative products are likely to be relatively low, whereas the prices of unrepresentative products are likely to be relatively high. The method proceeds by calculating two separate Jevons indices for each pair of countries. One Jevons index covers products that are representative in the first country, treated here as the base country. The other covers products that are representative in the second country. Of course, some products may be representative in both countries and included in both indices. The two indices may be described as *Jevons 1* and *Jevons 2* respectively." Peter Hill (2007a; 9).

Thus two bilateral Jevons type indexes are calculated for any two countries. Jevons 1 (2) compares only the price relatives of products that are representative in country 1 (2). The final bilateral index of prices between the two countries under consideration is a geometric mean of the two Jevons indexes.[11] Once all of these bilateral parities have been constructed over each pair of countries in the region, they can be harmonized by

[11] Note that prices which are not representative in both countries but are collected in both countries do not appear in the final bilateral index of prices between the two countries. This means that the EKS* procedure is not fully efficient in a statistical sense, whereas the CPRD procedure is fully efficient.

using the EKS procedure.[12] For further details of this method, the reader is referred to Hill (2007a).

A majority of the members of the Technical Advisory Group who provided advice to ICP 2005 favoured the CPRD method described in the previous section over the EKS* method described in this section for two reasons:

- The CPRD method used all of the available price information whereas EKS* did not and
- The CPRD method gave straightforward measures of the statistical precision of the estimated parities.

However, it appears that Eurostat price statisticians are locked into the EKS* method by legislation and thus the OECD/Eurostat region stuck by its EKS* method in the current European Comparison Program. More research is required in order to determine how much difference there would be between CPRD and EKS*. But without having this research in hand, I would certainly favor the use of CPRD over EKS*, mainly because the EKS* method throws away valuable information on some prices and this cannot be a statistically efficient procedure.

There is also the issue of choosing between the original CPD method and the enhanced CPRD method, which makes use of representativity information on the item prices. Hill (2007a) explains theoretically why the CPRD method should be preferred over the CPD method. However, in practice, national price collectors in all of the non OECD regions had great difficulty in deciding on which items were representative and which items were not. Thus when the CPRD regressions were run, the coefficients for the representative dummy variables had more or less random signs instead of the expected signs. This was the case even for the South American region, which used the CPRD method.[13] Thus at this stage of our knowledge of the various methods used to aggregate prices at the basic heading level, I would favor the use of the plain vanilla CPD method.

Having described the methods used to construct PPPs for the 155 basic headings for each country in a region, we now consider how to link these PPPs across regions.

3. The Comparison of Prices Across Regions at the Basic Heading Level

As noted in the introduction, a group of *ring countries* collected prices from a common list and this price information was used to link the regional basic heading prices across the 6 regions. However, since the CIS region was locked into the OECD/Eurostat region, in practice, there were only 5 regions to link, with the CIS, OECD and Eurostat countries forming a single region.

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[12] The EKS method is explained in more detail by Balk (1996), Diewert (1999) and Hill (2007a) (2008). The method is due to Gini (1924) (1931) and independently rediscovered by Eltetö and Köves (1964) and Szulc(1964).

[13] Personal communication from Yuri Dikhanov.

The methodology used to link basic heading prices across regions was developed by Diewert (2004b; 36-39) and we review that methodology here.[14] The model is basically an adaptation of the unweighted CPD model presented in section 2.1.

In order to set the stage for what was actually done in linking the regions, we first generalize the CPD model presented in section 2 to allow for a reorganization of the list of C countries into 5 regions and C(r) ring countries in each region r. Thus C(r) is not the total number of countries in region r; it is only the number of ring countries in each region because only the ring countries collected data on prices from a common international product list. With these changes, the basic model becomes:

(7) $p_{rcn} \approx a_r b_{rc} c_n$; $\quad r = 1,\ldots,5;\ c = 1,\ldots.,C(r);\ n = 1,\ldots,N;$
(8) $a_1 = 1;$
(9) $b_{r1} = 1;$ $\quad r = 1,\ldots,5.$

The normalization (8) means that we have to choose a numeraire region. The normalizations (9) mean that within each region, we need to choose a numeraire country in order to identify all of the parameters uniquely. Thus the parameters a_r and b_{rc} replace our initial model parameters a_c. Note that the total number of parameters remains unchanged when we group all of the countries in the comparison into regions and countries within the regions.

Taking logarithms of both sides of (7) and then adding error terms ε_{rcn} (with means 0) leads to the following regression model:

(10) $\ln p_{rcn} = \ln a_r + \ln b_{r\,c} + \ln c_n + \varepsilon_{rcn}$; $\quad r = 1,\ldots,5;\ c = 1,\ldots.,C(r);\ n = 1,\ldots,N;$
$\qquad = \alpha_r + \beta_{rc} + \gamma_n + \varepsilon_{rcnk}$

where we impose the following normalizations on the parameters in order to uniquely identify them:

(11) $\alpha_1 = 0$;
(12) $\beta_{r1} = 0$; $\quad r = 1,\ldots,5$

where $\alpha_r \equiv \ln a_r$, $\beta_{rc} \equiv \ln b_{rc}$, $\gamma_n \equiv \ln c_n$.

If all of the data collected for each regional comparison could be pooled and if there are product overlaps between the regions, then there will be 155 regressions of the form (10) to run, one for each basic heading category. In the above model, the interregional log parities (the α_r) are estimated along with the within region country log parities (the β_{rc}) and the product log price premiums (the γ_n). Call this the *first approach* to estimating the regional parities for each basic heading. It uses all of the available information in making comparisons between all of the countries.

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[14] The basic methodology is also described in Hill (2007d). However, Hill uses somewhat different normalizations than (11) and (12).

However, the above one big regression approach (for each basic heading) is *not consistent* with approaches that used only the regional data to determine the within region parities, the β_{rc} parameters, holding r fixed. But a principle of the current ICP methodology was that regions should be allowed to determine their own parities, independently of other regions. However, the regression model (10) can be modified to deal with this problem. If the regional log parities β_{rc} are known, then the term β_{rc} (which is equal to $\ln b_{rc}$) can be subtracted from both sides of (10), leading to the following regression model:

(13) $\ln p_{rcn} - \ln b_{rc} = \ln a_r + \ln c_n + \varepsilon_{rcn}$; $r = 1,\ldots,5;\ c = 1,....,C(r);\ n = 1,...,N;$

or

(14) $\ln [p_{rcn}/b_{rc}] = \alpha_r + \gamma_n + \varepsilon_{rcn}$;

where the normalization (8) still holds. Thus if the within region parities are known, then prices in each region p_{rcn} can be divided by the appropriate regional parity for that country in that region b_{rc}, and these regionally adjusted prices can be used as inputs into the usual CPD model that has now only the regional log parities α_r and the commodity adjustment factors γ_n as unknown parameters to be estimated. Call the model defined by (11) and (14) the *second approach* to estimating the regional parities for each basic heading. This second approach respects the within region parities that have been constructed by the regional price administrators. It is this second approach that was used in ICP 2005.[15]

We now turn our attention to the problems associated with aggregating up the basic heading PPP information (along with country expenditure information) in order to form aggregate country price and volume comparisons within a region.

4. Aggregate Price and Volume Comparisons Across Countries Within a Region

Once the 155 BH price parities for each of the K countries in a region have been constructed, aggregate measures of country prices and relative volumes can be constructed using a wide variety of multilateral comparison methods that have been suggested over the years. These aggregate comparisons assume that in addition to BH price parities for each country, national statisticians have provided country expenditures (in their home currencies) for each of the 155 BH categories for the reference year 2005. Then the 155 by K matrices of Basic Heading price parities and country expenditures are used to form average price levels across all commodities and relative volume shares for each country.

There are a large number of methods that can be used to construct these aggregate Purchasing Power Parities and relative country volumes and Hill (2007b) surveys the

[15] Yuri Dikhanov at the World Bank carried out the computations for the global linking.

main methods that have been used in previous rounds of the ICP and other methods that might be used.[16] Basically, only two multilateral methods have been used in previous rounds:

- The Gini-EKS (GEKS) method based on Fisher (1922) bilateral indexes and
- The Geary (1958) Khamis (1972) (GK) method, which is an additive method.

In the present ICP round, aggregate PPPs and relative country volumes for countries within each region were constructed for five of the six regions using the Gini-EKS method. However, the African region wanted to use an additive method and so this region used a relatively new additive method, the Iklé Dikhanov Balk (IDB) method, for constructing PPPs and relative volumes within the region.[17] These methods will be discussed in more detail below. However, at this point, it may be appropriate to comment briefly on the relative merits of the GEKS, GK and IDB methods. The GK and IDB methods are additive methods; i.e., the real output of each country can be expressed as a sum of the country's individual outputs but each output is weighted by an international price which is constant across countries. This feature of an additive method is tremendously convenient for users and so for many purposes, it is useful to have available a set of additive international comparisons.

4.1 The Gini EKS Method

It will be useful to introduce some notation at this point. Let N equal 155 and let K be the number of countries in the regional comparison for the reference year. Denote the regional PPP for country k and commodity category n by $p_n^k > 0$ and the corresponding expenditure (in local currency units) on commodity class n by country k in the reference year by e_n^k for $n = 1,...,N$ and $k = 1,...,K$. Given this information, we can define *implicit quantity levels* y_n^k for each Basic Heading category n and for each country k as the category expenditure deflated by the corresponding commodity PPP for that country:

$$(15)\quad y_n^k \equiv e_n^k/p_n^k\,; \qquad n = 1,...,N\,;\ k = 1,...,K.$$

It will be useful to define *country commodity expenditure shares* s_n^k as follows:

$$(16)\quad s_n^k \equiv e_n^k/\textstyle\sum_{i=1}^N e_i^k\,; \qquad n = 1,...,N\,;\ k = 1,...,K.$$

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[16] For additional methods, see Balk (1996), R.J. Hill (1997) (1999a) (1999b) (2001) (2004) and Diewert (1999).

[17] Iklé (1972; 203) proposed the equations for the method in a rather difficult to interpret manner and provided a proof for the existence of a solution for the case of two countries. Dikhanov (1994; 6-9) used the much more transparent equations (16) and (18), explained the advantages of the method over the GK method and illustrated the method with an extensive set of computations. Balk (1996; 207-208) used the Dikhanov equations and provided a proof of the existence of a solution to the sytem for an arbitrary number of countries. Van Ijzeren (1983; 42) also used Iklé's equations and provided an existence proof for the case of two countries.

Now define *country vectors of BH prices* as $p^k \equiv [p_1{}^k,...,p_N{}^k]$, *country vectors of BH quantities* as $y^k \equiv [y_1{}^k,...,y_N{}^k]$, *country expenditure vectors* as $e^k \equiv [e_1{}^k,...,e_N{}^k]$ and *country expenditure share vectors* as $s^k \equiv [s_1{}^k,...,s_N{}^k]$ for k = 1,...,K.

In order to define the GEKS parities $P^1,P^2,...,P^K$, we first need to define the *Fisher* (1922) *ideal bilateral price index* P_F between country j relative to k:[18]

(17) $P_F(p^k,p^j,y^k,y^j) \equiv [p^j\cdot y^j\ p^j\cdot y^k/p^k\cdot y^j\ p^k\cdot y^k]^{1/2}$; $\qquad$ j = 1,...,K ; k = 1,...,K.

The *aggregate PPP for country* j, P^j, is defined as follows:

(18) $P^j \equiv \prod_{k=1}^K [P_F(p^k,p^j,y^k,y^j)]^{1/K}$; $\qquad$ j = 1,...,K.

Once the GEKS P^j's have been defined by (18), the corresponding GEKS *country real outputs or volumes* Y^j can be defined as the country expenditures $p^j\cdot y^j$ in the reference year divided by the corresponding GEKS purchasing power parity P^j:

(19) $Y^j \equiv p^j\cdot y^j/P^j$; $\qquad$ j = 1,...,K.

If we divide all of the P^j defined by (18) by a positive number, α say, then we can multiply all of the Y^j defined by (19) by this same α without materially changing the GEKS multilateral method. If country 1 is chosen as the numeraire country in the region, then we set α equal to P^1 defined by (18) for j = 1 and then the price level P^j is interpreted as the number of units of country j's currency it takes to purchase 1 unit of country 1's currency and get an equivalent amount of utility and the rescaled Y^j is interpreted as the volume of output of country j in the currency units of country 1.

It is also possible to normalize the outputs of each country in common units (the Y^k) by dividing each Y^k by the sum $\sum_{j=1}^K Y^j$ in order to express each country's real output as a fraction or share of total regional output; i.e., we can define the country k's share of regional output, S^k, as follows:[19]

(20) $S^k \equiv Y^k/\sum_{j=1}^K Y^j$; $\qquad$ k = 1,...,K.

Of course, the country shares of regional real output, the S^k, remain unchanged after rescaling the PPPs by the scalar α.

This completes our brief overview of the Gini EKS method for making multilateral comparisons.[20]

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[18] Notation: $p\cdot y \equiv \sum_{n=1}^N p_ny_n$ denotes the inner product between the vectors p and y.

[19] There are several additional ways of expressing the GEKS PPP's and relative volumes; see Balk (1996) and Diewert (1999; 34-37).

[20] It should be noted that all of the multilateral methods that are described in this section can be applied to subaggregates of the 155 basic heading categories; i.e., instead of working out aggregate price and volume comparisons across all 155 commodity classifications, we could just choose to include the food categories

4.2 The Geary Khamis Method

The method was suggested by Geary (1958) and Khamis (1972) showed that the equations that define the method have a positive solution under certain conditions.

The GK system of equations involves K country price levels or PPPs, $P^1,...,P^K$, and N international commodity reference prices, $\pi_1,...,\pi_N$. The equations which determine these unknowns (up to a scalar multiple) are the following ones:

$$(21)\ \pi_n = \sum_{k=1}^K [y_n^k/\sum_{j=1}^K y_n^j][p_n^k/P^k]\ ; \qquad n = 1,...,N\ ;$$
$$(22)\ P^k = p^k \cdot y^k/\pi \cdot y^k\ ; \qquad k = 1,...,K$$

where $\pi \equiv [\pi_1,...,\pi_N]$ is the vector of GK regional average reference prices. It can be seen that if we have a solution to equations (21) and (22), then if we multiply all of the country parities P^k by a positive scalar λ say and divide all of the reference prices π_n by the same λ, then we obtain another solution to (21) and (22). Hence, the π_n and P^k are only determined up to a scalar multiple and we require an additional normalization such as

$$(23)\ P^1 = 1$$

in order to uniquely determine the parities. It can also be shown that only N + K − 1 of the N equations in (21) and (22) are independent. Once the parities P^k have been determined, the real output for country k, Y^k, can be defined as country k's *nominal value of output in domestic currency units*, $p^k \cdot y^k$, divided by its PPP, P^k; i.e., we have

$$(24)\ Y^k = p^k \cdot y^k/P^k\ ; \qquad k = 1,...,K$$
$$= \pi \cdot y^k \qquad \text{using (22).}$$

Finally, if we substitute equations (24) into the regional share equations (20), we find that country k's share of regional output is

$$(25)\ S^k = \pi \cdot y^k/\pi \cdot y \qquad k = 1,...,K$$

where the *region's total output vector* y is defined as the sum of the country output vectors; i.e., we have

$$(26)\ y \equiv \sum_{j=1}^K y^j\ .$$

Equations (24) show how convenient it is to have an additive multilateral comparison method: when country outputs are valued at the international reference prices, values are additive across both countries and commodities. However, additive multilateral methods are not really consistent with economic comparisons of utility across countries if the

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in our list of N categories and use the multilateral method to compare aggregate food consumption across the countries in the region.

number of countries in the comparison is greater than two; see Diewert (1999; 48-50) and the Appendix on this point.[21] In addition, looking at equations (41), it can be seen that large countries will have a larger contribution to the determination of the international prices π_n and thus these international prices will be much more representative for the largest countries in the comparison as compared to the smaller ones.[22] This leads us to the next method for making multilateral comparisons: an additive method that does not suffer from this problem of big countries having undue influence in the comparison.

4.3 The Iklé Dikhanov Balk Method

Iklé (1972; 202-204) suggested the method in a very indirect way, Dikhanov (1994) (1997) suggested the much clearer system (27)-(28) below and Balk (1996; 207-208) provided the first existence proof. Dikhanov's (1994; 9-12) equations that are the counterparts to the GK equations (21) and (22) are the following ones:

$$(27)\ \pi_n = [\textstyle\sum_{k=1}^{K} s_n^{k}\, [p_n^{k}/P^{k}]^{-1}/\sum_{j=1}^{K} s_n^{j}]^{-1}\ ; \qquad n = 1,...,N$$
$$(28)\ P^{k} = [\textstyle\sum_{n=1}^{N} s_n^{k}\, [p_n^{k}/\pi_n]^{-1}]^{-1} \qquad k = 1,...,K.$$

As in the GK method, equations (27) and (28) involve the K country price levels or PPPs, $P^1,...,P^K$, and N international commodity reference prices, $\pi_1,...,\pi_N$. Equations (27) tell us that the *nth international price*, π_n, is a *share weighted harmonic mean of the country k prices for commodity n*, p_n^k, deflated by country k's PPP, P^k. The country k share weights for commodity n, s_n^k, do not sum (over countries k) to unity but when we divide s_n^k by $\sum_{j=1}^{K} s_n^j$, the resulting normalized shares do sum (over countries k) to unity. Thus equations (27) are similar to the GK equations (21), except that now a harmonic mean of the deflated commodity n prices, p_n^k/P^k, is used in place of the old arithmetic mean and in the GK equations, country k's share of commodity n in the region, $y_n^k/\sum_{j=1}^{K} y_n^j$, was used as a weighting factor (and hence large countries had a large influence in forming these weights) but now the weights involve country expenditure shares and so each country in the region has an equal influence in forming the weighted average. Equations (28) tell us that P^k, the *PPP for country* k, P^k, is equal to a *weighted harmonic mean of the country k commodity prices*, p_n^k, deflated by the international price for commodity n, π_n, where we sum over commodities n instead of over countries k as in equations (27). The share weights in the harmonic means defined by (28), the s_n^k, of course sum to one when we

[21] "Figure 1.1 also illustrates the Gerschenkron effect: in the consumer theory context, countries whose price vectors are far from the 'international' or world average prices used in an additive method will have quantity shares that are biased upward. ... It can be seen that these biases are simply quantity index counterparts to the usual substitution biases encountered in the theory of the consumer price index. However, the biases will usually be much larger in the multilateral context than in the intertemporal context since relative prices and quantities will be much more variable in the former context. ... The bottom line on the discussion presented above is that the quest for an additive multilateral method with good economic properties (i.e., a lack of substitution bias) is a doomed venture: nonlinear preferences and production functions cannot be adequately approximated by linear functions. Put another way, if technology and preferences were always linear, there would be no index number problem and hundreds of papers and monographs on the subject would be superfluous!" W. Erwin Diewert (1999; 50).
[22] Dikhanov (1994; 5) made this point.

sum over n, so there is no need to normalize these weights as was the case for equations (27).

It can be seen that if we have a solution to equations (27) and (28), then if we multiply all of the country parities P^k by a positive scalar λ say and divide all of the reference prices π_n by the same λ, then we obtain another solution to (27) and (28). Hence, the π_n and P^k are only determined up to a scalar multiple and we require an additional normalization such as (23).

Although the IDB equations (28) do not appear to be related very closely to the corresponding GK equations (22), it can be shown that these two sets of equation are actually the same system. To see this, note that the country k expenditure share for commodity n, $s_n{}^k$, has the following representation:

$$(29)\ s_n{}^k = p_n{}^k y_n{}^k / p^k \cdot y^k\ ; \qquad n = 1,\ldots,N\ ;\ k = 1,\ldots,K.$$

Now substitute equations (29) into equations (28) to obtain the following equations:

$$\begin{aligned}(30)\ P^k &= 1/\textstyle\sum_{n=1}^N s_n{}^k\, [p_n{}^k/\pi_n]^{-1} \qquad k = 1,\ldots,K \\ &= 1/\textstyle\sum_{n=1}^N [p_n{}^k y_n{}^k/p^k\cdot y^k][\pi_n/p_n{}^k] \\ &= p^k\cdot y^k/\textstyle\sum_{n=1}^N \pi_n y_n{}^k \\ &= p^k\cdot y^k/\pi\cdot y^k.\end{aligned}$$

Thus equations (28) are equivalent to equations (22) and the IDB system is an additive system; i.e., equations (24)-(26) can be applied to the present method just as they were applied to the GK method for making international comparisons.

In the Appendix, we will obtain many different ways of representing the IDB system of parities and we will also establish fairly weak conditions for the existence and uniqueness of the IDB parities. We will also indicate how solutions to the equations can be found.

As was mentioned in the introduction, the Iklé Dikhanov Balk method was used by the African region in order to construct regional aggregates. Basically, this method appears to be an improvement over the GK method in that large countries no longer have a dominant influence on the determination of the international reference prices π_n and so if an additive method is required with more democratic reference prices, IDB appears to be "better" than GK. However, again, we caution the reader that additive multilateral methods will not generate very accurate relative volumes (from the viewpoint of the economic approach) if the number of countries is greater than three and there is heterogeneity in relative prices and quantities; see Diewert (1999; 50) and the final section in the Appendix.

We now turn our attention to the problem of linking the regions at higher levels of aggregation.

5. Aggregate Price and Volume Comparisons Across Regions

There are 146 countries in the ICP project and 155 basic headings. At this stage of the aggregation procedure, we assume that we have two 155 by 146 matrices of data: one matrix contains the PPPs for basic heading category n and country k, p_n^k, and the other contains country expenditures in each country's currency, e_n^k, so that the notation is basically the same as in the previous section but now k runs over all 146 countries instead of just the countries in a given region. At this stage, we could use any suitable multilateral method to aggregate up these data into a set of 146 country PPP's and volumes, such as the EKS or IDB methods explained in the previous section. Call this *Approach 1*. However, the problem with this approach is that the multilateral method to be used would not necessarily respect the regional PPP's unless it was restricted in some manner.

Thus we consider *Approach 2*, which will link the regions, while respecting the within region overall PPP's that the regions deem best for their purposes.[23] The first step is to reorganize the countries into 5 regions (we regard the OECD/Eurostat/CIS countries as forming one region). Consider region r which has C(r) countries in it. Let p_n^{rc} denote the within region PPP for basic heading class n and country c in region r[24] and let e_n^{rc} denote the corresponding expenditure in local currency. The total regional expenditure on commodity group n in currency units of country 1 in each region, E_n^r, is defined as follows:

$$(31)\ E_n^r \equiv p_n^{r1} \textstyle\sum_{c=1}^{C(r)} e_n^{rc}/p_n^{rc} ; \qquad r = 1,\ldots,5 ; n = 1,\ldots,155.$$

The corresponding regional PPPs by region and commodity, P_n^r, are defined to be the world BH parities for the numeraire country in each region:

$$(32)\ P_n^r \equiv p_n^{r1} ; \qquad r = 1,\ldots,5 ; n = 1,\ldots,155.$$

Now each region can be treated as if it were a single supercountry with supercountry expenditures E_n^r and basic heading PPPs P_n^r defined by (31) and (32) respectively for the 5 supercountries and any of the linking methods described in the previous section can be used to link the regions. Once the interregional price and volumes have been determined, the regional price and volume aggregates can be used to provide world wide price and volume comparisons for each individual country. This method necessarily preserves all regional relative parities. Hill (2007e) attempted to show that the overall procedure does not depend on the choice of numeraire countries, either within regions or between regions; i.e., the relative country parities will be the same no matter what the choices are for the numeraire countries. However, Sergeev (2009) noted that this method is not

[23] This approach was proposed by Diewert (2004b; 45-47). It is further described in much more detail by Hill (2007e).

[24] The parities p_n^{rc} are the interregionally consistent PPP's that were linked across regions as described in section 3 above; i.e., the p_n^{rc} are the estimated parameters $a_r b_{rc} c_n$ which occur on the right hand side of equations (7). Assuming that country 1 is the numeraire country in each region, then the p_n^{r1} are the parities that link the numeraire countries in each region.

invariant to the choice of the numeraire countries within the regions. Thus this method should probably not be used in the next round of the ICP in 2011.

Approach 2 in conjunction with the EKS method was used to link the regions in the current ICP round; i.e., the EKS method was used to link the 5 supercountry regions.

Hill (2007e) discusses other possible methods that could be used to link the regions and these various alternative methods should part of the research agenda for the next round of comparisons. In particular, at higher levels of aggregation, we need to use the results of the present round to evaluate whether regional fixity is a good idea or not. The problem with regional fixity is that countries are not homogeneous within each region. In principle, it makes sense to compare countries whose (relative) price structures are similar and whose (absolute) quantity structures are similar: index number comparisons of price and volumes will work best under these conditions. Thus roughly speaking, it makes sense to compare directly countries who are at the same stage of development and build up a complete set of multilateral comparisons by linking (bilaterally) countries who are most structurally similar.[25] R.J. Hill (1997) (1999a) (1999b) (2001) (2004) has developed methodology along these lines and it should be tested out using the detailed data generated by the present round.[26] It may well be that the fixity methodology developed in this round is not the most appropriate methodology for subsequent rounds.

6. Problem Areas and the Future Research Agenda

There are a number of problem areas associated with making international comparisons that require additional research and discussion before the next round of the ICP takes place:

- If a country experiences hyperinflation during the reference year, the average price concept may not be meaningful. A possible solution to this problem is to use within the year inflation rates to "discount" prices collected throughout the year to a single reference week or day.[27]
- The problem of pricing exports and imports.[28] At present, exchange rates are taken as the price of exports and imports. This is a reasonable approximation in some cases but the question is can we do anything better (that is not too costly)?

[25] See Diewert (2002) for a discussion on how to measure structural similarity.
[26] Another interesting issue is this: the present fixity imposed procedure is essentially a two stage GEKS procedure. At the first stage, countries are compared using GEKS within each region and then at the second stage, the five regions are linked together using another round of GEKS. Question: how does this two stage procedure compare to a single stage GEKS procedure using all 146 countries? The answer will probably be: they generate rather different parities. What then? What is the "truth"? We need criteria to determine "truth". We could look at the axiomatic properties of two stage methods as compared to single stage methods but I do not believe that this would resolve the issues. At this point, I would fall back on the spatial linking methodology: it makes sense to build up a path of linked comparisons where we link together the countries which are most similar in structure.
[27] See Hill (1996) for a discussion of the accounting problems when there is high inflation.
[28] See Heston and Summers (2008; 4) for a discussion of this problem. A first approach to the problem would be to coordinate the calculation of national unit value export and import indexes across countries. This is a separate exercise that should be started well before the next ICP round. O'Connor (2008)

- The problem of negative expenditure categories. This problem arises with the net export category and the net additions to inventory category. Typically, there is not a problem provided that we do not attempt to provide PPPs for a single category that could be positive or negative across countries.[29] If it is necessary to provide PPPs across countries for such a category, the problems can be avoided by providing separate PPPs for exports and imports or for starting and finishing inventory stocks and users can difference the results.
- Inaccurate expenditure weights can cause grave difficulties. In the next ICP round, it would be very desirable to have more accurate information on expenditures by basic heading available from participating countries.
- Methodological difficulties with hard to measure areas of the accounts. There are particular problems with the treatment of housing[30], financial services and nonmarket production.[31] These are problem areas for regular country accounts as well due to the lack of consensus on an appropriate methodology. Hopefully, international groups and academic economists interested in measurement problems will undertake additional research in these areas before the next ICP round.
- There is a very basic problem that makes international comparisons of prices and volumes very difficult and that is the lack of matching of products. The same problem occurs in the time series context due to the introduction of new products and the disappearance of "old" products but the lack of matching is much worse in the international context due to differences in tastes and big differences in the levels of development across countries, leading to very different consumption patterns. However, Structured Product Descriptions were introduced in the current ICP round and this does open up the possibility for undertaking hedonic regression exercises in the next round in order to improve the matching process. There are many problems to be addressed however,[32] and it would be wise to undertake experimental hedonic studies well in advance of the next round.
- The fact that the ring list of commodities was somewhat different from the regional lists means that there is the possibility of anomalies in the final results; i.e., if different products are priced in the ring list, we cannot be sure the relative ring price levels really match up with the relative prices within the regions. The ring list of commodities was not determined completely independently from the

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mentions that the problems associated with calculating export and import price indexes is getting worse over time due to increasing trade in multinational intermediate goods and the transfer price problem.

[29] Index number theory tends to break down if a value aggregate crosses zero or is equal to zero!

[30] One area that we have not addressed is the impact of different procedures in different regions. For example, Asia and Africa used different methods for making productivity adjustments for government outputs index and they also used a different method for measuring housing output as compared to the CIS and South American regions. These problems need to be addressed well in advance of the next ICP round.

[31] See Heston and Summers (2008), Giovannini (2008) and Bevacqua, Fantin, Quintslr and Ruiz (2008) for a discussion of these problems. The fact that current System of National Accounts conventions do not allow an imputed interest charge for capital that is used in the nonmarket sector tends to understate the contribution of this sector and the degree of understatement will not be constant across rich and poor countries.

[32] See Hill and Timmer (2006) for a discussion of the problem of differing degrees of product overlap across countries.

country lists and this is all to the good.[33] But in the next round, this integration of the ring product list with the regional product lists should be intensified with a best case scenario where the ring list becomes unnecessary.[34]

- It would be advisable to undertake some studies on alternative methods of aggregation at the higher levels of aggregation.[35] In particular, the program of making comparisons based on the degree of similarity of the price and quantity data being compared that was initiated by Robert Hill (1999a) (1999b) (2001) (2004) seems to be sensible but users have not embraced it, perhaps due to the instability of the method.[36] In any case, the World Bank now has a considerable data set based on the current ICP round that could be used to experiment with alternative methods of aggregation.
- Looking ahead into the more distant future, it would be desirable to integrate the ICP with the EU KLEMS project[37], which is assembling data on the producer side of the economy as opposed to the final demand side, which is the focus of the ICP. Producer data are required in order to calculate relative productivity levels across economies, a topic of great interest to policy makers. Thus in addition to comparing components of final demand across countries, it would be desirable to

[33] Yonas Biru who was responsible for organizing the ring list describes how this was done as follows: “The Global ring list was developed in close consultation with regional and country experts in an iterative processes. First, a consolidated global draft list was prepared that contained over 6,500 products from the five ICP regions and Eurostat-OECD comparison. Second, the list was then pruned by the Global Office to about 1,500 products, based on the country responses. The next step involved harmonizing product descriptions that originated from different regions and the list was sent back to the regions and a second round regional meetings were organized. A revised list was then created taking the second round comment from ring countries. The Global Office analyzed these second round country responses basic heading by basic heading to determine which products should be retained and which should be dropped. Key criteria for determining the final list included the number of regions and also the number of countries within each region where the product could be priced. A workshop was organized in Washington for regional coordinators and representatives of ring countries (one from each region) to go through the list and build consesnus on a global list. The workshop modified some products, dropped some and came up with the final list containing about 1200 products. In doing so the Global Office made sure that at least one product was represented from each region for each basic heading.” Other World Bank researchers who were involved with the ring project were Yuri Dikhanov, Ramgopal Erabelly, Nada Hamadeh, Farah Hussain, Jinsook Lee and Amy Lee.

[34] Yonas Biru and Virgina Romand are currently undertaking a study to improve the SPD process. Yonas Biru describes the study as as follows: “We are developing coding structures not only for products but also for product characteristics for the next generation SPDs. This would facilitate mathing products across regions. This means we will be able to determine ring countries and ring products after data collection based on maximum overlap of products. Both the number and the mix of countries will be determined basic heading by basic heading. Potentially, linking can be done based on diferent criteria, including by ICP regions, consumption and price similarity indecies, etc. The method would also facilitate hedonic type regression, particularly for equipment goods.”

[35] Alan Heston is currently undertaking such a study.

[36] Robert Hill’s methodology for linking countries via a path of bilateral links for countries which have similar price and quantity structures has mainly been applied at higher levels of aggregation. Using a statistical approach, Hill and Timmer (2006) extend this similarity methodology to lower levels of aggregation where only price information is available and they also take into account situations where the amount of overlap in pricing products differs across countries. This is an important practical problem and their methods need to be studied and tested.

[37] See van Ark, Maddison and Timmer (2008) on this topic.

compare outputs and inputs by industry across countries so that international comparisons of sectoral productivity levels could be undertaken.[38]

7. Conclusion

My overall conclusion is that the 2005 ICP round was a big success. The regions liked the idea that they could define their own list of products for international pricing and this improved the quality of the data. The new methodology to link prices across the regions using ring countries also seems to be a clear improvement over previous rounds. Finally, the use of hand held computers and the structured product description methodology led to improvements in the production of national price statistics in many cases.[39]

One issue that has not been entirely satisfactorily resolved is the issue of disclosure of the data; i.e., a great deal of effort has gone in to collecting PPPs for 155 categories for 146 countries but only data on 15 highly aggregated PPPs will be released. Why the reluctance to release the data? Probably because at lower level of aggregation, the results can be quite unreliable. Still one would think that more than 15 categories could be released.[40]

As indicated in the previous section, some challenges remain but hopefully, these problems will be addressed before the next round takes place.

Appendix: The Properties of the Iklé Dikhanov Balk Multilateral System

A.1 Introduction and Overview

Unfortunately, multilateral index number theory is much more complicated than bilateral index number theory. Thus a rather long appendix is required in order to investigate the axiomatic and economic properties of the IDB multilateral system, particularly when we allow some prices and quantities to be zero.[41] A brief overview of this appendix follows.

There are many equivalent ways of expressing the equations that define the IDB parities. Section 2 lists these alternative systems of equations that can be used to define the method. Section 3 provides proofs of the existence and uniqueness of solutions to the IDB equations. Section 4 considers various special cases of the IDB equations. When there are only two countries so that K = 2, we obtain a bilateral index number formula and this case is considered along with the case where N = 2, so that we have only 2 commodities. These special cases cast some light on the structure of the general indexes. Section 5 explores the axiomatic properties of the IDB method while section 6 looks at

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[38] See O'Connor (2008) for a similar long run proposal for the direction of the ICP. O'Connor also advocates making wealth comparisons across countries, which is feasible once we generate measures of capital input.

[39] See Trewin (2008) and Fenwick and Whitestone (2008) on the externalities created by the ICP program.

[40] Heston and Summers (2008; 5) and O'Connor (2008) discuss this issue.

[41] Balk (1996; 207-208) has the most extensive published discussion of the properties of the IDB system but he considered only the case of positive prices and quantities for all commodities across all countries and he did not discuss the economic properties of the method.

the system's economic properties. Finally, section 6 concludes by calculating a numerical example.

Throughout this appendix, we assume that the number of countries K and the number of commodities N is equal to or greater than two.

A.2 Alternative Representations

A.2.1 The P^k, π_n Representation

Recalling equations (15) and (16) in the main text, the basic data for the multilateral system are the prices and quantities for commodity n in country k at the basic heading level, p_n^k and y_n^k respectively, for n = 1,...,N and k = 1,...,K where the number of basic heading categories $N \geq 2$ and the number of countries $K \geq 2$. The N by 1 vectors of prices and quantities for country k are denoted by p^k and y^k and their inner product is $p^k \cdot y^k$ for k = 1,...,K. The share of country k expenditure on commodity n is denoted by $s_n^k \equiv p_n^k y_n^k / p^k \cdot y^k$ for k = 1,...,K and n = 1,...,N.

We will assume that for each n and k, either p_n^k, y_n^k and s_n^k are all zero or p_n^k, y_n^k and s_n^k are all positive. Thus we allow for the possibility that some countries do not consume all of the basic heading commodities. This complicates the representations of the equations since division by zero prices, quantities or shares leads to difficulties and complicates proofs of existence.[42] For now, we make the following assumptions:

(A1) For every basic heading commodity n, there exists a country k such that p_n^k, y_n^k and s_n^k are all positive so that each commodity is demanded by some country.

(A2) For every country k, there exists a commodity n such that p_n^k, y_n^k and s_n^k are all positive so that each country demands at least one basic heading commodity.

In section A.2, we will strengthen the above assumptions in order to ensure that the IDB equations have unique, positive solutions.

Recall that the IDB multilateral system was defined by the Dikhanov equations (27) and (28) (plus one normalization such as (23)). Taking into account the division by zero problem, these equations can be rewritten as follows:[43]

(A3) $\pi_n = [\sum_{j=1}^K s_n^j]/[\sum_{k=1}^K (y_n^k P^k / p^k \cdot y^k)]$; n = 1,...,N
(A4) $P^k = p^k \cdot y^k / \pi \cdot y^k$; k = 1,...,K

where π is a vector whose components are $\pi_1,...,\pi_N$.

[42] Balk's (1996; 208) existence proof assumed that all prices and quantities were strictly positive.
[43] Equations (A3) are equivalent to Balk's (1996; 207) equations (38a) in the case where all price p_n^k are positive and equations (A4) are Balk's equations (38b).

Using assumptions (A1) and (A2), it can be seen that equations (A3) and (A4) will be well behaved even if some p_n^k and y_n^k are zero. Equations (A3) and (A4) (plus a normalization on the P^k or π_n such as $P^1 = 1$ or $\pi_1 = 1$) provide our second representation of the IDB multilateral equations.[44]

In order to find a solution to equations (A3) and (A4), one can start by assuming that $\pi = 1_N$, a vector of ones and then use equations (A4) to determine a set of P^k. These P^k can then be inserted into equations (A3) in order to determine a new π vector. Then this new π vector can be inserted into equations (A4) in order to determine a new set of P^k. And so on; the process can be continued until convergence is achieved.

A.2.2 An Alternative P^k, π_n Representation using Biproportional Matrices

It can be seen that equations (A3) and (A4) can be rewritten in the following manner:

(A5) $\sum_{k=1}^{K} y_n^k\, [p^k \cdot y^k]^{-1} \pi_n P^k = \sum_{j=1}^{K} s_n^j$; $\quad n = 1,\ldots,N;$
(A6) $\sum_{n=1}^{N} y_n^k\, [p^k \cdot y^k]^{-1} \pi_n P^k = \sum_{n=1}^{N} s_n^j = 1$; $\quad k = 1,\ldots,K.$

Define the N by K *normalized quantity matrix* A which has element a_{nk} in row n and column k where

(A7) $a_{nk} \equiv y_n^k / p^k \cdot y^k$; $\quad n = 1,\ldots,N$; $k = 1,\ldots,K.$

Define the N by K *expenditure share matrix* S which has the country k expenditure share for commodity n, s_n^k in row n and column k. Let 1_N and 1_K be vectors of ones of dimension N and K respectively. Then equations (A5) and (A6) can be written in matrix form as follows:[45]

(A8) $\hat{\pi} AP = S1_K$;
(A9) $\pi^T A \hat{P} = 1_N^T S$

where $\pi \equiv [\pi_1,\ldots,\pi_N]$ is the vector of IDB international prices, $P \equiv [P^1,\ldots,P^K]$ is the vector of DI country PPPs, $\hat{\pi}$ denotes an N by N diagonal matrix with the elements of the vector π along the main diagonal and $\hat{P}$ denotes an K by K diagonal matrix with the elements of the vector P along the main diagonal. There are N equations in (A8) and K equations in (A9). However, examining (A8) and (A9), it is evident that if $N+K-1$ of these equations are satisfied, then the remaining equation is also satisfied. Equations (A8) and (A9) are a special case of the *biproportional matrix fitting model* due to Deming and Stephan (1940) in the statistics context and to Stone (1962) in the economics context (the RAS method). Bacharach (1970; 45) studied this model in great detail and gave rigorous conditions for the existence of a unique positive π, P solution set to (A8), (A9)

[44] Equations (27) and (28) provide a first representation in the case where all prices and quantities are positive.
[45] Notation: when examining matrix equations, vectors such as π and P are to be regarded as column vectors and π^T and P^T denote their row vector transposes.

and a normalization such as $P^1 = 1$ or $\pi_1 = 1$.[46] In section A.2 below, we will use Bacharach's analysis in order to provide simple sufficient conditions for the existence and uniqueness of a solution to equations (A8) and (A9) (plus a normalization).

In order to find a solution to (A8) and (A9), one can use the procedure suggested at the end of section A.2.1, since equations (A3) and (A4) are equivalent to (A5) and (A6).[47] Experience with the RAS method has shown that this procedure tends to converge quite rapidly.

A.2.3 The Y^k, π_n Representation

The above representations of the IDB system are in terms of a system of equations involving the N international reference prices π_n and the K country PPPs, P^k. It is useful to substitute equations (24) in the main text, $Y^k = p^k \cdot y^k/P^k$, which define the country volumes or aggregate quantities Y^k in terms of the country k price and quantity vectors p^k and y^k and the country k aggregate PPP, P^k, into equations (A3) and (A4) in order to obtain the following representation of the IDB multilateral system in terms of the Y^k and the π_n:

$$\text{(A10)} \quad \pi_n = [\textstyle\sum_{j=1}^{K} s_n^{\,j}]/[\sum_{k=1}^{K} (y_n^{\,k}/Y^k)] \,; \qquad n = 1,\ldots,N$$
$$\text{(A11)} \quad Y^k = \pi \cdot y^k \,; \qquad k = 1,\ldots,K.$$

Of course, we need to add a normalization such as $Y^1 = 1$ or $\pi_1 = 1$ in order to obtain a unique positive solution to (A10) and (A11).[48] Obviously, a biproportional iteration process could be set up to find a solution to equations (A10) and (A11) along the lines suggested at the end of section A.2.1, except that now the Y^k are determined rather than the P^k.

A.2.4 The Y^k Representation

If we substitute equations (A10) into equations (A11), we obtain the following K equations involving only the country volumes, $Y^1,\ldots,Y^K$:

$$\text{(A12)} \quad Y^k = \textstyle\sum_{n=1}^{N} \{[s_n^{\,1} + \ldots + s_n^{\,K}]\, y_n^{\,k} / [(y_n^{\,1}/Y^1) + \ldots + (y_n^{\,K}/Y^K)]\} \,; \qquad k = 1,\ldots,K.$$

---

[46] It is obvious that if the positive vectors π and P satisfy (A8) and (A9), then $\lambda\pi$ and $\lambda^{-1}P$ also satisfy these equations where λ is any positive scalar. Dikhanov (1997; 12-13) also derived conditions for the existence and uniqueness of the solution set using a different approach.

[47] Bacharach (1970; 46) calls this method of solution the biproportional process. Bacharach (1970; 46-59) establishes conditions for the existence and uniqueness of a solution to the biproportional process; i.e., for the convergence of the process. The normalization (say $P^1 = 1$ or $\pi_1 = 1$) can be imposed at each iteration of the biproportional process or it can be imposed at the end of the process when convergence has been achieved.

[48] It can be verified that if N+K−1 of the equations (A10) and (A11) are satisfied, then the remaining equation is also satisfied; equations (A12) may be used to establish this result.

We need a normalization on the Y^k in order to obtain a unique solution, such as $Y^1 = 1$. It also can be seen that the K equations (A12) are not independent; i.e., if we divide both sides of equation k in (A12) by Y^k for each k and then sum the resulting equations, we obtain the identity K equals K, using the fact that $\sum_{n=1}^N s_n^k = 1$ for each k. Thus once any K−1 of the K equations in (A12) are satisfied, the remaining equation is also satisfied.

Equations (A12) can be used in an iterative fashion in order to obtain a $Y^1,...,Y^K$ solution; i.e., make an initial guess at these volume parities and calculate the right hand side of each equation in (A12). This will generate a new set of volume parities, which can then be normalized to satisfy say $\sum_{k=1}^K Y^k$ equals 1. Then these new volume parities can again be inserted into the right hand sides of equations (A12) and so on.[49]

A.2.5 The P^k Representation

If we substitute the equations $Y^k = p^k \cdot y^k/P^k$ into equations (A12), we obtain the following K equations involving only the country PPPs, $P^1,...,P^K$:

$$\text{(A13)}\ (P^k)^{-1} = \sum_{n=1}^N \{[s_n^1 + ... + s_n^K][y_n^k/p^k \cdot y^k]/[(P^1 y_n^1/p^1 \cdot y^1) + ... + (P^K y_n^K/p^K \cdot y^K)]\}\ ; \qquad k = 1,...,K.$$

As usual, we need a normalization on the P^k in order to obtain a unique solution, such as $P^1 = 1$. It also can be seen that the K equations (A13) are not independent; i.e., if we multiply both sides of equation k in (A13) by P^k for each k and then sum the resulting equations, we obtain the identity K equals K, using the fact that $\sum_{n=1}^N s_n^k = 1$ for each k. Thus once any K−1 of the K equations in (A13) are satisfied, the remaining equation is also satisfied.

Equations (A13) can be used iteratively in order to find a solution in a manner similar to the method described at the end of section A.2.4.

Equations (A12) and (A13) are difficult to interpret at this level of generality but when we look at axiomatic properties and we study special cases of these general equations, it will be seen that the IDB parities have good axiomatic properties.

A.2.6 The π_n Representation

Finally, we substitute equations (A4) into equations (A3) in order to obtain the following system of N equations which characterize the IDB international prices π_n:

$$\text{(A14)}\ \sum_{k=1}^K [\pi_n y_n^k/\pi \cdot y^k] = \sum_{k=1}^K s_n^k\ ; \qquad n = 1,...,N.$$

[49] When this method was tried on the data for the numerical example in Diewert (1999; 79) (see the last section of this appendix), we found that convergence was very slow. The iterative methods described in section A.2.1 converged much more quickly.

It can be seen that equations (A14) are homogeneous of degree 0 in the components of the π vector and so we require a normalization such as $\pi_1 = 1$ in order to obtain a unique positive solution. It also can be seen that if we sum the N equations in (A14), we obtain the identity K equals K and so if any N−1 of the N equations in (A14) are satisfied, then so is the remaining equation.

Equations (A14) can be rewritten as follows:

$$\text{(A15)}\quad \pi_n = [\textstyle\sum_{k=1}^{K} s_n^{k}]/[\sum_{k=1}^{K} y_n^{k}/\pi\cdot y^{k}]\,; \qquad n = 1,\ldots,N.$$

Equations (A15) can be used iteratively in the usual manner in order to obtain a solution to equations (A14).

Equations (A14) have an interesting interpretation. Using the international reference prices π_n, we can define *country k's expenditure share for commodity n using these international prices* as:

$$\text{(A16)}\quad \sigma_n^{k} \equiv \pi_n y_n^{k}/\pi\cdot y^{k}\,; \qquad k = 1,\ldots,K\,;\ n = 1,\ldots,N.$$

Substituting (A16) into (A14) leads to the following system of equations:

$$\text{(A17)}\quad \textstyle\sum_{k=1}^{K} \sigma_n^{k} = \sum_{k=1}^{K} s_n^{k}\,; \qquad n = 1,\ldots,N.$$

Thus for each basic heading commodity group n, the international prices π_n are chosen by the IDB method to be such that the sum over countries expenditure shares for commodity n using these international reference prices, $\sum_{k=1}^{K} \sigma_n^{k}$, is equal to the corresponding sum over countries expenditure shares using domestic prices in each country, $\sum_{k=1}^{K} s_n^{k}$, and this equality holds for all commodity groups n.[50]

A.3 Conditions for the Existence and Uniqueness of Solutions to the IDB Equations

In order to find conditions for positive solutions to any set of the IDB equations, we will use the biproportional matrix representation that was explained in section A.2.2 above.[51]

Bacharach (1970; 43-59) provides very weak sufficient conditions for the existence of a strictly positive solution $\pi_1,\ldots,\pi_N$, $P^1,\ldots,P^K$ to equations (A5) and (A6), assuming that (A1) and (A2) also hold. Bacharach's conditions involve the concept of *matrix connectedness*. Let A be an N by K matrix. Then Bacharach (1970; 44) defines A to be *disconnected* if after a possible reordering of its rows and columns, it can be written in the following *block rectangular* form:

[50] Dividing both sides of (A17) by K means that for each commodity group, the average (over countries) expenditure share using the IDB international prices is equal to the corresponding average expenditure share using the domestic prices prevailing in each country.
[51] Once the existence and uniqueness of a positive solution to any one of our representations of the IDB equations has been established, using assumptions (A1) and (A2), it is straightforward to show that a unique positive solution to the other representations is also implied.

$$\text{(A18) } A = \begin{bmatrix} A_{nxk} & 0_{nx(K-k)} \\ 0_{(N-n)xk} & A_{(N-n)x(K-k)} \end{bmatrix}$$

where $1 \le n < N$, $1 \le k < K$, $A_{n\times k}$ and $A_{(N-n)\times(K-k)}$ are submatrices of A of dimension n by k and N−n by K−k respectively and $0_{n\times(K-k)}$ and $0_{(N-n)\times(K-k)}$ are n by K−k and N−n by K−k matrices of zeros. As Bacharach (1970; 47) notes, the concept of disconnectedness is a generalization to rectangular matrices of the concept of decomposability which applies to square matrices. Bacharach (1970; 47) defines A to be *connected* if it is not disconnected (and it can be seen that this is a generalization of the concept of indecomposibility which applies to square matrices). Bacharach (1970; 47-55) goes on to show that if the matrix A defined by (A7) is connected, assumptions (A1) and (A2) hold, and we add a normalization like $\pi_1 = 1$ or $P^1 = 1$ to equations (A5) and (A6), then these equations have a unique positive solution which can be obtained by using the biproportional procedure suggested at the end of section A.2.1, which will converge.

It is useful to have somewhat simpler conditions on the matrix A defined by (A7) which will imply that it is connected. It can be seen that either of the following two simple conditions will imply that A is connected (and hence, we have sufficient conditions for the existence of unique positive solutions to any representation of the IDB equations):

(A19) There exists a commodity n which is demanded by all countries; i.e., there exists an n such that $y_n^k > 0$ for k = 1,...,K;

(A20) There exists a country k which demands all commodities; i.e., there exists a k such that $y_n^k > 0$ for n = 1,...,N.

Conditions (A19) and (A20) are easy to check. We will make use of these assumptions in the following section.

A.4 Special Cases

In this section, we will specialize some of the general N and K representations of the IDB equations to cases where the number of commodities N or the number of countries K is equal to two.

A.4.1 The Two Country, Many Commodity Quantity Index Case

Suppose that the number of countries K is equal to 2. Set the country 1 volume equal to 1 so that Y^1 equals one and the first equation in (A12) becomes:

$$\text{(A21) } \textstyle\sum_{n=1}^{N} \{[s_n^1 + s_n^2]\, y_n^1 / [y_n^1 + (y_n^2/Y^2)]\} = 1.$$

Equation (A21) is one equation in the one unknown Y^2 and it implicitly determines Y^2. It can be seen that Y^2 can be interpreted as a Fisher (1922) type bilateral quantity index,

$Q_{IDB}(p^1,p^2,y^1,y^2)$, where p^k and y^k are the price and quantity vectors for country k. Thus in what follows for this section, we will replace Y^2 by Q.

At this point, we will assume that the data for country 1 satisfy assumption (A20) (so that y^1, p^1 and s^1 are all strictly positive vectors), which guarantees a unique positive solution to (A21). With this assumption, the quantity relatives r_n are well defined as follows:

$$(A22)\ r_n \equiv y_n^2/y_n^1 \geq 0\ ; \qquad n = 1,...,N.$$

Assumption (A2) implies that at least one quantity relative r_n is positive. Since each y_n^1 is positive and letting Q equal Y^2, we can rewrite (A21) using definitions (A22) as follows:[52]

$$(A23)\ \sum_{n=1}^{N} \{[s_n^1 + s_n^2] / [1 + (r_n/Q)]\} = 1.$$

Define the *vector of quantity relatives* r as $[r_1,...,r_N]$. Then the function on the left hand side of (A23) can be defined as $F(Q,r,s^1,s^2)$, where s^k is the expenditure share vector for country k for k = 1,2. Note that $F(Q,r,s^1,s^2)$ is a continuous, monotonically increasing function of Q for Q positive. Recall that we are assuming that the components of y^1 and hence s^1 are all positive. We now compute the limits of $F(Q,r,s^1,s^2)$ as Q tends to plus infinity:

$$(A24)\ \lim_{Q\to+\infty} F(Q,r,s^1,s^2) = \sum_{n=1}^{N} \{[s_n^1 + s_n^2] = 2.$$

In order to compute the limit of $F(Q,r,s^1,s^2)$ as Q tends to 0, we need to consider two cases. For the first case, assume that both countries consume all commodities so that $y^2 >> 0_N$ (in addition to our earlier assumption that $y^1 >> 0_N$). In this case, it is easy to verify that:

$$(A25)\ \lim_{Q\to0} F(Q,r,s^1,s^2) = 0.$$

For the second case, assume that one or more components of y^2 are zero and let N^* be the set of indexes n such that y_n^2 equals 0. In this case, we have:

$$(A26)\ \lim_{Q\to0} F(Q,r,s^1,s^2) = \sum_{n\in N^*} s_n^1 < 1$$

where the inequality in (A26) follows from the fact that we are assuming that all s_n^1 are positive and the sum of all of the s_n^1 is 1.

The fact that $F(Q,r,s^1,s^2)$ is a continuous, monotonically increasing function of Q along with (A24)-(A26) implies that a finite positive Q solution to the equation $F(Q,r,s^1,s^2) = 1$ exists and is unique. Denote this solution as

$$(A27)\ Q = G(r,s^1,s^2).$$

[52] (A23) shows that Q depends only on the components of two N dimensional vectors, r and $s^1 + s^2$.

Now use the Implicit Function Theorem to show that $G(r,s^1,s^2)$ is a continuously differentiable function which is increasing in the components of r; i.e., we have:

$$\text{(A28)}\quad \partial G(r,s^1,s^2)/\partial r_n = [s_n^1 + s_n^2]\,[1 + (r_n/Q)]^{-2}Q/\{\sum_{i=1}^N [s_i^1 + s_i^2][1 + (r_i/Q)]^{-2}r_i\} > 0\,; \qquad n = 1,\ldots,N$$

where Q satisfies (A27). However, the inequalities in (A28) do not imply that the IDB bilateral index number formula $Q_{IDB}(p^1,p^2,y^1,y^2)$ is increasing in the components of y^2 and decreasing in the components of y^1, since the derivatives in (A28) were calculated under the hypothesis that r_n equal to y_n^2/y_n^1 increased but the share vectors s^1 and s^2 were held *constant* as r_n was increased. In fact, it is *not* the case that $Q_{IDB}(p^1,p^2,y^1,y^2)$ is globally increasing in the components of y^2 and globally decreasing in the components of y^1.[53]

It is clear that $Q_{IDB}(p^1,p^2,y^1,y^2)$ satisfies the *identity test*; i.e., if $y^1 = y^2$ so that all quantity relatives r_n equal 1, then the only Q which satisfies (A23) is Q = 1. It is also clear that if $y^2 = \lambda y^1$ for $\lambda > 0$, then $Q_{IDB}(p^1,p^2,y^1,\lambda y^1)$ equals λ.[54]

Define $\alpha \geq 0$ as the minimum over n of the quantity relatives, $r_n = y_n^2/y_n^1$ and define $\beta > 0$ as the maximum of these quantity relatives. Then using the monotonicity properties of the function $F(Q,r,s^1,s^2)$ defined by the left hand side of (A23), it can be shown that

$$\text{(A29)}\quad \alpha \leq Q_{IDB}(p^1,p^2,y^1,y^2) \leq \beta$$

with strict inequalities in (A29) if the r_n are not all equal. Thus the IDB bilateral quantity index satisfies the usual *mean value test* for bilateral quantity indexes.[55]

It is possible to develop various approximations to $Q_{IDB}(p^1,p^2,y^1,y^2)$ that cast some light on the structure of the index. Recall that (A23) defined Q_{IDB} in implicit form. This equation can be rewritten as a *weighted harmonic mean* equal to 2 as follows:

$$\text{(A30)}\quad \{\sum_{n=1}^N w_n\,[1 + (r_n/Q)]^{-1}\}^{-1} = 2$$

where the *weights* w_n in (A30) are defined as follows:

$$\text{(A31)}\quad w_n \equiv (1/2)[s_n^1 + s_n^2]\,; \qquad n = 1,\ldots,N.$$

[53] This negative monotonicity result also applies to the Törnqvist Theil bilateral index number formula, Q_T; see Diewert (1992; 221). The logarithm of Q_T is defined as $\ln Q_T = \sum_{n=1}^N (1/2)[s_n^1 + s_n^2] \ln r_n$.

[54] It is also clear from (A23) that $Q_{IDB}(p^1,p^2,y^1,y^2)$ satisfies the following four homogeneity tests $Q_{IDB}(p^1,p^2,y^1,\lambda y^2) = \lambda Q_{IDB}(p^1,p^2,y^1,y^2)$, $Q_{IDB}(p^1,p^2,\lambda y^1,y^2) = \lambda^{-1}Q_{IDB}(p^1,p^2,y^1,y^2)$, $Q_{IDB}(\lambda p^1,p^2,y^1,y^2) = Q_{IDB}(p^1,p^2,y^1,y^2)$ and $Q_{IDB}(p^1,\lambda p^2,y^1,y^2) = Q_{IDB}(p^1,p^2,y^1,y^2)$ for all $\lambda > 0$. Equations (A21) or (A23) can be used to show that $Q_{IDB}(p^1,p^2,y^1,y^2)$ satisfies the first eleven of Diewert's (1999; 36) thirteen tests for a bilateral quantity index, failing only the monotonicity in the components of y^1 and y^2 tests. Thus the axiomatic properties of the IDB bilateral quantity index are rather good.

[55] See Diewert (1992) for the history of these bilateral tests.

Now approximate the weighted harmonic mean on the left hand side of (A30) by the corresponding weighted arithmetic mean and we obtain the following approximate version of equation (A30):

(A31) $\sum_{n=1}^{N} w_n [1 + (r_n/Q)] \approx 2$.

Using the fact that the weights w_n sum up to one, (A31) implies that $Q = Q_{IDB}$ is approximately equal to the following expression:

(A32) $Q_{IDB}(r,w) \approx \sum_{n=1}^{N} w_n r_n = \sum_{n=1}^{N} (1/2)[(p_n{}^1 y_n{}^1/p^1\cdot y^1) + (p_n{}^2 y_n{}^2/p^2\cdot y^2)][y_n{}^2/y_n{}^1]$.

If we further approximate the weighted arithmetic mean on the right hand side of (A32) by the corresponding weighted geometric mean, then we find that $Q_{IDB}(r,w)$ is approximately equal to the following expression:

(A33) $Q_{IDB}(r,w) \approx \prod_{n=1}^{N} r_n^{w_n} \equiv Q_T(r,w)$

where Q_T is the logarithm of the Törnqvist Theil quantity index defined as $\ln Q_T = \sum_{n=1}^{N} w_n \ln r_n$. If all of the quantity relatives r_n are equal to the same positive number, λ say, then the approximations in (A31)-(A33) will be exact and under these conditions where y^2 is equal to λy^1, we will have

(A34) $Q_{IDB}(\lambda 1_N,w) = Q_T(\lambda 1_N,w) = \lambda$.

In the more general case, where the quantity relatives r_n are approximately equal to the same positive number so that y^2 is approximately proportional to y^1, then the Törnqvist Theil quantity index $Q_T(r,w)$ will provide a good approximation to the implicitly defined IDB quantity index, $Q_{IDB}(r,w)$.[56] However, in the international comparison context, it is frequently the case that quantity vectors are far from being proportional and in this nonproportional case, Q_{IDB} can be rather far from Q_T and other superlative indexes as we shall see in the final section of this Appendix.

A.4.2 The Two Country, Many Commodity Price Index Case

Again, suppose that the number of countries K is equal to 2. Set the country 1 PPP, P^1, equal to 1 and the first equation in (A13) becomes:

(A35) $\sum_{n=1}^{N} \{[s_n{}^1 + s_n{}^2](y_n{}^1/p^1\cdot y^1)/[(y_n{}^1/p^1\cdot y^1) + (P^2 y_n{}^2/p^2\cdot y^2)]\} = 1$.

Equation (A35) is one equation in the one unknown P^2 (the country 2 PPP) and it implicitly determines P^2. It can be seen that P^2 can be interpreted as a Fisher (1922) type

[56] If we regard $Q_{IDB}(r)$ and $Q_T(r)$ as functions of the vector of quantity relatives, then it can be shown directly that $Q_{IDB}(r)$ approximates $Q_T(r)$ to the second order around the point $r = 1_N$.

bilateral price index, $P_{IDB}(p^1,p^2,y^1,y^2)$, where p^k and y^k are the price and quantity vectors for country k. Thus in what follows, we will replace P^2 by P.

We again assume that the data for country 1 satisfy assumption (A20) (so that y^1, p^1 and s^1 are all strictly positive vectors), which guarantees a unique positive solution to (A35). It is convenient to define the *country k normalized quantity vector* u^k as the country k quantity vector divided by the value of its output in domestic currency, $p^k \cdot y^k$:

$$(A36)\quad u^k \equiv y^k/p^k \cdot y^k \; ; \qquad k = 1,2.$$

Since y^1 is strictly positive, so is u^1. Hence definitions (A36) can be substituted into (A35) in order to obtain the following equation, which implicitly determines $P^2 = P = P_{IDB}$:

$$(A37)\quad \sum_{n=1}^{N} \{[s_n^1 + s_n^2]/[1 + P(y_n^2/y_n^1)(p^1 \cdot y^1/p^2 \cdot y^2)]\} = \sum_{n=1}^{N} \{[s_n^1 + s_n^2]/[1 + P(u_n^2/u_n^1)]\} = 1.$$

If we define $r_n \equiv u_n^2/u_n^1$ for n = 1,...,N and rewrite P as 1/Q, it can be seen that equation (A37) becomes equation (A23) in the previous section and so the analysis surrounding equations (A23)-(A29) can be repeated to give the existence of a positive solution $P(r,s^1,s^2)$ to (A37) along with some of the properties of the solution.

Equation (A37) can be used to show that the IDB bilateral price index P, which is the solution to (A37), regarded as a function of the price and quantity data pertaining to the two countries, $P_{IDB}(p^1,p^2,y^1,y^2)$, satisfies the first eleven of the thirteen bilateral tests listed in Diewert (1999; 36)[57], failing only the monotonicity in the components of p^1 and p^2 tests; i.e., it is not necessarily the case that $P_{IDB}(p^1,p^2,y^1,y^2)$ is decreasing in the components of p^1 and increasing in the components of p^2. Thus the axiomatic properties of the IDB bilateral price index are rather good.

The bounds on the IDB bilateral quantity index given by (A29) do not have exactly analogous price counterparts. To develop counterparts to the bounds (A29), it is convenient to assume that all of the price and quantity data pertaining to both countries are positive and then we can define the following N *implicit partial price indexes* ρ_n:

$$(A38)\quad \rho_n \equiv [p^2 \cdot y^2/y_n^2]/[p^1 \cdot y^1/y_n^1] = [p^2 \cdot y^2/p^1 \cdot y^1]/[y_n^2/y_n^1] \; ; \qquad n = 1,...,N.$$

An implicit bilateral price index is defined as the value ratio, $p^2 \cdot y^2/p^1 \cdot y^1$, divided by a quantity index, say $Q(p^1,p^2,y^1,y^2)$, where Q is generally some type of weighted average of the individual quantity relatives, y_n^2/y_n^1. Thus each quantity relative, y_n^2/y_n^1, can be regarded as a partial quantity index and hence the corresponding implicit quantity index, which is the value ratio divided by the quantity relative, can be regarded as an implicit

[57] The role of prices and quantities must be interchanged; i.e., Diewert's (1999; 36) tests referred to quantity indexes whereas we are now considering price indexes.

partial price index. Substitution of definitions (A38) into (A37) leads to the following equation which implicitly determines P equal to $P_{IDB}(p^1,p^2,y^1,y^2)$:

(A39) $\sum_{n=1}^{N} \{[s_n^1 + s_n^2]/[1 + (P/\rho_n)]\} = 1.$

Define α as the minimum over n of the partial price indexes ρ_n and define β as the maximum of these partial price indexes. Then the monotonicity properties of the function defined by the left hand side of (A39) can be used in order to establish the following inequalities:

(A40) $\alpha \leq P_{IDB}(p^1,p^2,y^1,y^2) \leq \beta$

with strict inequalities in (A40) if the ρ_n are not all equal.

An approximate explicit formula for P_{IDB} can readily be developed. Recall that (A39) defined P_{IDB} in implicit form. This equation can be rewritten as a *weighted harmonic mean* equal to 2 as follows:

(A41) $\{\sum_{n=1}^{N} w_n [1 + (P/\rho_n)]^{-1}\}^{-1} = 2$

where the *weights* w_n in (A41) are the average expenditure shares, $(1/2)[s_n^1 + s_n^2]$ for n = 1,...,N. Now approximate the weighted harmonic mean on the left hand side of (A41) by the corresponding weighted arithmetic mean and we obtain the following approximate version of equation (A30):

(A42) $\sum_{n=1}^{N} w_n [1 + (P/\rho_n)] \approx 2.$

Using the fact that the weights w_n sum up to one, (A42) implies that $P = P_{IDB}$ is approximately equal to the following expression:

$$
\begin{aligned}
\text{(A43)}\ P_{IDB}(\rho,w) &\approx [\textstyle\sum_{n=1}^{N} w_n (\rho_n)^{-1}]^{-1} \\
&= \{\textstyle\sum_{n=1}^{N} (1/2)[(p_n^1 y_n^1/p^1\cdot y^1) + (p_n^2 y_n^2/p^2\cdot y^2)][y_n^2/y_n^1][p^1\cdot y^1/p^2\cdot y^2]\}^{-1}
\end{aligned}
$$

where $\rho \equiv [\rho_1,...,\rho_N]$ and $w \equiv [w_1,...,w_N]$. Thus the IDB bilateral price index P_{IDB} is *approximately* equal to a weighted harmonic mean of the N partial price indexes ρ_n defined earlier by (A38).[58]

A.4.3 The Many Country, Two Commodity Case

We now consider the case where there are K countries but only two commodities so that N = 2. Recall that equations (A4) and (A11) determine the IDB country PPPs, P^k, and the country volumes, Y^k, in terms of the country price and quantity vectors, p^k and y^k, and

[58] The expressions involving the reciprocals of the ρ_n require that y^2 be strictly positive (in addition to our maintained assumption that y^1 be strictly positive). Equations (A35) and (A37) require only that y^1 be strictly positive.

a vector of international reference prices $\pi \equiv [\pi_1,...,\pi_N]$. Thus once π is determined, the P^k and Y^k can readily be determined. In this section, we assume that N = 2, so that there are only 2 commodities and K countries. In order to ensure the existence of a solution to the IDB equations, we will assume that commodity 1 is consumed by all countries; i.e., we assume that:

$$\text{(A44)}\ y_1^k > 0\ ; \qquad\qquad k = 1,...,K.$$

We are allowed to normalize one of the two international prices so we will set the first price equal to one:

$$\text{(A45)}\ \pi_1 = 1.$$

The equations which determine the π_n are equations (A14) but since N = 2, we can drop the second equation in (A14). Using the normalization (A45), the first equation in (A14) becomes the following equation:

$$\text{(A46)}\ \sum_{k=1}^K y_1^k/[y_1^k + \pi_2 y_2^k] = \sum_{k=1}^K s_1^k$$

which is one equation which determines the international price for commodity 2, π_2.

Using assumptions (A44), the *country k commodity relatives* R^k (the quantity of commodity 2 relative to 1 in country k) are well defined as follows:

$$\text{(A47)}\ R^k \equiv y_2^k/y_1^k \geq 0\ ; \qquad\qquad k = 1,...,K.$$

Assumption (A1) implies that at least one quantity relative R^k is positive. Since each y_1^k is positive, we can rewrite (A46) using definitions (A47) as follows:[59]

$$\text{(A48)}\ \sum_{k=1}^K 1/[1 + \pi_2 R^k] = \sum_{k=1}^K s_1^k \equiv s_1$$

where s_1 is defined to be the sum over countries k of the expenditure share of commodity 1 in country k, s_1^k.[60] Define the *vector of country quantity relatives* R as $[R^1,...,R^K]$. Then the function on the left hand side of (A48) can be defined as $F(\pi_2,R,s_1)$.[61] Note that $F(\pi_2,R,s_1)$ is a continuous, monotonically decreasing function of π_2 for π_2 positive, since the R^k are nonnegative with at least one R^k positive. We now compute the limits of $F(\pi_2,R,s_1)$ as π_2 tends to zero:

$$\text{(A49)}\ \lim_{\pi_2 \to 0} F(\pi_2,R,s_1) = K > \sum_{k=1}^K s_1^k = s_1.$$

[59] (A23) shows that Q depends only on the components of two N dimensional vectors, r and $s^1 + s^2$.
[60] Note that s_1 satisfies the inequalities $0 < s_1 < K$.
[61] Thus the π_2 solution to (A48) depends only on the vector of country quantity relatives, R, and the sum across countries k of the expenditure shares on commodity 1, s_1^k. Alternatively, π_2 depends on the K dimensional vector R and the sum across countries commodity share vector, $s^1 +...+ s^K$, which is a two dimensional vector in the present context where N = 2.

In order to compute the limit of $F(\pi_2,R,s_1)$ as π_2 tends to plus infinity, we need to consider two cases. For the first case, assume that all countries consume both commodities so that $R >> 0_N$.

(A50) $\lim_{\pi_2 \to +\infty} F(\pi_2,R,s_1) = 0 < \sum_{k=1}^{K} s_1^{k} = s_1$.

For the second case, assume that one or more components of R are zero and let K^* be the set of indexes k such that R^k equals 0. In this case, we have:

(A51) $\lim_{\pi_2 \to +\infty} F(\pi_2,R,s_1) = \sum_{k \in K^*} s_n^{1} < \sum_{k=1}^{K} s_1^{k} = s_1$.

The fact that $F(\pi_2,R,s_1)$ is a continuous, monotonically decreasing function of π_2 along with (A49)-(A51) implies that a finite positive π_2 solution to equation (A48) exists and is unique. Denote this solution as $\pi_2 = G(R,s_1)$. It is straightforward to verify that G is decreasing in the components of R and decreasing in s_1.

Suppose that all country quantity relatives R^k are positive and define α and β to be the minimum and maximum over k respectively of these quantity relatives. Then it is also straightforward to verify that π_2 satisfies the following bounds:[62]

(A52) $[(s_1/K)^{-1} - 1]/\beta \leq \pi_2 \leq [(s_1/K)^{-1} - 1]/\alpha$.

Thus if all of country quantity relatives $R^k = y_2^{k}/y_1^{k}$ are equal to the same positive number λ, then the bounds in (A52) collapse to the common value $[(s_1/K)^{-1} - 1]/\lambda$.

In the case where prices and quantities are positive across all countries (so that all R^k are positive), then it is possible to rewrite the basic equation (A48) in a more illuminating form as follows:

$$
\begin{aligned}
\text{(A53)}\ \sum_{k=1}^{K} s_1^{k} &= \sum_{k=1}^{K} 1/[1 + \pi_2 R^k] \\
&= \sum_{k=1}^{K} \{s_1^{k}/[s_1^{k} + \pi_2 s_1^{k}(y_2^{k}/y_1^{k})]\} \\
&= \sum_{k=1}^{K} \{s_1^{k}/[s_1^{k} + \pi_2 s_2^{k}(p_2^{k}/p_1^{k})^{-1}]\}.
\end{aligned}
$$

Equation (A53) shows that the π_2 which solves the equation is a function of the K country share vectors, $s^1,...,s^K$ (each of which is of dimension 2) and the vector of K country price relatives, $[p_2^{1}/p_1^{1},..., p_2^{K}/p_1^{K}]$. It can be seen that if all of these country price relatives are equal to a common ratio, say $\lambda > 0$, then the solution to (A53) is $\pi_2 = \lambda$. In the case where all of these country price relatives are positive, let α^* and β^* to be the minimum and maximum over k respectively of these price relatives. Then it is straightforward to verify that π_2 satisfies the following bounds:

[62] It can be verified that $0 < s_1 < K$ so that $(s_1/K)^{-1} > 1$ so that the bounds in (A52) are positive when $R >> 0_N$. In the case where $R > 0_N$, the lower bound is still valid but the upper bound becomes $+\infty$.

(A54) $\alpha^* \le \pi_2 \le \beta^*$.

A.4.4 The Two Country, Two Commodity Case

In this section, we assume that K = 2 (two countries) and that N = 2 (two commodities). In this case, it is possible to obtain an explicit formula for the country 2 volume Y^2 relative to relative to the country 1 volume Y^1 which is set equal to one; i.e., we can obtain an explicit formula for the IDB bilateral quantity index, $Y^2 = Q = Q_{IDB}(p^1,p^2,y^1,y^2)$. Our starting point is equation (A21) which determines Q implicitly. In the case where N equals 2, this equation becomes:

$$\text{(A55)}\ \{[s_1^1 + s_1^2]y_1^1/[y_1^1 + (y_1^2/Q)]\} + \{[(1-s_1^1) + (1-s_1^2)]y_2^1/[y_2^1 + (y_2^2/Q)]\} = 1.$$

As usual, we assume that the data for country 1 are positive so that $y_1^1 > 0$ and $y_2^1 > 0$. Thus the two quantity relatives, $r_n \equiv y_n^2/y_n^1$ for n = 1,2, are well defined nonnegative numbers. We assume that at least one of the relatives r_1 and r_2 are strictly positive. Substitution of these quantity relatives into (A55) leads to the following equation for Q:

$$\text{(A56)}\ \{[s_1^1 + s_1^2]Q/[Q + r_1]\} + \{[(1-s_1^1) + (1-s_1^2)]Q/[Q + r_2]\} = 1.$$

The above equation simplifies into the following quadratic equation:[63]

$$\text{(A57)}\ Q^2 + [s_1^1+s_1^2-1][r_2-r_1]Q - r_1r_2 = 0.$$

In the case where both r_1 and r_2 are positive, there is a negative and a positive root for (A57). The positive root is the desired bilateral quantity index and it is equal to the following expression:

$$\text{(A58)}\ Q_{IDB}(p^1,p^2,y^1,y^2) = -(1/2)(s_1^1+s_1^2-1)(r_2-r_1) + (1/2)[(s_1^1+s_1^2-1)^2 (r_2-r_1)^2 + 4r_1r_2]^{1/2}.$$

Suppose $r_1 = y_1^2/y_1^1 = 0$ so that $y_1^1 > 0$ and $y_1^2 = 0$. Then $s_1^2 = 0$ as well and using (A57), we have:

$$\text{(A59)}\ Q = [1-s_1^1]r_2 = [1-s_1^1][y_2^2/y_2^1].$$

Formula (A59) makes sense in the present context. Remember that Q is supposed to reflect the country 2 volume or average quantity relative to country 1. If, as a preliminary estimate of this relative volume, we set Q equal to the single nonzero quantity relative, r_2, then this would overestimate the average volume of country 2 relative to 1 since country 2 has a zero amount of commodity 1 while country 1 has the positive amount y_1^1. Thus we scale down r_2 by multiplying it by one minus country 1's share of commodity 1, s_1^1. The bigger is this share, the more we downsize the preliminary volume ratio r_2.

---

[63] This equation can be utilized to show that $Q_{IDB}(p^1,p^2,y^1,y^2)$ is not necessarily monotonically increasing in the components of y^1 or monotonically decreasing in the components of y^1.

Now suppose that $r_2 = y_2^2/y_2^1 = 0$ so that $y_2^1 > 0$ and $y_2^2 = 0$. Then $s_1^2 = 1$ and using (A57), we have:

(A60) $Q = s_1^1 r_1 = [1-s_2^1][y_1^2/y_1^1]$.

Again, formula (A60) makes sense in the present context. If we set Q equal to the single nonzero quantity relative, r_1, then this would overestimate the average volume of country 2 relative to 1 since country 2 has a zero amount of commodity 2 while country 1 has the positive amount y_2^1. Thus we scale down r_1 by multiplying it by one minus country 1's share of commodity 2, s_2^1. The bigger is this share, the more we downsize the preliminary volume ratio r_1.

Two other special cases of (A57) are of interest. Consider the cases where the following conditions hold:

(A61) $r_1 = r_2$;
(A62) $s_1^1+s_1^2 = 1$.

If either of the above two special cases hold, then Q equals $(r_1r_2)^{1/2}$, the geometric mean of the two quantity relatives. This first result is not surprising since this result is implied by our earlier N commodity results for two countries; i.e., see (A29). The second result is more interesting. Note that if (A62) holds, so that the sum of the two country expenditure shares on commodity 1 is equal to 1, then the sum of the two country expenditure shares on commodity 2 is also equal to 1; i.e., we also have $s_2^1+s_2^2 = 1$ and the IDB quantity index is equal to the geometric mean of the two quantity relatives, $(r_1r_2)^{1/2}$.[64]

We turn now to a discussion of the axiomatic or test properties of the IDB multilateral system.

A.5 The Axiomatic Properties of the Iklé Dikhanov Balk Multilateral System

Balk (1996; 207-212) developed the axiomatic properties of the IDB multilateral method using a set of nine axioms based on the earlier work of Diewert (1988).[65] Diewert (1999; 16-20) further refined his set of axioms and below, we will list the thirteen "reasonable" axioms he proposed for a multilateral system. A slightly different system of notation will be used in the present section: $P \equiv [p^1,...,p^K]$ will signify an N by K matrix which has the domestic price vectors $p^1,...,p^K$ as its K columns and $Y \equiv [y^1,...,y^K]$ will signify an N by K matrix which has the country quantity vectors $y^1,...,y^K$ as its K columns.

[64] Under these conditions, it is also the case that all prices and quantities are positive in the two countries since we also assumed that y^1 is strictly positive and y^2 is nonnegative and nonzero; i.e., $y^1 >> 0_2$ and $y^2 > 0_2$.

[65] Balk's axioms were somewhat different from those proposed by Diewert since Balk also introduced an extra set of country weights into Diewert's axioms. We will not follow Balk's example since it is difficult to determine precisely these country weights.

Equations (A12) plus a normalization determine the country aggregate products, $Y^1,...,Y^K$. These country volumes Y^k can be regarded as functions of the data matrices P and Y, so equations (A12) plus a normalization determine the functions, $Y^k(P,Y)$ for $k = 1,...,K$. Once these functions $Y^k(P,Y)$ have been determined, then *country k's share of world output*, $S^k(P,Y)$, can be defined as follows:

(A63) $S^k(P,Y) \equiv Y^k(P,Y)/[Y^1(P,Y) + ... + Y^K(P,Y)]$; $\qquad k = 1,...,K.$

Both Balk (1996) and Diewert (1988) (1999) used the system of world share equations $S^k(P,Y)$ as the basis for their axioms. Alternatively, instead of using equations (A12) and (A63) to define the share functions $S^k(P,Y)$, these functions can be determined as the functions $Y^k(P,Y)$ which solve equations (A12) if we add the following normalization to equations (A12):

(A64) $Y^1 + ... + Y^K = 1.$

We will now list 11 of Diewert's (1999; 16-20) system of 12 tests or axioms for a multilateral share system, $S^1(P,Y),...,S^K(P,Y)$.[66] The domain of definition for these functions is the set of price and quantity vectors, P,Y, which satisfy the assumptions listed in the first paragraph of section A.2.1 plus it is assumed that at least one of assumptions (A1) or (A2) hold.

T1: *Share Test*: There exist K continuous, positive functions, $S^k(P,Y)$, $k = 1,...,K$, such that $\sum_{k=1}^K S^k(P,Y) = 1$ for all P,Y in the appropriate domain of definition.

T2: *Proportional Quantities Test*: Suppose that $y^k = \beta_k y$ for some $y >> 0_N$ and $\beta_k > 0$ for $k = 1,...,K$ with $\sum_{k=1}^K \beta_k = 1$ Then $S^k(P,Y) = \beta_k$ for $k = 1,...,K$.

T3: *Proportional Prices Test*: Suppose that $p^k = \alpha_k p$ for $p >> 0_N$ and $\alpha_k > 0$ for $k = 1,...,K$. Then $S^k(P,Y) = p\cdot y^k/[p\cdot\sum_{i=1}^K y^i]$ for $k = 1,...,K$.

T4: *Commensurability Test*: Let $\delta_n > 0$ for $n = 1,...,N$ and let D denote the N by N diagonal matrix with the δ_n on the main diagonal. Then $S^k(DP,D^{-1}Y) = S^k(P,Y)$ for $k = 1,...,K$.

This test implies that the country shares $S^k(P,Y)$ are invariant to changes in the units of measurement.

T5: *Commodity Reversal Test*: Let Π denote an N by N permutation matrix. Then $S^k(\Pi P,\Pi Y) = S^k(P,Y)$ for $k = 1,...,K$.

[66] We omit Diewert's (1999; 18) bilateral consistency in aggregation test (which the IDB system does not satisfy) since this test depends on choosing a "best" bilateral quantity index and there may be no consensus on what this "best' functional form is.

This test implies that each country's share of world output remains unchanged if the ordering of the N commodities is changed.

T6: *Multilateral Country Reversal Test*: Let $S(P,Y)$ denote a K dimensional column vector that has the country shares $S^1(P,Y),.., S^K(P,Y)$ as components and let Π^* be a K by K permutation matrix. Then $S(P\Pi^*,Y\Pi^*) = S(P,Y)\Pi^*$.

This test implies that countries are treated in a symmetric manner; i.e., the country shares of world output are not affected by a reordering of the countries. The next two tests are homogeneity tests.

T7: *Monetary Units Test*: Let $\alpha_k > 0$ for $k = 1,...,K$. Then $S^k(\alpha_1 p^1,...,\alpha_K p^K,Y) = S^k(p^1,...,p^K,Y) = S^k(P,Y)$ for $k = 1,...,K$.

This test implies that the absolute scale of domestic prices in each country does not affect each country's share of world output; i.e., only relative prices within each country affect the multilateral volume parities.

T8: *Homogeneity in Quantities Test*: For $i = 1,...,K$, let $\lambda_i > 0$ and let j denote another country not equal to country i. Then $S^i(P,y^1,...,\lambda_i y^i,...,y^K)/S^j(P,y^1,...,\lambda_i y^i,...,y^K) = \lambda_i S^i(P, y^1,..., y^i,...,y^K)/S^i(P,y^1,..., y^i,...,y^K) = \lambda_i S^i(P,Y)/S^j(P,Y)$.

This test says that the output share of country i relative to country j is linearly homogeneous in the components of the country i output vector y^i.

T9: *Monotonicity Test in Quantities Test*: For each k, $S^k(P,y^1,...,y^{i-1},y^k,y^{i+1},...,y^K) = S^k(P,Y)$ is increasing in the components of y^k.

This test says that country k's share of world output increases as any component of the country k quantity vector y^k increases.

T10: *Country Partitioning Test*: Let A be a strict subset of the indexes (1,2,...,K) with at least two members. Suppose that for each $i \in A$, $p^i = \alpha_i p^a$ for $\alpha_i > 0$, $p^a >> 0_N$ and $y^i = \beta_i p^a$ for $\beta_i > 0$, $y^a >> 0_N$ with $\sum_{i\in A} \beta_i = 1$. Denote the subset of $\{1,2,...,K\}$ that does not belong to A by B and denote the matrices of country price and quantity vectors that belong to B by P^b and Y^b respectively. Then: (i) for $i \in A$, $j \in A$, $S^i(P,Y)/S^j(P,Y) = \beta_i/\beta_j$ and (ii) for $i \in B$, $S^i(P,Y) = S^{i*}(p^a,P^b,y^a,Y^b)$ where $S^{k*}(p^a,P^b,y^a,Y^b)$ is the system of share functions that is obtained by adding the group A aggregate price and quantity vectors, p^a and y^a respectively, to the group B price and quantity data, P^b,Y^b.

Thus if the aggregate quantity vector for the countries in group A were distributed proportionally among its members (using the weights β_i) and each group A country faced prices that were proportional to p^a, then part (i) of T10 requires that the group A share functions reflect this proportional allocation. Part (ii) of T10 requires that the group B share functions give the same values no matter whether we use the original share system or a new share system where all of the group A countries have been aggregated up into the single country which has the price vector p^a and the group A aggregate quantity vector y^a. Conversely, this test can be viewed as a consistency in aggregation test if a single group A country is partitioned into a group of smaller countries.

T11: *Additivity Test*: For each set of price and quantity data, P,Y, belonging to the appropriate domain of definition, there exists a set of positive world reference prices $\pi >> 0_N$ such that $S^k(P,Y) = \pi \cdot y^k/[\pi \cdot \sum_{i=1}^K y^i]$ for k = 1,...,K.

The IDB multilateral system obviously satisfies the additivity test T11.

Proposition 1: Assume that the country price and quantity data P,Y satisfy assumptions (A1), (A2) and at least one of the assumptions (A19) and (A20). Then the IDB multilateral system fails only Tests 9 and 10 in the above list of 11 tests.

Proof: We have already discussed the existence and uniqueness of a solution to any one of our representations of the IDB equations in section A.3 above. The continuity (and once continuous differentiability) of the IDB share functions $S^k(P,Y)$ in the data follow using the Implicit Function Theorem on the system of equations (A8) and (A9) (plus a normalization) by adapting the arguments in Bacharach (1970; 67-68). This establishes T1.

The proofs of tests T2 and T4-T8 follow by straightforward substitution into equations (A12).

The proof of T3 follows by setting $\pi = p$ and then showing that this choice of π satisfies equations (A14). Once π has been determined as p, then the Y^k are determined as $\pi \cdot y^k$ for k = 1,...,K and finally the share functions are determined using (A63).

The results in section A.4.4 can be used to show that the monotonicity test T9 fails.

Finally, it can be seen that the "democratic" nature of the IDB system (each country's shares are treated equally in forming the reference prices π) leads to a failure of test T10.[67]

The main text showed that the IDB method satisfied the additivity test T11. Q.E.D.

It is useful to contrast the IDB method with the other additive method that has been used in ICP and that is the Geary Khamis system. Both methods satisfy tests T1-T7 and T11 and both methods fail the monotonicity in quantities test T9. Thus the tests that discriminate between the two methods are T8 and T10: the IDB multilateral system passes the homogeneity test T8 and fails the country partitioning test T10 and vice versa

[67] Diewert (1999; 27) showed that the GK system satisfied all of the 11 tests except the homogeneity test T8 and the monotonicity test T9. The GK system is a "plutocratic" method where the bigger countries have a greater influence in the determination of the international price vector π.

for the GK system.[68] There has been more discussion about test T10 than test T8. Proponents of the GK system like the fact that it has good aggregation (across countries) properties and the fact that big countries have more influence on the determination of the world reference price vector π is regarded as a reasonable price to pay for these good aggregation properties. On the other hand, proponents of the IDB method like the fact that the world reference prices are more democratically determined and they place less weight on having good aggregation properties. Finally, proponents of the economic approach to multilateral systems (like myself) are not enthusiastic about any additive method, since if there are more than two countries and relative prices and quantities differ substantially across countries, additive methods tend to give rather different parities than those obtained using approaches that have a strong economic rationale (such as the GEKS system).

A.6 The Economic Properties of the Iklé Dikhanov Balk Multilateral System

An economic approach to bilateral index number theory was initiated by Diewert (1976) and generalized to multilateral indexes in Diewert (1999; 20-23). We will explain this approach in the present section and examine the properties of the IDB system in this economic framework.

The basic assumption in the economic approach to multilateral indexes is that the country k quantity vector y^k is a solution to the following country k utility maximization problem:

$$\text{(A65)}\quad \max_y \{f(y) : p^k{\cdot}y = p^k{\cdot}y^k\} = u_k = Y^k$$

where $u^k \equiv f(y^k)$ is the utility level for country k which can also be interpreted as the country's volume Y^k, $p^k \gg 0_N$ is the vector of positive prices for outputs that prevail in country k, for $k = 1,...,K$[69] and f is a linearly homogeneous, increasing and concave utility function that is assumed to be the same across countries. The utility function has a *dual unit cost or expenditure function* c(p) which is defined as the minimum cost or expenditure required to achieve unit utility level if the consumer faces the positive commodity price vector p.[70] Since consumers in country k are assumed to face the prices $p^k \gg 0_N$, we have the following equalities:

$$\text{(A66)}\quad c(p^k) \equiv \min_y \{p^k{\cdot}y : f(y) \geq 1\} \equiv e_k = P^k\,; \qquad k = 1,...,K$$

where e_k is the (unobserved) minimum expenditure that is required for country k to achieve unit utility level when it faces its prices p^k, which can also be interpreted as country k's PPP, P^k. Under assumptions (A65), it can be shown[71] that the country k data satisfies the following equation:

[68] Balk (1996; 212) also compares the performance of the two methods (along with other multilateral methods) using his axiomatic system.
[69] In this section, we will assume that all country prices and quantities are positive so that $p^k \gg 0_N$ and $y^k \gg 0_N$ for $k = 1,...,K$.
[70] The unit cost function c(p) is an increasing, linearly homogeneous and concave function in p for $p \gg 0_N$.
[71] See Diewert (1974) for material on duality theory and unit cost functions.

(A67) $p^k \cdot y^k = c(p^k)f(y^k) = e_k u_k = P^k Y^k$ $\quad k = 1,...,K.$

In order to make further progress, we assume that either the utility function f(y) is once continuously differentiable with respect to the components of y *or* the unit cost function c(p) is once continuously differentiable with respect to the components of p (or both).

In the case where f is assumed to be differentiable, the first order necessary conditions for the utility maximization problems in (A65), along with the linear homogeneity of f, imply the following relationships between the country k price and quantity vectors, p^k and y^k respectively, and the country unit expenditures, e_k defined in (A66):[72]

(A68) $p^k = \nabla f(y^k)e_k$; $\quad k = 1,...,K$

where $\nabla f(y^k)$ denotes the vector of first order partial derivatives of f with respect to the components of y evaluated at the country k quantity vector, y^k.

In the case where c(p) is assumed to be differentiable, then Shephard's Lemma implies the following equations:

(A69) $y^k = \nabla c(p^k)u_k = \nabla c(p^k)Y^k$ $\quad k = 1,...,K$

where $u_k = f(y^k) = Y^k$ denotes the utility level for country k and $\nabla c(p^k)$ denotes the vector of first order partial derivatives of the unit cost function c with respect to the components of p evaluated at the country k price vector p^k.

If f(y) or c(p) are differentiable, then since both of these functions are assumed to be linearly homogeneous, Euler's Theorem on homogeneous functions implies the following relationships:

(A70) $f(y^k) = \nabla f(y^k) \cdot y^k = \sum_{n=1}^{N} [\partial f(y^k)/\partial y_n] y_n^k$; $\quad k = 1,...,K;$
(A71) $c(p^k) = \nabla c(p^k) \cdot p^k = \sum_{n=1}^{N} [\partial c(p^k)/\partial p_n] p_n^k$; $\quad k = 1,...,K.$

Recall that the expenditure share on commodity n for country k was defined as $s_n^k \equiv p_n^k y_n^k / p^k \cdot y^k$. In the case where f(y) is differentiable, substitution of (A68) and (A70) into these shares leads to the following expressions:

(A72) $s_n^k = y_n^k f_n(y^k)/f(y^k)$; $\quad n = 1,...,N ; k = 1,...,K$

where $f_n(y^k) \equiv \partial f(y^k)/\partial y_n$. In the case where c(p) is differentiable, substitution of (A69) and (A71) into the expenditure shares s_n^k leads to the following expressions:

(A73) $s_n^k = p_n^k c_n(p^k)/c(p^k)$; $\quad n = 1,...,N ; k = 1,...,K$

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[72] See Diewert (1999; 21) for more details on the derivation of these equations.

where $c_n(p^k) \equiv \partial c(p^k)/\partial p_n$. With the above preliminaries laid out, we are now ready to attempt to determine what classes of preferences (i.e., differentiable functional forms for f or c) are consistent with the IDB system of equations (A12).

We start out by considering the case of a differentiable utility function, f(y), which is positive, increasing, linearly homogeneous and concave for $y >> 0_N$.[73] Let $y^k >> 0_N$, $Y^k = f(y^k)$ for k = 1,...,K and substitute these equations and (A72) into equations (A12). We find that f must satisfy the following system of K functional equations:

$$\text{(A74)}\ \sum_{n=1}^{N}\{[y_n^1 f_n(y^1)/f(y^1)]+...+[y_n^K f_n(y^K)/f(y^K)]\}[y_n^k/f(y^k)]/\{[y_n^1/f(y^1)]+...+[y_n^K/f(y^K)]\} = 1 \ ; \quad k = 1,...,K.$$

Note that all of the terms in the above system of K equations are the same in each equation except the terms $y_n^k/f(y^k)$ in the middle of equation k. Now suppose that f(y) is a *linear function* of y; i.e., we have:

$$\text{(A75)}\ f(y) = f(y_1,...,y_N) = a_1y_1 + ... + a_Ny_N \ ; \ a_1 > 0 , ... , a_N > 0.$$

It is straightforward to verify that the linear function f(y) defined by (A75) satisfies our maintained hypotheses on f and it also satisfies the system of functional equations (A75). Thus the IDB multilateral system is consistent with linear preferences.

We now consider the case of a differentiable unit cost function c(p), which is positive, increasing, linearly homogeneous and concave for $p >> 0_N$. Let $p^k >> 0_N$, $e_k = c(p^k)$ for k = 1,...,K and substitute these equations and (A69) into equations (A12). We find that c must satisfy the following system of K functional equations:

$$\text{(A76)}\ \sum_{n=1}^{N} \{[p_n^1 c_n(y^1)/c(p^1)] +...+ [p_n^K c_n(p^K)/c(p^K)]\}c_n(p^k)/\{c_n(p^1) +...+ c_n(p^K)\} = 1 \ ; \quad k = 1,...,K.$$

Note that all of the terms in the above system of K equations are the same in each equation except the partial derivative terms $c_n(p^k)$ in the middle of equation k. Now suppose that c(p) is a *linear function* of p; i.e., we have:

$$\text{(A77)}\ c(p) = c(p_1,...,p_N) = b_1y_1 + ... + b_Ny_N \ ; \ b_1 > 0 , ... , b_N > 0.$$

It is straightforward to verify that the linear function c(p) defined by (A77) satisfies our maintained hypotheses on c and it also satisfies the system of functional equations (A76). Thus the IDB multilateral system is consistent with Leontief (no substitution) preferences.

The above computations show that the IDB multilateral system is consistent with preferences that exhibit perfect substitutability between commodities (the linear utility function case) and with preferences that exhibit no substitution behavior as prices change (the case of Leontief preferences where the unit cost function is linear). It turns out that

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[73] We will say that f or c are regular if they satisfies these regularity conditions.

if the number of countries is three or greater, these are the only (differentiable) preferences that are consistent with the IDB system; i.e., we have the following result:

Proposition 2: The linear utility function defined by (A75) is the only regular differentiable utility function that is consistent with the IDB equations (A74) and the preferences that are dual to the linear unit cost function defined by (A77) are the only differentiable dual preferences that are consistent with the IDB equations (A76).

Proof: Let $K \geq 3$ and let $y^k >> 0_N$ for $k = 1,...,K$. Then the first two equations in (A74) can be rearranged to give us the following equation:

(A78) $f(y^2) - f(y^1)$
$= \sum_{n=1}^N \{[y_n^1 f_n(y^1)/f(y^1)]+...+[y_n^K f_n(y^K)/f(y^K)]\}[y_n^2 - y_n^1]/\{[y_n^1/f(y^1)]+...+[y_n^K/f(y^K)]\}.$

Now fix n let the components of y^1 and y^2 satisfy the following assumptions:

(A79) $y_n^2 \neq y_n^1$; $y_i^2 = y_i^1$ for $i \neq n$.

Now look at the equation (A78) when assumptions (A79) hold. The left hand side is independent of the components of y^3 and hence the right hand side of (A78) must also be independent of y^3. Using the linear homogeneity of f, this is sufficient to show that $f_n(y^3)$ must be a constant for any $y^3 >> 0_N$; i.e., for all $y >> 0_N$, we have $f_n(y)$ equal to a constant a_N, which must be positive under our regularity conditions on f. This proof works for n = 1,...,N, which completes the proof of the first part of the proposition.

Again, let $K \geq 3$ and let $p^k >> 0_N$ for $k = 1,...,K$. Then equations (A76) can be rewritten as follows:

(A80) $\sum_{n=1}^N \rho_n(p^1,...,p^K)c_n(p^k) = 1$; $\qquad k = 1,...,K$

where the coefficients $\rho_n(p^1,...,p^K)$ in (A80) are defined for n = 1,...,N as follows:

(A81) $\rho_n(p^1,...,p^K) \equiv \{[p_n^1 c_n(y^1)/c(p^1)] +...+ [p_n^K c_n(p^K)/c(p^K)]\}/\{c_n(p^1) +...+ c_n(p^K)\}.$

The first two equations in (A80) can be subtracted from each other to give the following equation:

(A82) $\sum_{n=1}^N \rho_n(p^1,...,p^K)[c_n(p^2) - c_n(p^1)] = 0.$

Define the vector $\rho(p^1,...,p^K) \equiv [\rho_1(p^1,...,p^K), ..., \rho_N(p^1,...,p^K)]$. Since $K \geq 3$, looking at definitions (A81), it can be seen that we can vary the components of p^3 (holding the remaining price vectors constant) so that we can find N linearly independent $\rho(p^1,...,p^K)$ vectors. Substitution of these linearly independent vectors into equation (A82) implies that

(A83) $\nabla c(p^2) = \nabla c(p^1).$

Since equations (A83) hold for all positive p^1 and p^2, the partial derivatives of c(p) are constant, which completes the proof of the proposition. Q.E.D.

Thus the IDB multilateral system suffers from the same defect as the GK system; both of these additive systems are not consistent with an economic approach that allows consumer preferences to be represented by flexible functional forms, whereas the GEKS system is consistent with preferences that are representable by flexible functional forms.[74]

A.7 A Numerical Example

We compare the IDB price and volume parities with other multilateral systems in a simple numerical example that was used in Diewert (1999; 79-84). This was a three country, two commodity example. The price and quantity vectors for the three countries were as follows:

(A84) $p^1 \equiv [1,1]$; $p^2 \equiv [10, 1/10]$; $p^3 \equiv [1/10,10]$; $y^1 \equiv [1,2]$; $y^2 \equiv [1,100]$; $y^3 \equiv [1000,10]$.

Note that the geometric average of the prices in each country is 1, so that average price levels are roughly comparable across countries, except that the price of commodity 1 is very high and the price of commodity 2 is very low in country 2 and vice versa for country 3. As a result of these price differences, consumption of commodity 1 is relatively low and consumption of commodity 2 is relatively high in country 2 and vice versa in country 3. Country 1 can be regarded as a tiny country, with total expenditure (in national currency units) equal to 3, country 2 is a medium country with total expenditure equal to 20 and country 3 is a large country with expenditure equal to 200.

The Fisher (1922) quantity index Q_F can be used to calculate the volume Y^k of each country k relative to country 1; i.e., we can calculate Y^k/Y^1 as $Q_F(p^1,p^k,y^1,y^k) \equiv [p^1\cdot y^k p^k\cdot y^k/p^1\cdot y^1 p^k\cdot y^1]^{1/2}$ for k = 2,3. We can then set $Y^1 = 1$ and then Y^2 and Y^3 are determined and these volumes using country 1 as the base or star country are reported in the Fisher 1 column of Table 1. In a similar manner, we can use country 2 as the base and use the Fisher formula to calculate Y^1, $Y^2 = 1$ and Y^3. We then divide these numbers by Y^1 and we obtain the numbers listed in the Fisher 2 column of Table 1. Finally, we can use country 3 as the base and use the Fisher formula to calculate Y^1, Y^2 and $Y^3 =1$. We then divide these numbers by Y^1 and we obtain the numbers listed in the Fisher 3 column of Table 1. Ideally, these Fisher star parities would all coincide but since they do not, we take the geometric mean of them and obtain the GEKS parities which are listed in the fourth column of Table 1. Thus for this example, an economic approach to forming

[74] See Diewert (1999; 46) for descriptions of multilateral methods that have good economic properties; i.e., methods that are consistent with maximizing behavior on the part of consumers with preferences represented by flexible functional forms. See Diewert (1976) for the concept of a flexible functional form and the economic approach to index number theory. In addition to the GEKS system, the Own Share and van Ijzeren's (1983) weighted and unweighted balanced methods have good economic properties.

multilateral quantity indexes leads to the volumes of countries 2 and 3 to be equal to 7.26 and 64.81 times the volume of country 1.[75]

Table 1: Fisher Star, GEKS, GK and IDB Relative Volumes for Three Countries

	Fisher 1	Fisher 2	Fisher 3	GEKS	GK	IDB
Y^1	1.00	1.00	1.00	1.00	1.00	1.00
Y^2	8.12	8.12	5.79	7.26	47.42	33.67
Y^3	57.88	81.25	57.88	64.81	57.35	336.67

The GK parities for P^k and π_n can be obtained by iterating between equations (21) and (22) until convergence has been achieved.[76] Once these parities have been determined, the Y^k can be determined using equations (24). These country volumes were then normalized so that $Y^1 = 1$. The resulting parities are listed in the GK column in Table 1. It can be seen that the GK parity for Y^3/Y^1, 57.35, is reasonable but the parity for Y^2/Y^1, 47.42, is much too large to be reasonable from an economic perspective. The cause of this unreasonable estimate for Y^2 is the fact that the GK international price vector, $[\pi_1,\pi_2]$, is equal to [1, 9.00] so that these relative prices are closest to the structure of relative prices in country 3, the large country. Thus the relatively large consumption of commodity 2 in country 2 gets an unduly high price weight using the GK vector of international reference prices, leading to an exaggerated estimate for its volume, Y^2. This illustrates a frequent criticism of the GK method: the structure of international prices that it gives rise to is "biased" towards the price structure of the biggest countries.

We now calculate the IDB parities for our numerical example in order to see if the method can avoid the unreasonable results generated by the GK method. The parities for P^k and π_n can be obtained by iterating between equations (27) and (28) until convergence has been achieved.[77] Once these parities have been determined, the Y^k can be determined using equations (24). These country volumes were then normalized so that $Y^1 = 1$. The resulting parities are listed in the IBD column in Table 1. It can be seen that the GK parity for Y^2/Y^1 is 33.67 which is well outside our suggested reasonable range of 5 to 9 and the GK parity for Y^3/Y^1 is 336.7 which is well outside our suggested reasonable range of 50 to 90. What is the cause of these problematic parities?

The problematic IDB volume estimates are not caused by an unrepresentative vector of international prices since the IBD international price vector, $[\pi_1,\pi_2]$, is equal to [1, 1], which in turn is equal to the vector of (equally weighted) geometric mean commodity prices across countries. The problem is due to the fact that *any* additive method cannot

[75] Since the Fisher star parities are not all equal, we need to recognize that the GEKS parities are only an approximation to the "truth". Thus we would expect that an economic approach would lead to a Y^2/Y^1 parity in the 5 to 9 range and to a Y^3/Y^1 parity in the 50 to 90 range. Note that the IDB parities are well outside these ranges and the GK parity for Y^2/Y^1 is well outside our suggested range.

[76] Only 5 iterations were required for convergence.

[77] Since all of the prices and quantities are positive in our example, equations (27) and (28) can be used instead of the more robust (to zero entries) equations (A3) and (A4). Eighteen iterations were required for convergence.

take into account the problem of declining marginal utility as consumption increases if there are 3 or more countries in the comparison. Thus the IBD vector of international prices $\pi = [1,1]$ is exactly equal to the country 1 price vector $p^1 = [1,1]$ and so the use of these international prices leads to an accurate volume measure for country 1. But the structure of the IBD international prices is far different from the prices facing consumers in country 2, where the price vector is $p^2 \equiv [10, 1/10]$. The very low relative price for commodity 2 leads consumers to demand a relatively large amount of this commodity (100 units) and the relatively high price for commodity 1 leads to a relatively low demand for this commodity (1 unit). Thus at international prices, the output of country 2 is $\pi \cdot y^2$ which is equal to 101 as compared to its nominal output $p^2 \cdot y^2$ which is equal to 20. Thus the use of international prices overvalues the output of country 2 relative to country 1 because the international price of commodity 2 is equal to 1 which is very much larger than the actual price of commodity 2 in country 2 (which is 1/10). Thus Y^2/Y^1 is equal to $\pi \cdot y^2/\pi \cdot y^1 = 101/3 = 33.67$, an estimate which fails to take into account the declining marginal utility of the relatively large consumption of commodity 2 in country 1. A similar problem occurs when we compare the outputs of countries 1 and 3 using international prices except in this case, the use of international prices tremendously overvalues country 3's consumption of commodity 1. The problem of finding international reference prices that are "fair" for two country comparisons can be solved[78] but the problem cannot be solved in general if there are three or more countries in the comparison.

Our conclusion here is that additive methods for making international price and quantity comparisons where there are tremendous differences in the structure of prices and quantities across countries are likely to give rather different answers than methods that are based on economic approaches. Hence although additive methods are very convenient, they are likely to lead to biased comparisons from the viewpoint of the economic approach to index number theory.

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[78] See Diewert (1996; 246) for examples of superlative indexes that are additive if there are only two countries or observations.

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