On the Tang and Wang Decomposition of Labour Productivity Growth into Sectoral Effects

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Abstract

Tang and Wang provided a decomposition of economy wide labour productivity into sectoral contribution effects. The present note reworks their methodology to provide a more transparent and simple decomposition. This new decomposition is then related to another decomposition due to Gini and analyzed by Balk. Overall growth in labour productivity is due to three factors: (i) growth in the labour productivity of individual sectors; (ii) changes in real output prices of the sectors and (iii) changes in the allocation of labour across sectors.

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Keywords

Index numbers, labour productivity, decompositions of aggregate labour productivity into sectoral effects.

1. Introduction

Jianmin Tang and Weimin Wang (2004; 426) provided an interesting decomposition for economy wide labour productivity into sectoral contribution effects. However, the interpretation of the individual terms in their decomposition is not completely clear and so in section 2, we rework their methodology in order to provide a more transparent and simple decomposition. In section 3, we pursue a somewhat different approach which is due to Gini (1937) and is a generalization of the Fisher (1922) ideal index number methodology to aggregates that are products of three factors rather than two.

2. The Tang and Wang Methodology Reworked

Let there be N sectors or industries in the economy. Suppose that for period t = 0,1, the *output* (or real value added) of sector n is Y_n^t with corresponding period t *price* P_n^t and

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labour input L_n^t for n = 1,...,N. We assume that these labour inputs can be added across sectors and that the *economy wide labour input* in period t is L^t defined as

$$(1)\ L^t \equiv \sum_{n=1}^{N} L_n^{\,t}\ ; \qquad t = 0,1.$$

Industry n labour productivity in period t, X_n^t, is defined as industry n output divided by industry n labour input:

$$(2)\ X_n^{\,t} \equiv Y_n^{\,t}/L_n^{\,t}\ ; \qquad t = 0,1\ ;\ n = 1,...,N.$$

It is not entirely clear how aggregate labour productivity should be defined since the outputs produced by the various industries are measured in heterogeneous, noncomparable units. Thus we need to weight these heterogeneous outputs by their prices, sum the resulting period t values and then divide by a *general output price index*, say P^t for period t, in order to make the economy wide nominal value of aggregate output comparable in real terms across periods. Thus with an appropriate choice for the aggregate output price index P^t, the period t *economy wide labour productivity*, X^t, is defined as follows:[2]

$$(3)\ X^t \equiv \sum_{n=1}^{N} P_n^{\,t}\, Y_n^{\,t} / P^t L^t\ ; \qquad t = 0,1.$$

We can simplify the expression for aggregate labour productivity in period 0, X^0, by a judicious choice of units of measurement for each industry output. We will choose to measure each industry's output in terms of the number of units of the industry's output that can be purchased by one dollar in period 0. The effect of these choices for the units of measurement is to set the price of each industry's output equal to unity in period 0; i.e., we have:[3]

$$(4)\ P_n^{\,0} \equiv 1\ ; \qquad n = 1,...,N.$$

We will also normalize the economy wide price index to equal unity in period 0; i.e., we have:[4]

$$(5)\ P^0 \equiv 1.$$

Using definition (3) for t = 0 along with the normalizations (4) and (5), it can be seen that the period 0 economy wide labour productivity X^0 is equal to the following expression:

$$\begin{aligned}(6)\ X^0 &= \sum_{n=1}^{N} Y_n^{\,0} / L^0 \\ &= \sum_{n=1}^{N} X_n^{\,0}\, L_n^{\,0} / L^0 && \text{using definitions (2)}\end{aligned}$$

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[2] This follows the methodological approach taken by Tang and Wang (2004; 425).

[3] In reality, each industry will be producing many products and so P_n^1 will be say the Fisher (1922) price index for all of the industry n products going from period 0 to 1.

[4] Typically, P^1 will be the Fisher price index going from period 0 to 1 where the period 0 and 1 price and quantity vectors are the period t industry price and quantity vectors, $[P_1^t,...,P_N^t]$ and $[Y_1^t,...,Y_N^t]$ respectively for t = 0,1.

$$= \sum_{n=1}^{N} s_{Ln}^{0} X_n^{0}$$

where the share of labour used by industry n in period t, s_{Ln}^{t}, is defined in the obvious way as follows:

$$(7)\ s_{Ln}^{t} \equiv L_n^{t} / L^{t} ; \qquad n = 1,...,N ; t = 0,1.$$

Thus aggregate labour productivity for the economy in period 0 is a (labour) share weighted average of the sectoral labour productivities, a quite sensible result.

Using definition (3) for t = 1 and the definitions (7) for t = 1 leads to the following expression for aggregate labour productivity in period 1:[5]

$$\begin{aligned}(8)\ X^1 &\equiv \sum_{n=1}^{N} P_n^{1} Y_n^{1} / P^1 L^1 \\ &= \sum_{n=1}^{N} [P_n^{1}/P^1] [Y_n^{1}/L_n^{1}] [L_n^{1}/L^1] \\ &= \sum_{n=1}^{N} [P_n^{1}/P^1] X_n^{1} s_{Ln}^{1} \qquad \text{using definitions (2) and (7) for } t = 1 \\ &= \sum_{n=1}^{N} p_n^{1} s_{Ln}^{1} X_n^{1}\end{aligned}$$

where the *period t industry n real output price*, p_n^{t}, is defined as the industry t output price P_n^{t}, divided by the aggregate output price index for period t, P^t; i.e., we have the following definitions:[6]

$$(9)\ p_n^{t} \equiv P_n^{t}/P^{t} ; \qquad n = 1,...,N ; t = 0,1.$$

Thus economy wide labour productivity in period 1, X^1, is not equal to the (labour) share weighted average of the sectoral labour productivities, $\sum_{n=1}^{N} s_{Ln}^{1} X_n^{1}$. Instead, X^1 is equal to $\sum_{n=1}^{N} p_n^{1} s_{Ln}^{1} X_n^{1}$, so that the labour productivity of say sector n which has experienced a real output price increase (so that p_n^{1} is greater than one) gets a weight that is greater than its straightforward labour share weighted contribution, $s_{Ln}^{1} X_n^{1}$; i.e., sector n gets the weight $p_n^{1} s_{Ln}^{1} X_n^{1}$.

Up to this point, our analysis follows that of Tang and Wang (2004; 425-426) except that Tang and Wang did not bother with the normalizations (4) and (5). However, in what follows, we hopefully provide some additional value added to their analysis.

First, we define the *value added or output share of industry* n *in period* 0, s_{Yn}^{0}, as follows:

$$\begin{aligned}(10)\ s_{Yn}^{0} &\equiv P_n^{0} Y_n^{0} / \sum_{i=1}^{N} P_i^{0} Y_i^{0} ; \qquad n = 1,...,N \\ &= Y_n^{0} / \sum_{i=1}^{N} Y_i^{0} \qquad \text{using the normalizations (4).}\end{aligned}$$

Note that the product of the sector n labour share in period 0, s_{Ln}^{0}, with the sector n labour productivity in period 0, X_n^{0}, equals the following expression:

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[5] Equation (8) corresponds to equation (2) in Tang and Wang (2004; 426).
[6] These definitions follow those of Tang and Wang (2004; 425).

$$(11)\ s_{Ln}^0 X_n^0 = [L_n^0/L^0][Y_n^0/L_n^0]\ ; \qquad n = 1,...,N$$
$$= Y_n^0/L^0\ .$$

Using (11), we can establish the following equalities:

$$(12)\ s_{Ln}^0 X_n^0 / \textstyle\sum_{i=1}^N s_{Li}^0 X_i^0 = [Y_n^0/L^0] / \sum_{i=1}^N [Y_i^0/L^0]\ ; \qquad n = 1,...,N$$
$$= Y_n^0 / \textstyle\sum_{i=1}^N Y_i^0$$
$$= s_{Yn}^0 \qquad \text{using (10).}$$

Now we are ready to develop an expression for the rate of growth of economy wide labour productivity. Using expressions (6) and (8), we have:

$$(13)\ X^1/X^0 = \textstyle\sum_{n=1}^N p_n^1 s_{Ln}^1 X_n^1 / \sum_{n=1}^N s_{Ln}^0 X_n^0$$
$$= \textstyle\sum_{n=1}^N p_n^1 [s_{Ln}^1/s_{Ln}^0] [X_n^1/X_n^0] s_{Ln}^0 X_n^0 / \sum_{n=1}^N s_{Ln}^0 X_n^0$$
$$= \textstyle\sum_{n=1}^N p_n^1 [s_{Ln}^1/s_{Ln}^0] [X_n^1/X_n^0] s_{Yn}^0 \qquad \text{using (12)}$$
$$= \textstyle\sum_{n=1}^N [p_n^1/p_n^0] [s_{Ln}^1/s_{Ln}^0] [X_n^1/X_n^0] s_{Yn}^0 \qquad \text{using (4) and (5).}$$

Thus overall economy wide labour productivity growth, X^1/X^0, is an output share (see the term s_{Yn}^0 in (13) above) weighted average of three growth factors associated with industry n. The three growth factors are:

- X_n^1/X_n^0, (one plus) the rate of growth in the labour productivity of industry n;
- s_{Ln}^1/s_{Ln}^0, (one plus) the rate of growth in the share of labour being utilized by industry n and
- $p_n^1/p_n^0 = [P_n^1/P_n^0]/[P^1/P^0]$ which is (one plus) the rate of growth in the real output price of industry n.

Thus in looking at the contribution of industry n to overall (one plus) labour productivity growth, we start with a straightforward share weighted contribution factor, $s_{Yn}^0[X_n^1/X_n^0]$, which is the period 0 output or value added share of industry n in period 0, s_{Yn}^0, times the industry n (one plus) rate of labour productivity growth, X_n^1/X_n^0. This straightforward contribution factor will be augmented if real output price growth is positive (if p_n^1/p_n^0 is greater than one) and if the share of labour used by industry n is growing (if s_{Ln}^1/s_{Ln}^0 is greater than one). The decomposition of overall labour productivity growth given by the last line of (13) seems to be intuitively reasonable and fairly simple as opposed to the rather complex decomposition obtained by Tang and Wang (2004; 426).

3. An Alternative Decomposition due to Gini

Rewrite (13), making use of (4), (5) and (9) as follows:

$$(14)\ X^1/X^0 = \textstyle\sum_{n=1}^N p_n^1 s_{Ln}^1 X_n^1 / \sum_{n=1}^N p_n^0 s_{Ln}^0 X_n^0\ .$$

Suppose that we want to decompose X^1/X^0, the overall change in aggregate productivity, into the product of three effects:

- One effect that holds constant the sectoral labour shares s_{Ln}^t and the sectoral productivities X_n^t and just gives us the effects of the changes in the real prices p_n^1;
- Another effect that holds constant the real prices p_n^1 and the sectoral productivities X_n^t and gives us the effects of the changes in the sectoral labour shares s_{Ln}^t and
- A final effect that holds constant the individual labour shares s_{Ln}^t and real prices p_n^1 and gives us the effects of the changes in the sectoral productivities X_n^t.

This is a well known problem that has been studied extensively by Balk (2002/3) and Balk and Hoogenboom-Spijker (2003) and by many others. In particular, the generalization of the Fisher (1922) ideal index to an aggregate that is the product of 3 different factors made by Gini (1937; 72) seems to be appropriate for the present situation.

A relatively simple way to derive Gini's formula is as follows. X^1/X^0 is equal to the ratio $\sum_{n=1}^N p_n^1\, s_{Ln}^1\, X_n^1 / \sum_{n=1}^N p_n^0\, s_{Ln}^0\, X_n^0$. Let us write this ratio as a product of three similar ratios, where in each of these three ratios, one of the factors in the numerator is set equal to either p_n^1 or s_{Ln}^1 or X_n^1 and the same factor in the denominator is set equal to either p_n^0 or s_{Ln}^0 or X_n^0. The remaining factors in the numerator and denominator are constant. There are only 6 ways this can be done and the resulting decompositions are as follows:

(15) $$\frac{X^1}{X^0} = \left[\frac{\sum_{n=1}^N p_n^1 s_{Ln}^1 X_n^1}{\sum_{n=1}^N p_n^0 s_{Ln}^1 X_n^1}\right]\left[\frac{\sum_{n=1}^N p_n^0 s_{Ln}^1 X_n^1}{\sum_{n=1}^N p_n^0 s_{Ln}^0 X_n^1}\right]\left[\frac{\sum_{n=1}^N p_n^0 s_{Ln}^0 X_n^1}{\sum_{n=1}^N p_n^0 s_{Ln}^0 X_n^0}\right] \equiv \mathrm{P(1)S(1)X(1)}\ ;$$

(16) $$\frac{X^1}{X^0} = \left[\frac{\sum_{n=1}^N p_n^1 s_{Ln}^1 X_n^1}{\sum_{n=1}^N p_n^0 s_{Ln}^1 X_n^1}\right]\left[\frac{\sum_{n=1}^N p_n^0 s_{Ln}^1 X_n^0}{\sum_{n=1}^N p_n^0 s_{Ln}^0 X_n^0}\right]\left[\frac{\sum_{n=1}^N p_n^0 s_{Ln}^1 X_n^1}{\sum_{n=1}^N p_n^0 s_{Ln}^1 X_n^0}\right] \equiv \mathrm{P(2)S(2)X(2)}\ ;$$

(17) $$\frac{X^1}{X^0} = \left[\frac{\sum_{n=1}^N p_n^1 s_{Ln}^0 X_n^0}{\sum_{n=1}^N p_n^0 s_{Ln}^0 X_n^0}\right]\left[\frac{\sum_{n=1}^N p_n^1 s_{Ln}^1 X_n^0}{\sum_{n=1}^N p_n^1 s_{Ln}^0 X_n^0}\right]\left[\frac{\sum_{n=1}^N p_n^1 s_{Ln}^1 X_n^1}{\sum_{n=1}^N p_n^1 s_{Ln}^1 X_n^0}\right] \equiv \mathrm{P(3)S(3)X(3)}\ ;$$

(18) $$\frac{X^1}{X^0} = \left[\frac{\sum_{n=1}^N p_n^1 s_{Ln}^0 X_n^0}{\sum_{n=1}^N p_n^0 s_{Ln}^0 X_n^0}\right]\left[\frac{\sum_{n=1}^N p_n^1 s_{Ln}^1 X_n^1}{\sum_{n=1}^N p_n^1 s_{Ln}^0 X_n^1}\right]\left[\frac{\sum_{n=1}^N p_n^1 s_{Ln}^0 X_n^1}{\sum_{n=1}^N p_n^1 s_{Ln}^0 X_n^0}\right] \equiv \mathrm{P(4)S(4)X(4)}\ ;$$

(19) $$\frac{X^1}{X^0} = \left[\frac{\sum_{n=1}^N p_n^1 s_{Ln}^1 X_n^0}{\sum_{n=1}^N p_n^0 s_{Ln}^1 X_n^0}\right]\left[\frac{\sum_{n=1}^N p_n^0 s_{Ln}^1 X_n^0}{\sum_{n=1}^N p_n^0 s_{Ln}^0 X_n^0}\right]\left[\frac{\sum_{n=1}^N p_n^1 s_{Ln}^1 X_n^1}{\sum_{n=1}^N p_n^1 s_{Ln}^1 X_n^0}\right] \equiv \mathrm{P(5)S(5)X(5)}\ ;$$

(20) $$\frac{X^1}{X^0} = \left[\frac{\sum_{n=1}^N p_n^1 s_{Ln}^0 X_n^1}{\sum_{n=1}^N p_n^0 s_{Ln}^0 X_n^1}\right]\left[\frac{\sum_{n=1}^N p_n^1 s_{Ln}^1 X_n^1}{\sum_{n=1}^N p_n^1 s_{Ln}^0 X_n^1}\right]\left[\frac{\sum_{n=1}^N p_n^0 s_{Ln}^0 X_n^1}{\sum_{n=1}^N p_n^0 s_{Ln}^0 X_n^0}\right] \equiv \mathrm{P(6)S(6)X(6)}$$

where P(1) is defined as the price index which is the first term in brackets on the right hand side of (15), S(1) is defined as the share index which is the second term in brackets on the right hand side of (15) and X(1) is the productivity index which is the third term in brackets on the right hand side of (15) and so on for the definitions in (16)-(20). All of the decompositions of the ratio X^1/X^0 are equally valid so it seems sensible to define an overall index of price change, say P, as a symmetric average of the individual price indexes P(1)-P(6) which appeared in (15)-(20). It is also natural to follow the example of Fisher (1922) and Gini (1937; 72) and take geometric means so that the indexes will satisfy the time reversal test and also preserve the exact decomposition of X^1/X^0 into the product of three explanatory factors. Hence letting p^t, s^t and X^t be the N dimensional vectors of the real prices in period t, p_n^t, the labour shares in period t, s_{Ln}^t, and the sectoral productivities in period t, X_n^t, respectively, we have the following expression for the *Gini price change contribution factor to overall labour productivity growth*:

$$(21)\ P(p^0,s^0,X^0,p^1,s^1,X^1) \equiv [P(1)P(2)...P(6)]^{1/6}$$
$$= \{\left[\frac{\sum_{n=1}^N p_n^1 s_{Ln}^1 X_n^1}{\sum_{n=1}^N p_n^0 s_{Ln}^1 X_n^1}\right]^2 \left[\frac{\sum_{n=1}^N p_n^1 s_{Ln}^0 X_n^0}{\sum_{n=1}^N p_n^0 s_{Ln}^0 X_n^0}\right]^2 \left[\frac{\sum_{n=1}^N p_n^1 s_{Ln}^1 X_n^0}{\sum_{n=1}^N p_n^0 s_{Ln}^1 X_n^0}\right] \left[\frac{\sum_{n=1}^N p_n^1 s_{Ln}^0 X_n^1}{\sum_{n=1}^N p_n^0 s_{Ln}^0 X_n^1}\right]\}^{1/6}.$$

In a similar manner, we can derive the following expression for the *Gini labour share change contribution factor to overall labour productivity growth*:

$$(22)\ S(p^0,s^0,X^0,p^1,s^1,X^1) \equiv [S(1)S(2)...S(6)]^{1/6}$$
$$= \{\left[\frac{\sum_{n=1}^N p_n^1 s_{Ln}^1 X_n^1}{\sum_{n=1}^N p_n^1 s_{Ln}^0 X_n^1}\right]^2 \left[\frac{\sum_{n=1}^N p_n^0 s_{Ln}^1 X_n^0}{\sum_{n=1}^N p_n^0 s_{Ln}^0 X_n^0}\right]^2 \left[\frac{\sum_{n=1}^N p_n^0 s_{Ln}^1 X_n^1}{\sum_{n=1}^N p_n^0 s_{Ln}^0 X_n^1}\right] \left[\frac{\sum_{n=1}^N p_n^1 s_{Ln}^1 X_n^0}{\sum_{n=1}^N p_n^1 s_{Ln}^0 X_n^0}\right]\}^{1/6}.$$

Finally, we can derive the following expression for the *Gini pure productivity change contribution factor to overall labour productivity growth* (which holds constant the effects of changing real output prices and changing sector labour shares):

$$(23)\ X(p^0,s^0,X^0,p^1,s^1,X^1) \equiv [X(1)X(2)...X(6)]^{1/6}$$
$$= \{\left[\frac{\sum_{n=1}^N p_n^1 s_{Ln}^1 X_n^1}{\sum_{n=1}^N p_n^1 s_{Ln}^1 X_n^0}\right]^2 \left[\frac{\sum_{n=1}^N p_n^0 s_{Ln}^0 X_n^1}{\sum_{n=1}^N p_n^0 s_{Ln}^0 X_n^0}\right]^2 \left[\frac{\sum_{n=1}^N p_n^1 s_{Ln}^0 X_n^1}{\sum_{n=1}^N p_n^1 s_{Ln}^0 X_n^0}\right] \left[\frac{\sum_{n=1}^N p_n^0 s_{Ln}^1 X_n^1}{\sum_{n=1}^N p_n^0 s_{Ln}^1 X_n^0}\right]\}^{1/6}.$$

Balk (2002/3; 210) suggests some axioms that index number formulae of the type defined by (21)-(23) should satisfy.[7] It can be verified that the above Gini indexes satisfy all of Balk's suggested tests.

Another interesting aspect of the Gini formulae is that if the labour shares are constant across the two periods, so that $s^0 = s^1$, then the labour share contribution factor $S(p^0,s^0,X^0,p^1,s^1,X^1)$ defined by (22) is unity, the real price change contribution factor $P(p^0,s^0,X^0,p^1,s^1,X^1)$ defined by (21) reduces to the ordinary Fisher price index, P_F, and the

[7] Balk (2002/3; 211) also notes with approval the Gini formulae defined by (21)-(23) and gives additional historical references to the literature.

pure productivity change contribution factor $X(p^0,s^0,X^0,p^1,s^1,X^1)$ defined by (23) reduces to the ordinary Fisher quantity index Q_F, where P_F and Q_F are defined as follows:

(24) $P_F(p^0,p^1,X^0,X^1) \equiv [p^1\cdot X^0\ p^1\cdot X^1/p^0\cdot X^0\ p^0\cdot X^1]^{1/2}$;
(25) $Q_F(p^0,p^1,X^0,X^1) \equiv [p^0\cdot X^1\ p^1\cdot X^1/p^0\cdot X^0\ p^1\cdot X^0]^{1/2}$.

Similarly, if the real prices are constant across the two periods, then the real price change contribution factor $P(p^0,s^0,X^0,p^1,s^1,X^1)$ is unity, the labour share contribution factor $S(p^0,s^0,X^0,p^1,s^1,X^1)$ collapses to the Fisher index $[s^1\cdot X^0\ s^1\cdot X^1/s^0\cdot X^0\ s^0\cdot X^1]^{1/2}$ and the pure productivity change contribution factor $X(p^0,s^0,X^0,p^1,s^1,X^1)$ reduces to the Fisher quantity index $[s^0\cdot X^1\ s^1\cdot X^1/s^0\cdot X^0\ s^1\cdot X^0]^{1/2}$.

Each of the contribution factors defined by (21)-(23) has an interpretation as an index of change of prices, labour shares and sectoral labour productivities, holding constant the other two factors. However the interpretation of (21) and (22) is not completely straightforward (as it is the case of normal index number theory) since shares by definition cannot *all* grow from one period to the next and so the interpretation of (22) as a weighted average of the individual share growth rates, s_n^1/s_n^0, while valid does not seem to be very intuitive. Similarly, the interpretation of (21) as a weighted average of the growth rates of the sectoral real output prices, p_n^1/p_n^0, also seems to lack intuitive appeal since the average of the real prices p_n^t for each period t will necessarily be close to one, and hence, it will not be possible for *all* of the relative prices, p_n^1/p_n^0, to exceed unity under normal conditions. Fortunately, it is possible to reinterpret each of the contribution factors defined by (21) and (22) as indicators of *structural change* as we will now show.

In order to derive these alternative interpretations of (21) and (22), it is first necessary to develop an identity that was used by Bortkiewicz (1923; 374-375) in an index number context. Suppose that we have two N dimensional vectors $x \equiv [x_1,...,x_N]$ and $y \equiv [y_1,...,y_N]$ and an N dimensional vector of positive share weights $s \equiv [s_1,...,s_N]$.[8] We use these shares in order to define the *share weighted averages* of x and y, x^* and y^* respectively and the *share weighted covariance* between x and y, Cov (x,y;s):

(26) $x^* \equiv \sum_{n=1}^N s_n x_n$; $y^* \equiv \sum_{n=1}^N s_n y_n$; $\mathrm{Cov}(x,y;s) \equiv \sum_{n=1}^N s_n (x_n - x^*)(y_n - y^*)$.

It is straightforward to use the above definitions in order to derive the following identity:

(27) $\sum_{n=1}^N s_n\, x_n\, y_n = \mathrm{Cov}(x,y;s) + x^* y^*$.

Now consider a generic share index of the type defined by S(1) to S(6) in (15)-(20). We have the following decomposition of such an index, which we label as S:[9]

(28) $S \equiv \sum_{n=1}^N p_n\, s_n^1 X_n / \sum_{n=1}^N p_n\, s_n^0 X_n$

[8] We assume that the shares sum to unity; i.e., $\sum_{n=1}^N s_n = 1$.
[9] The generic sector n real output price p_n will be equal to p_n^0 or p_n^1 and the generic sector n labour productivity level X_n will be equal to X_n^0 or X_n^1.

$$= \sum_{n=1}^{N} (s_n^{1}/s_n^{0})\, s_n^{0}\, p_n X_n \,/\, \sum_{n=1}^{N} s_n^{0}\, p_n X_n$$
$$= \sum_{n=1}^{N} s_n^{0}\, x_n\, (z_n/z^*) \qquad \text{defining } x_n \equiv s_n^{1}/s_n^{0}\,;\; z_n \equiv p_n X_n \text{ and } z^* \equiv \sum_{n=1}^{N} s_n^{0}\, z_n$$
$$= \sum_{n=1}^{N} s_n^{0}\, x_n\, y_n \qquad \text{defining } y_n \equiv z_n/z^* \text{ for } n = 1,\ldots,N.$$

Note that the s_n^0 share weighted means of the x_n and y_n are both equal to one; i.e., we have:

$$(29)\; x^* \equiv \sum_{n=1}^{N} s_n^{0}\, x_n = \sum_{n=1}^{N} s_n^{0}\, (s_n^{1}/s_n^{0}) = \sum_{n=1}^{N} s_n^{1} = 1\,;$$
$$(30)\; y^* \equiv \sum_{n=1}^{N} s_n^{0}\, y_n = \sum_{n=1}^{N} s_n^{0}\, (z_n/z^*) = z^*/z^* = 1.$$

Note that each z_n is equal to the product of the generic real output price in sector n, p_n, which will typically be close to one, times the generic productivity level of sector n, X_n. Thus z^* is the s_n^0 weighted average of these sector n real price weighted productivity levels, $\sum_{n=1}^{N} s_n^{0}\, p_n X_n$. Hence $y_n = p_n X_n / \sum_{j=1}^{N} s_j^{0}\, p_j X_j$ is the real price weighted generic productivity level of sector n relative to a s_n^0 weighted average of these same price weighted productivity levels. Now apply the identity (27) to the last line in (28) and we obtain the following decomposition for the generic S:

$$(31)\; S = \sum_{n=1}^{N} s_n^{0}\, (x_n - x^*)(y_n - y^*) + x^* y^*$$
$$= \sum_{n=1}^{N} s_n^{0}\, (x_n - 1)(y_n - 1) + 1 \qquad \text{using (29) and (30)}$$
$$= \mathrm{Cov}\,(x,y;s^0) + 1.$$

Thus the generic labour share change contribution factor to overall labour productivity growth S defined by the first equation in (28) will be greater than one if and only if the $\mathrm{Cov}(x,y;s^0)$ is positive. Thus if the s^0 share weighted correlation between the $x_n \equiv s_n^1/s_n^0$ (one plus the rate of change of the sectoral labour shares) and the sectoral real price weighted productivity levels relative to their s^0 share weighted average levels $y_n = p_n X_n / \sum_{j=1}^{N} s_j^{0}\, p_j X_j$ is positive, then S will be greater than one. Put another way, if the labour shares going from period 0 to 1 change in such a way that higher shares go to higher productivity sectors, then the contribution factor S to overall labour productivity growth will be positive. Thus the Gini labour share contribution factor $S(p^0,s^0,X^0,p^1,s^1,X^1)$ defined by (22) will be greater than one if all 6 of the covariances $\mathrm{Cov}\,(x,y,s^0)$ of the type defined in (31) are positive for the specific indexes defined by S(1) to S(6). Thus the Gini labour share contribution factor can be interpreted as a measure of structural shifts of labour across industries of varying productivity levels.

Now consider a generic real output price index of the type defined by P(1) to P(6) in (15)-(20). We have the following decomposition of such an index, which we label as P:[10]

$$(32)\; P \equiv \sum_{n=1}^{N} p_n^{1}\, s_n X_n / \sum_{n=1}^{N} p_n^{0}\, s_n X_n$$
$$= \sum_{n=1}^{N} (p_n^{1}/p_n^{0})\, s_n\, p_n^{0} X_n \,/\, \sum_{n=1}^{N} s_n\, p_n^{0} X_n$$
$$= \sum_{n=1}^{N} s_n\, x_n\, (z_n/z^*) \qquad \text{defining } x_n \equiv p_n^{1}/p_n^{0}\,;\; z_n \equiv p_n^{0} X_n \text{ and } z^* \equiv \sum_{n=1}^{N} s_n\, z_n$$
$$= \sum_{n=1}^{N} s_n\, x_n\, y_n \qquad \text{defining } y_n \equiv z_n/z^* \text{ for } n = 1,\ldots,N.$$

[10] The generic sector n labour share s_n will be equal to s_n^0 or s_n^1 and the generic sector n labour productivity level X_n will be equal to X_n^0 or X_n^1.

Note that the s_n share weighted mean of the y_n is equal to one but we cannot establish the same equality for the mean of the x_n; i.e., we have:

(33) $x^* \equiv \sum_{n=1}^N s_n x_n = \sum_{n=1}^N s_n (p_n^1/p_n^0)$;
(34) $y^* \equiv \sum_{n=1}^N s_n y_n = \sum_{n=1}^N s_n (z_n/z^*) = z^*/z^* = 1$.

Note that each z_n is equal to the product of the period 0 real output price in sector n, p_n^0, which will typically be close to one, times the generic productivity level of sector n, X_n. Thus z^* is the generic s_n weighted average of these sector n real price weighted productivity levels, $\sum_{n=1}^N s_n p_n^0 X_n$. Hence $y_n = p_n^0 X_n / \sum_{j=1}^N s_j p_j X_j$ is the real price weighted generic productivity level of sector n relative to a s_n weighted average of these same price weighted productivity levels. Now apply the identity (27) to the last line in (32) and we obtain the following decomposition for the generic P:

$$\begin{aligned}(35)\ P &= \textstyle\sum_{n=1}^N s_n (x_n - x^*)(y_n - y^*) + x^* y^* \\ &= \textstyle\sum_{n=1}^N s_n (x_n - x^*)(y_n - 1) + x^* \qquad \text{using (34)} \\ &= \mathrm{Cov}\,(x,y;s) + x^*.\end{aligned}$$

The interpretation of (35) is not as straightforward as was the interpretation of (31). The price contribution factor P defined by (32) will be greater than one if the sum of the covariance term Cov (x,y;s), equal to $\sum_{n=1}^N s_n (x_n - x^*)(y_n - 1)$, and the mean real price change x^*, equal to $\sum_{n=1}^N s_n (p_n^1/p_n^0)$, is greater than one. The interpretation of the x^* term is straightforward. P is equal to this straightforward effect (which will generally be close to one) plus the covariance term, $\sum_{n=1}^N s_n (x_n - x^*)(y_n - 1)$. Recalling that x_n is equal to (one plus) the rate of growth of the sector n real output price, p_n^1/p_n^0, and that y_n is the productivity level of sector n relative to an average productivity level, it can be seen that this covariance will be positive if the sectors which have high rates of growth of real output prices are associated with sectors that have high relative productivity levels.

4. Conclusion

Our conclusion at this point is that the Gini (1937) decomposition of aggregate labour productivity into sectoral contribution factors and the associated structural shifts seems promising. In terms of simplicity, the decomposition given by (13) also seems attractive. But it appears that there are many very reasonable decompositions and at this stage it is difficult to say which is "best". It appears that there is room for additional research in this area, particularly in developing the axiomatic approach to the topic, an approach which was initiated by Balk (2002/3). An economic approach may also be useful in indicating what a "best" decomposition might be.

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