Reinterpreting Mixed Strategy Equilibria: A Unification of the Classical and Bayesian Views*

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Abstract

We provide a new interpretation of mixed strategy equilibria that incorporates both von Neumann and Morgenstern's classical concealment role of mixing as well as the more recent Bayesian view originating with Harsanyi. For any two-person game, G, we consider an incomplete information game, $\mathcal{IG}$, in which each player's type is the probability he assigns to the event that his mixed strategy in G is "found out" by his opponent. We show that, generically, any regular equilibrium of G can be approximated by an equilibrium of $\mathcal{IG}$ in which almost every type of each player is strictly optimizing. This leads us to interpret i's equilibrium mixed strategy in G as a combination of deliberate randomization by i together with uncertainty on j's part about which randomization i will employ. We also show that such randomization is not unusual: For example, i's randomization is nondegenerate whenever the support of an equilibrium contains cyclic best replies.

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1. Introduction

The purpose of this paper is to better understand mixed strategy Nash equilibria in finite two-person games. In particular, we show that a player's equilibrium mixture can be usefully understood partly in terms of deliberate randomization by the player, and partly as an expression of the opponent's uncertainty about which randomization the player will employ. This allows us to unify the otherwise sharply distinct views of mixed strategies proposed by von Neumann and Morgenstern (1944) for zero-sum games and by Harsanyi (1973) for nonzero-sum games.

Von Neumann and Morgenstern (vNM (1944)), when focusing on two-person zero-sum games, unequivocally interpret mixing as an act of deliberate randomization whose purpose is to conceal. They point out that each player strictly prefers any one of his equilibrium strategies over any other strategy if he is certain that the mixed strategy he chooses will be found out by his opponent prior to his opponent's choice.

For example, in Matching Pennies player 1 strictly prefers the fifty-fifty mixture over every other mixed strategy when he knows that player 2 will find out the mixed strategy he chooses. VNM conclude from this that there is a defensive or concealment rationale for mixing in zero-sum games:

Thus one important consideration for a player in such a game is to protect himself against having his intentions found out by his opponent. Playing several such strategies at random, so that only their probabilities are determined is a very effective way to achieve a degree of such protection: By this device the opponent cannot possibly find out what the player's strategy is going to be, since the player does not know it himself (vNM (1944, p. 146)).

According to the *classical* rationale, then, a mixed strategy represents deliberate randomization on a player's part. Everyone, including the player himself, is uncertain about that player's pure choice. However, because it is based on the desirability of concealment, the classical rationale for mixing runs into difficulties in nonzero-sum games. As Schelling notes:

The essence of randomization in a two-person zero-sum game is to preclude the adversary from gaining intelligence about one's mode of play... In games that mix conflict with common interest, however, randomization plays no such central role... (Schelling (1960, p. 175)).

Consider, for example, the mixed equilibrium in the Battle of the Sexes. In this equilibrium, *neither* player can be thought of as deliberately randomizing to conceal his pure choice because, in this game, *each* player prefers to reveal his pure choice, whatever it is, to the other player. Thus, the classical rationale is inappropriate here.

But if concealment is not the rationale for mixing in general games, what is? Harsanyi (1973) provides an ingenious answer. He shows that virtually any mixed equilibrium can be viewed as an equilibrium of a nearby game of incomplete information in which small private variations in the players' payoffs lead them to strictly prefer one of their pure strategies. Thus, from Harsanyi's point of view no player ever actually randomizes and a player i is uncertain about another player j's pure choice only because it varies with j's type, which is private information. The significant conceptual idea introduced by Harsanyi is that a player's mixed strategy expresses the ignorance of the *other* players—*not of the player himself*—about that player's pure strategy choice.[1]

Aumann (1987; see especially Section 6) takes Harsanyi's idea further. By eliminating the payoff perturbations altogether, he directly interprets a player's mixed strategy solely as an expression of the other players' uncertainty about that player's pure strategy choice. This view is now widespread.[2] Indeed, as Aumann and Brandenburger (1995) remark:

In recent years, a different view of mixing has emerged. According to this view, players do not randomize; each player chooses some definite action. But other players need not know which one, and the mixture represents their uncertainty, their conjecture about his choice.

Thus, the view of mixing that has emerged, the *Bayesian view* let us call it, bears no resemblance to the classical view that mixing represents a deliberate decision to randomize in order to conceal one's choice. The concealment role of mixing has been entirely left behind.

In contrast, we argue here that the intuitively appealing classical view can be incorporated into a general interpretation of mixed equilibria. In fact, the approach we propose is tied to *both* the Bayesian and classical views.

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[1] We thank Bob Aumann for suggesting to us that this *conceptual* contribution by Haranyi was at least as important as his formal purification theorem.

[2] For example, see Armbruster and Boege (1979), Tan and Werlang (1988) and Brandenburger and Dekel (1989).

Our approach is tied to the Bayesian view by incorporating Harsanyi's idea that a player's private information can lead to uncertainty about that player's choice from the opponent's perspective. However, our approach differs crucially from Harsanyi's in terms of the precise nature of the players' private information. In our setup, there is *no uncertainty about payoffs.* Rather, each player is concerned that his opponent might find out his choice of mixed strategy, and it is *the level of this concern* that is private information. As in Harsanyi, such private information can make the opponent more uncertain about a player's choice than is the player himself. But, in our approach, because each player is concerned that his mixed strategy might be found out by his opponent, he may benefit from the concealment effect of deliberate randomization. This simultaneously ties our approach to the classical view.

More precisely, we interpret equilibria of any finite two-person game G as limits of equilibria of certain sufficiently nearby games of incomplete information, $\mathcal{IG}$. Each incomplete information game, $\mathcal{IG}$, is derived from G as follows. Nature moves first by independently choosing, for each player i, a type, $t_i \in [0, 1]$, according to some continuous distribution. Each player is privately informed of his own type, which is his assessment of the probability that the opponent will find out his *mixed strategy* before the opponent moves. We shall be concerned with the equilibria of $\mathcal{IG}$ as the type distributions become concentrated around zero and so as the players' concerns for being found out vanish.

As in Harsanyi, our model provides the players with strict incentives. Indeed, as we show, any regular equilibrium of G can be approximated by an equilibrium of $\mathcal{IG}$ in which almost every type of each player is strictly optimizing. But there is an important difference. Harsanyi's players strictly prefer to use only pure strategies, while our players in general strictly prefer to use *mixed* strategies. When our players mix, they do so deliberately, because the benefits of concealment make this strictly optimal, not because the equilibrium requires them to make the other player indifferent. Conversely, when our players choose pure strategies, they do so because randomization is harmful and they actively wish to reveal their choice to the other player.

For example, in the unique equilibrium of our incomplete information perturbation, $\mathcal{IG}$, of Matching Pennies, all types of both players strictly optimize by choosing the fifty-fifty mixture (see Section 2). Thus, neither player's behavior depends upon his private information and each player deliberately randomizes. Such randomization is strictly beneficial because each player believes the other might find out his mixed strategy. Our approach therefore supports the classical view of the mixed equilibrium in Matching Pennies, namely, that each player deliberately randomizes fifty-fifty and is certain that his opponent will do so as well. Indeed, Theorem 4.1 generalizes this to all zero-sum games.

On the other hand, all equilibria of $\mathcal{IG}$ near the completely mixed equilibrium of Battle of the Sexes require almost every type of each player to employ one of his two pure strategies (see Section 2). Concealment is shown to play no role in this equilibrium precisely because each player prefers to reveal his pure choice in this game. Moreover, our rationale for the mixed equilibrium here coincides with the Bayesian view: Each player i employs one or the other pure strategy; player j does not know which pure strategy i will employ, but assigns some probability to each one, where these probabilities are given by i's equilibrium mixture. This is generalized in Theorem 6.4 which states that if, starting from any cell in G's payoff matrix, both players' payoffs increase whenever either one of them switches to a best reply, then every equilibrium of our incomplete information perturbation requires almost every type of each player to employ a pure strategy.

So, our interpretation coincides with the classical view in zero-sum games, and it coincides with the Bayesian view in a class of coordination games. But what about the vast majority of games lying between these two extremes? As shown by example in Section 2, our interpretation will typically differ from both the Bayesian and classical views. The example is a 3x3 nonzero-sum game with a unique completely mixed equilibrium, m^*. In our incomplete information perturbation, no type of either player chooses his completely mixed equilibrium strategy, yet no type of either player chooses a pure strategy either. Instead, almost every type of each player i strictly optimizes by using one of three mixed strategies, m_{i1}, m_{i2}, or m_{i3}, each of which gives positive weight to just two pure strategies. Each randomization, m_{ik}, benefits i by optimally concealing the two pure strategies in its support. Further, if μ_{ik} denotes the fraction of player i's types using m_{ik} in the limit as the players' concerns for being found out converge to zero, then $m_i^* = \mu_{i1}m_{i1} + \mu_{i2}m_{i2} + \mu_{i3}m_{i3}$.

The above three games serve to exemplify the following general interpretation of any equilibrium, m^*, of the original game G :

Each player i's equilibrium mixture, m_i^*, can be expressed as a convex combination of a fixed finite set of i's mixed strategies, m_{ik}, say. Each mixed strategy in the convex combination represents a strategy that i might deliberately employ, while the weight on that mixed strategy represents the opponent's belief that i will employ it.

Such convex combinations reveal the role of deliberate randomization. In our perturbed game, where players are slightly concerned that their mixed strategy might be found out, the strategies m_{ik} are strictly optimal for the types employing them and, when the m_{ik} are non-degenerate, they optimally conceal the pure strategies in their support.

One might wonder when at least one of the m_{ik} above is nondegenerate, because then our interpretation involves deliberate randomization. Theorem 6.2 states that if the support of an equilibrium, m^*, of any game G contains a best-reply cycle, then a positive (and bounded away from zero) measure of both players' types must use non-degenerate mixed strategies, m_{ik}, in any approximating equilibrium of $\mathcal{IG}$. Hence, the presence of best reply cycles in the support of an equilibrium of a two-person game indicates a role for deliberate randomization in that equilibrium.

From the perspective offered here, the classical and Bayesian views are extreme. On the one hand, our interpretation coincides with the classical view only when the above convex combination is degenerate, placing full weight on i's equilibrium mixed strategy, as in matching pennies. On the other, our interpretation coincides with the Bayesian view only when every mixed strategy in the above convex combination is pure, with weights given by i's equilibrium mixture, as in the Battle of the Sexes. In general, our interpretation differs from both the classical and Bayesian views, as typified by the third example above.

A strength of our interpretation is that it eliminates the sharp distinction between zero-sum and nonzero-sum games insofar as the role of randomization is concerned. For example, according to our view, when moving from matching pennies to the battle of the sexes through continuous payoff changes, the role played by deliberate randomization in their mixed equilibria continuously diminishes to zero.

We restrict attention to two-person games. Additional issues arise with three or more players. For example, one must then specify how many opponents find out a player's mixed strategy. There does not appear to be a single natural choice here. However, there is no reason to doubt that any reasonable choice will yield strict incentives to mix in some games, as we obtain here.

In addition to the work cited above, a rich literature on purification has grown out of Harsanyi's (1973) seminal contribution. (See, for example, Radner and Rosenthal (1982) and Aumann et al. (1983).) The central issue in this literature is whether every mixed strategy equilibrium of an incomplete information game is (perhaps approximately) equivalent to some pure strategy equilibrium. In our model, this is not an issue because, as we shall show, all equilibria of $\mathcal{IG}$ are pure, generically. But note that a pure strategy in $\mathcal{IG}$ allows the players' to choose non-degenerate mixed strategies from G.

More closely related are Rosenthal (1991) and Robson (1994).[3] Both papers are concerned with the robustness of equilibria of two-person games to changes in the information structure. Rosenthal observes that equilibria of some nonzero-sum games remain equilibria even when the opponent is sure to find out one's mixed strategy choice.

Robson perturbs arbitrary two-person games by supposing that each player's pure or mixed strategy is found out by the opponent with a common known probability. Equilibria that survive arbitrarily small perturbations of this kind are called "informationally robust." Robson shows that informationally robust equilibria exist and refine Nash equilibria.[4] He also observes that informational robustness with respect to mixed strategies yields strict incentives to mix in some 2x2 nonzero-sum examples. However, in a typical informationally robust equilibrium, the players will not have strict incentives and they randomize in order to make the opponent indifferent.

The remainder of the paper is organized as follows. Section 2 contains three leading examples illustrating the main ideas. Section 3 describes our incomplete information perturbation of an arbitrary two-person game. Section 4 provides our results concerning zero-sum games, while Section 5 analyzes the more challenging nonzero-sum case and contains our main approximation theorem. Section 6 provides conditions under which our interpretation necessarily involves the classical view that players deliberately randomize, as well as a condition under which our interpretation involves only the Bayesian view in which no player randomizes. Section 7 provides an example showing the potential significance of unused strategies. Finally, Section 8 briefly discusses how our static model, in which a player's *mixed* strategy is revealed with some probability, is the reduced form of a dynamic game in which only a player's past history of *pure* actions is ever revealed.

[3] Less closely related is Matsui (1989). He considers a repeated game with a small probability that one player's entire supergame strategy will be revealed to the other player. In contrast, in a repeated game interpretation of our model (see Section 8), an opponent observes, at most, one's history of past actions, not one's entire supergame strategy.

[4] Our results here imply that, generically, the sets of informationally robust equilibria and Nash equilibria coincide.

2. Three Leading Examples

The scope of the present approach can be demonstrated by considering three examples: Matching Pennies, Battle of the Sexes, and Modified Rock-Scissors-Paper. To each of these normal form games, G, say, we associate a nearby game of incomplete information, $\mathcal{IG}$, which we now describe informally.

For $0 \leq \underline{\varepsilon} < \bar{\varepsilon} \leq 1$, the players' types, t_1 for player 1 and t_2 for player 2, are drawn independently and uniformly from $[\underline{\varepsilon}, \bar{\varepsilon}]$. The players choose a mixed strategy in G as a function of their type. With probability $1 - t_i$ player i receives the payoff in G from the pair of mixed strategies chosen, whereas with probability t_i he receives the payoff in G resulting from his mixed strategy choice together with a best reply for j against it.[5] More precisely, letting u_i denote i's payoff in G, if i chooses the mixed strategy m_i from G and j chooses m_j, then i's payoff in $\mathcal{IG}$ when his type is t_i is

$$(1 - t_i)u_i(m_1, m_2) + t_i v_i(m_i),$$

where $v_i(m_i) = \max_{x_j \in B_j(m_i)} u_i(m_i, x_j)$ and $B_j(m_i)$ is the set of best replies for j against m_i.

Accordingly, we interpret a player's type to be the probability he assigns to the event that the opponent finds out his mixed strategy and best replies to it. However, note that neither player believes he will find out the opponent's mixed strategy.[6] Hence each type of each player makes only the single decision of choosing a mixed strategy in G. Note also that player i cares only about the overall distribution over pure strategies in G induced by the opponent's strategy in $\mathcal{IG}$. This is because, from i's point of view, the opponent's strategy in $\mathcal{IG}$ is relevant for determining i's payoff only when the opponent does not find out i's mixed strategy.

We are interested in the limiting equilibria of $\mathcal{IG}$ as $\bar{\varepsilon}$ and $\underline{\varepsilon}$ tend to zero, so that $\mathcal{IG}$ tends to the original game G.

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[5] If there are multiple best replies for j against i's mixed strategy, then one that is best for i is employed. See Section 3.

[6] It would be equivalent to consider an extensive form game in which it is common knowledge that each player might find out the other's mixed strategy (see Appendix A). A player's single decision in $\mathcal{IG}$ corresponds to his only nontrivial decision in the extensive form, arising when he does not find out the opponent's mixed strategy.

	H	T
H	1,-1	-1, 1
T	-1, 1	1,-1

Figure 2.1: Matching Pennies

2.1. Matching Pennies

Recall vNM's observation that in Matching Pennies (Figure 2.1) the players strictly prefer the fifty-fifty mixture when they are sure to be found out. We shall show that for every $\bar{\varepsilon} > \underline{\varepsilon} > 0$, including those near zero, $\mathcal{IG}$ has an equilibrium in which every type of each player chooses to randomize fifty-fifty over H and T and that this non-degenerate mixture is strictly optimal.

So, suppose that every type of player 2 uses the fifty-fifty mixture. Consider player 1's payoff as a function of the probability, p, that his mixed strategy assigns to H, given that player 2 finds out this mixed strategy. Player 1's payoff is negative both for $p \in [0, 1/2)$, where 2's best reply is H, and for $p \in (1/2, 1]$ where 2's best reply is T, and so is uniquely maximized at $p = 1/2$, where it is zero, regardless of 2's reply.

Now consider player 1 in $\mathcal{IG}$ when his type, the probability he assigns to being found out, is $t_1 \in [\underline{\varepsilon}, \bar{\varepsilon}]$. Because the other player is mixing equally, any type of player 1 is indifferent among all his mixed strategies *conditional on not being found out*. Hence, because player 1 of type $t_1 > 0$ assigns positive probability to the event that he *is* found out, the fifty-fifty mixture is the uniquely optimal choice for every type of player 1, as claimed. A similar argument holds when the players' roles are reversed. Thus, the incomplete information game associated with Matching Pennies captures the classical point of view that mixing is a deliberate attempt to conceal one's choice.

2.2. Battle of the Sexes

Consider next the Battle of the Sexes (henceforth BoS; see Figure 2.2) and the mixed equilibrium in which each player chooses his favorite pure strategy with

	L	R
T	2,1	0,0
B	0,0	1,2

Figure 2.2: Battle of the Sexes

probability 2/3. (A player's "favorite" pure strategy yields him a payoff of 2 if coordination is achieved.)

For this example, set $\underline{\varepsilon} = 0$, so that in $\mathcal{IG}$ the player types are drawn independently and uniformly from $[0, \bar{\varepsilon}]$. We will show first that, regardless of the value of $\bar{\varepsilon} > 0$, almost every type of each player has a unique optimal pure strategy in every equilibrium of $\mathcal{IG}$ and second that, for $\bar{\varepsilon} > 0$ small enough, there exists an equilibrium of $\mathcal{IG}$ in which approximately 2/3 of each player's types choose that player's favorite BoS pure strategy and the remainder choose the other pure strategy.[7] Taken together, this leads to a purely Bayesian interpretation of the strictly mixed equilibrium in BoS.

So, let us begin by consulting Figure 2.3. The solid lines in the figure show player 1's payoff as a function of the probability p he places on T, given that player 2 finds out player 1's mixed strategy. When $p \in [0, 2/3)$, player 2's best reply is R and 1's payoff is decreasing in p. When $p \in (2/3, 1]$, player 2's best reply is L and 1's payoff is increasing in p. When $p = 2/3$, player 2 is indifferent between L and R, but player 1 strictly prefers that player 2 choose L, which accounts for the discontinuity.

Now, a mixed strategy in $\mathcal{IG}$ specifies, for each of a player's types, a probability distribution, or "lottery," over the player's mixed strategies in G. Such a lottery therefore determines which mixed strategy, m_i from the game G, the player's type will employ in $\mathcal{IG}$. Consequently, if player i of type t_i uses such a lottery, then he assigns probability t_i to the event that his opponent finds out the mixed strategy m_i that is the outcome of this lottery. To avoid confusion, we will refer to mixed strategies in $\mathcal{IG}$ as "lotteries," reserving the term "mixed strategies" for strategies

[7] Because Battle of the Sexes has multiple equilibria, so does its associated incomplete information game $\mathcal{IG}$.

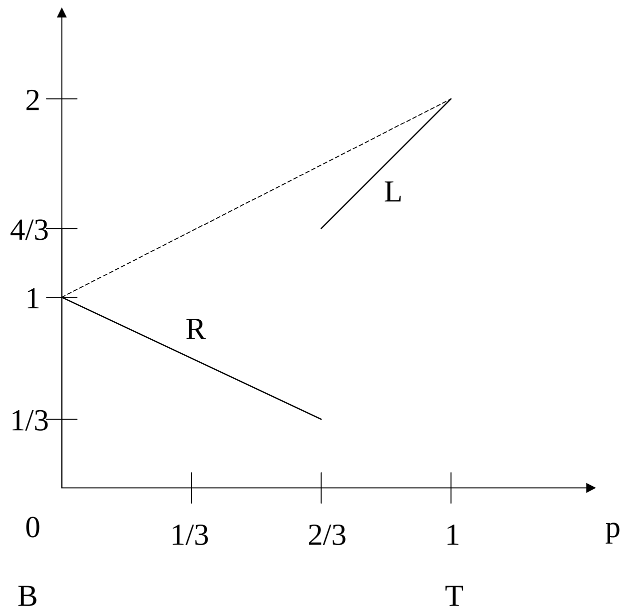

Figure 2.3: Player 1 Found Out in Battle of the Sexes

m_i in G.

Consider now the dotted line in Figure 2.3 connecting player 1's payoffs when he chooses the two pure strategies B ($p = 0$) and T ($p = 1$). We claim that, conditional on being found out, any payoff along the dotted line can be achieved by employing an appropriate lottery over the degenerate mixed strategies B and T. In particular, if player 1 of type t_1 uses the lottery that chooses T with probability π and B with probability $1 - \pi$, then player 1's payoff is $2\pi + 1(1 - \pi)$, conditional on being found out. That is, because player 2 finds out the outcome of the lottery, 2's best reply always yields coordination, giving player 1 an average of his payoffs along the diagonal.

The dotted line lies above player 1's payoff in Figure 2.3 and so player 1 prefers such a lottery $\pi \in (0, 1)$ to the mixed strategy giving T the same probability $p = \pi$, conditional on being found out. In contrast to the lottery, the mixed strategy, when combined with the opponent's best reply, leads to miscoordination with positive probability.

On the other hand, conditional on not being found out, the lottery π yields the *same* payoff as does the mixed strategy $p = \pi$. Altogether then, every positive type of player 1 must strictly prefer the lottery $\pi = p$ to the mixed strategy p, for any $p \in (0, 1)$.

Thus, regardless of player 2's strategy, every positive type of player 1 strictly prefers at least one of the two pure strategies T or B to any mixed strategy $p \in (0, 1)$. Furthermore, because T and B yield distinct payoffs conditional on being found out, the linearity of 1's payoff in his type implies that at most one of his types can be indifferent between T and B. We conclude that in every equilibrium of $\mathcal{IG}$, all but perhaps one of player 1's positive types strictly optimizes by employing a pure strategy. Since a similar argument applies to player 2, we have shown that almost every player type employs a unique optimal pure strategy in every equilibrium of $\mathcal{IG}$.

We now show that if $\bar{\varepsilon} > 0$ is small enough, $\mathcal{IG}$ has an equilibrium whose distribution over the pure strategies in G is arbitrarily close to the strictly mixed equilibrium of BoS. Given what we have already shown, we may restrict attention to strategies in $\mathcal{IG}$ in which player 1 chooses either T or B, and player 2 chooses either L or R. The equilibrium of $\mathcal{IG}$ we seek is such that roughly 2/3 of player 1's types choose T and roughly 2/3 of player 2's types choose R. This equilibrium is determined by a critical type for each player i, namely $\hat{t}_i = \alpha\bar{\varepsilon}$ for α near 1/3, where type t_1 of player 1 chooses

$$\text{B if } t_1 < \hat{t}_1\text{; and T if } t_1 \geq \hat{t}_1 \tag{2.1}$$

and type t_2 of player 2 chooses

$$\text{L if } t_2 < \hat{t}_2\text{; and R if } t_2 \geq \hat{t}_2. \tag{2.2}$$

Note that larger types, who assign a higher probability to being found out, choose their favorite pure BoS strategy.

For this to be an equilibrium, α must be such that the critical type $\hat{t}_i$ is indifferent between his two pure BoS strategies, T and B. A straightforward calculation yields

$$\alpha = \frac{1}{3} - \frac{\sqrt{4\bar{\varepsilon}^2 + 9} - 3}{6\bar{\varepsilon}},$$

which for $\bar{\varepsilon}$ small is close to 1/3. By symmetry, this value of α also makes player 2's critical type $\hat{t}_2 = \alpha\bar{\varepsilon}$ indifferent between L and R.[8]

[8]From (2.1) and (2.2), the payoff to player 1's critical type $\hat{t}_1 = \alpha\bar{\varepsilon}$ from choosing T is $\alpha 2 + (1-\alpha)0$, if he is not found out, since a fraction α of player 2's types choose L, and 2, if he is found out. Hence, player $\hat{t}_1$'s payoff from T is $\pi_{\hat{t}_1}(\mathrm{T}) = (1-\hat{t}_1)(\alpha 2 + (1-\alpha)0) + 2\hat{t}_1 = 2(-\bar{\varepsilon}\alpha^2 +$

Now, all types of player 1 below the critical type strictly prefer B to T, whereas all types above strictly prefer T to B. Indeed, for a typical type t_1 of player 1, the difference in payoff from choosing T versus B is

$$\pi_{t_1}(\mathrm{T}) - \pi_{t_1}(\mathrm{B}) = (1 - t_1)(3\alpha - 1) + t_1,$$

which, for α close enough to $1/3$ (for $\bar{\varepsilon}$ close enough to zero) is strictly increasing in t_1, and vanishes at $\hat{t}_1$. A similar preference holds between L and R for player 2.

Therefore, the strategies (2.1) and (2.2) form an equilibrium of $\mathcal{IG}$. Consequently, our interpretation of the mixed equilibrium of Battle of the Sexes is Bayesian: Each player chooses some particular pure strategy, yet the opponent is unsure of which one. The probabilities associated with a player's equilibrium mixture represent the opponent's beliefs about which pure strategy the player will choose.

Thus our incomplete information perturbation is, like Harsanyi (1973), able to rationalize the mixed equilibria of Matching Pennies and Battle of the Sexes as strict equilibria. But the interpretations of the two models are quite distinct. The player types in our perturbation optimally choose whether to reveal or to conceal their choices, choosing to conceal them in Matching Pennies (producing a classical interpretation) and to reveal them in Battle of the Sexes (producing a Bayesian interpretation); whereas in Harsanyi, almost all player types always use only pure strategies.

Our final example leads to a new interpretation of mixed strategy equilibria.

2.3. Modified Rock-Scissors-Paper

Consider the nonzero-sum modification of the zero-sum game Rock-Scissors-Paper shown in Figure 2.4, where $a < b < c < 1$ and a is close to 1. Modified Rock-Scissors-Paper (MRSP) differs from the usual version in two respects. First the game is no longer zero-sum because each player receives a payoff near -1 along the diagonal. Second, the off-diagonal payoffs have been perturbed slightly.

The new diagonal payoffs add an element of common interest in that both players now wish to avoid the diagonal. The perturbation of the off-diagonal payoffs avoids a particular non genericity, clarified below.[9]

$(1+\bar{\varepsilon})\alpha)$. Similarly, $\hat{t}_1$'s payoff from B is $\pi_{\hat{t}_1}(\mathrm{B}) = (1-\hat{t}_1)(\alpha 0 + (1-\alpha)1) + 1\hat{t}_1 = \bar{\varepsilon}\alpha^2 - \alpha + 1$. Equating the two gives the value of α.

[9] The remaining coincidences in payoffs are unimportant.

	L	C	R
T	-a,-a	1,-c	-b, 1
M	-c, 1	-b,-b	1,-a
B	1,-b	-a, 1	-c,-c

Figure 2.4: Modified Rock-Scissors-Paper

If $a = b = c = 1$, then MRSP has a unique equilibrium in which both players choose each of their pure strategies with probability 1/3. Moreover, because $a < b < c < 1$ and a is near 1, there is a unique equilibrium in which each pure strategy is chosen with probability near 1/3 and in which each player's equilibrium payoff is near $-1/3$.

Figure 2.5 shows player 1's payoff in the incomplete information game $\mathcal{IG}$ as a function of his mixed strategy, conditional on being found out. Triangle TMB in the figure is player 1's simplex of mixed strategy choices. Its vertices are labelled with the pure strategies, T, M and B, they represent. The hyperplanes above the triangle depict player 1's payoff, conditional on player 2 finding out his mixed strategy and choosing a best reply. Each hyperplane is labelled with the 2's best reply. Because a is almost equal to one, the three hyperplanes almost meet at player 1's equilibrium strategy in the center of the figure, yielding player 1 a payoff there close to $-1/3$, regardless of 2's best reply.

If player 1 were sure that his strategy would be found out, he would not choose a pure strategy, which would result in a payoff close to -1; neither would he choose the equilibrium mixture, which yields a payoff near $-1/3$. Instead, player 1 would choose the mixed strategy placing probability 1/2 on T and 1/2 on B. It is then a best reply for player 2 to choose C resulting in a positive payoff of $(1-a)/2$ for player 1.[10] Evidently, the fifty-fifty mixture reveals enough so that player 2 can avoid the diagonal, which is in their common interest, but it still conceals 1's final

[10] Player 2 is indifferent between C and R, but breaks this tie in player 1's favor by choosing C.

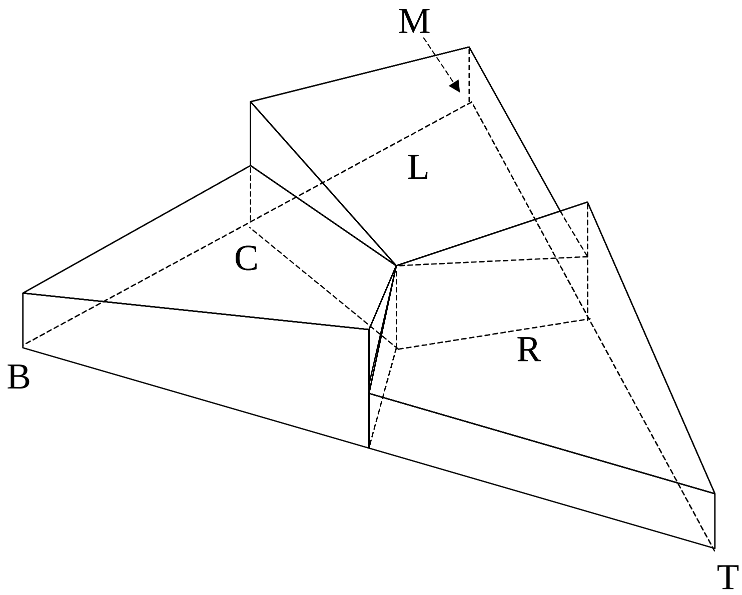

Figure 2.5: Player 1 Found Out in Modified Rock-Scissors-Paper

pure choice, reflecting the conflict of interest off the diagonal.

Indeed, the three mixed strategies 1/2-1/2 on T-B; 1/2-1/2 on T-M; 1/2-1/2 on M-B all yield player 1 a positive payoff conditional on being found out. Figure 2.5 shows that these strategies yield attractive payoffs relative to all other mixed strategies, when player 2 chooses a best reply.

If $\bar{\varepsilon}$ is small enough and $\underline{\varepsilon}$ is close enough to $\bar{\varepsilon}$, then these particular three mixed strategies yield an equilibrium of $\mathcal{IG}$, as follows. For each player i there are two critical types, $\hat{t}_{i1} < \hat{t}_{i2}$. Player 1 chooses

$$\begin{aligned} \text{1/2-1/2 on M-B if } t_1 &\in [\underline{\varepsilon}, \hat{t}_{11}) \\ \text{1/2-1/2 on T-M if } t_1 &\in [\hat{t}_{11}, \hat{t}_{12}) \\ \text{1/2-1/2 on T-B if } t_1 &\in [\hat{t}_{12}, \bar{\varepsilon}] \end{aligned}$$

and player 2 chooses

$$\begin{aligned} \text{1/2-1/2 on C-R if } t_2 &\in [\underline{\varepsilon}, \hat{t}_{21}) \\ \text{1/2-1/2 on L-C if } t_2 &\in [\hat{t}_{21}, \hat{t}_{22}) \\ \text{1/2-1/2 on L-R if } t_2 &\in [\hat{t}_{22}, \bar{\varepsilon}]. \end{aligned}$$

Moreover, each of these intervals of types occurs with probability approximately 1/3. Each player's strategy therefore induces a probability near 1/3 for each of the original pure strategies, and so approximates the mixed equilibrium of MRSP.

Thus, we are led to the following interpretation of the completely mixed equilibrium of MRSP: Each player i deliberately randomizes by choosing one of the mixed strategies that place probability one-half on each of two pure strategies. The opponent, player j, unaware of which one of the three possible fifty-fifty randomizations player i will employ, assigns probability roughly one-third to each possibility. Player i's equilibrium mixture is obtained by combining the three randomized strategies i might employ according to the weights j's beliefs assign to those strategies.

Let us emphasize the strategic benefits of the above strategies. By choosing a fifty-fifty mixture, enough information is revealed so that, if this mixture is found out, the opponent can successfully avoid the diagonal but cannot take undue advantage. Hence, our analysis uncovers the manner in which players strike a balance between revealing information and concealing it in nonzero-sum games.

Finally, because a, b, and c are distinct, it can be shown that these equilibrium mixed strategies are strictly optimal. That is, all types of each player except the two critical types strictly prefer their fifty-fifty equilibrium mixture to any other strategy.[11]

We now proceed with the formal analysis of the general case and also explore conditions under which concealment is helpful—as in Matching Pennies and Modified Rock-Scissors-Paper—and conditions under which it is not—as in Battle of the Sexes.

3. The Incomplete Information Perturbation

Let $G = (u_i, X_i)_{i=1,2}$ be a finite two-person normal form game in which player i's finite pure strategy set is X_i, his mixed strategy set is M_i, and his vNM payoff function is $u_i : X_1 \times X_2 \to \mathbb{R}$. We wish to capture the idea that each player is concerned that the other player might find out his mixed strategy, where the extent to which each player is concerned is private information. Ultimately, we shall be interested in the players' limiting behavior as these concerns vanish.

Given the game G, consider the following associated game of incomplete in-

[11] This is why we perturbed the off-diagonal payoffs.

formation.

$$\mathcal{IG} = (U_1, U_2, M_1, M_2, F_1, F_2):$$

- Each F_i is a cdf with $F_i(0) = 0$ and support $T_i \subseteq [0, 1]$.
- Player i's type set is T_i.
- Types are drawn independently according to F_1 and F_2.
- Player i's pure action set is M_i, his set of mixed strategies in G.
- When i's type is t_i and the vector of actions is (m_1, m_2), player i's payoff is

$$U_i(m_1, m_2, t_i) = (1 - t_i)u_i(m_1, m_2) + t_i v_i(m_i),$$

  where $v_i(m_i)$ is i's payoff in G resulting from m_i together with a best reply against it. If there are multiple best replies for j against m_i, one that is best for i is employed.[12]

Thus, $U_i(m_1, m_2, t_i)$ is the payoff i would receive in G when he plays m_i and his opponent plays m_j with probability $1 - t_i$ and plays a best reply to m_i with probability t_i. Player i's type t_i can therefore be interpreted as the probability he assigns to the event that his choice of mixed strategy in G will be found out by the opponent.

The above definition actually yields a collection of incomplete information games indexed by the distribution functions F_1 and F_2. We shall often be concerned with *atomless* cdf's. Such cdf's, F_i, in addition to satisfying $F_i(0) = 0$, are continuous on $[0, 1]$. Note that the incomplete information game approaches the original game G as the measure on each player's types tends to a mass point at zero.

3.1. Strategies, Lotteries and Induced Distributions

A strategy for player i in $\mathcal{IG}$ is a measurable map from T_i into $\Delta(M_i)$, where $\Delta(M_i)$ denotes the set of Borel probability measures on M_i. We shall refer to elements of M_i as mixed strategies in G, and to elements of $\Delta(M_i)$ as *lotteries*

[12] That is, $v_i(m_i) = \max_{x_j} u_i(m_i, x_j)$, s.t. $x_j \in \arg\max_{x'_j \in X_j} u_j(m_i, x'_j)$. So defined, $v_i(\cdot)$ is upper semicontinuous. The tie-breaking rule is innocuous because generically, some m'_i near m_i leaves j with a unique best reply and gives i a payoff near $v_i(m_i)$.

on M_i. So, in the incomplete information game $\mathcal{IG}$, a strategy specifies for each type of each player a lottery over that player's mixed strategies in G. Each player believes that, with the probability given by his type, his opponent finds out the mixed strategy in G that is the outcome of his type's lottery. Pure strategies in the incomplete information game are then degenerate lotteries and so specify a mixed strategy in G for each of a player's types.

An equilibrium of $\mathcal{IG}$ is a pair of strategies that constitute a Nash equilibrium from the ex-ante perspective. Equivalently, an equilibrium strategy pair must be such that given the other player's strategy, the element of $\Delta(M_i)$ chosen by t_i must be optimal for i conditional on t_i, for F_i-almost every t_i.

Let $\sigma_i(\cdot|\cdot)$ be a strategy for player i in $\mathcal{IG}$. Hence, $\sigma_i(\cdot|t_i)$ is for each t_i in T_i a lottery on M_i. Because each m_i in M_i induces a distribution over i's set of pure strategies X_i in G, $\sigma_i(\cdot|\cdot)$ gives x_i in X_i the probability

$$\int_{T_i}\int_{M_i} m_i(x_i)d\sigma_i(m_i|t_i)dF_i(t_i).$$

Let us denote this induced probability by $\bar{\sigma}_i(x_i)$, and the induced mixed strategy in M_i by $\bar{\sigma}_i$.

Because player i's payoff in $\mathcal{IG}$ does not directly depend upon j's type, and because j's strategy σ_j matters to i only when i's strategy is not found out, i's payoff depends only on the induced distribution $\bar{\sigma}_j$ over X_j and not otherwise on σ_j.

This can be seen by considering player i's payoff when his type is t_i and he chooses m_i while his opponent employs the strategy σ_j. Player i's payoff is then

$$(1-t_i)\int_{T_j}\int_{M_j} u_i(m_i,m_j)d\sigma_j(m_j|t_j)dF_j(t_j)+t_iv_i(m_i),$$

which, owing to the linearity of u_i in m_j is equal to

$$(1-t_i)u_i(m_i,\bar{\sigma}_j)+t_iv_i(m_i).$$

4. Zero-Sum Games

In our informal analysis of Matching Pennies in Section 2, we claimed that the equilibrium of $\mathcal{IG}$ in which every type of each player chooses the fifty-fifty mixture is the essentially unique equilibrium. This is a consequence of a more general result for zero-sum games that is given below.

Recall that a maxmin strategy in a zero-sum game is one that yields a player his value if the opponent employs a best reply. We then have the following result, whose proof can be found in Appendix B.

Theorem 4.1. *Suppose that G is a zero-sum game. Then a joint strategy in $\mathcal{IG}$ is an equilibrium if and only if almost every type of each player employs, with probability one, a maxmin strategy for G. Furthermore, in every equilibrium of $\mathcal{IG}$, every type of each player is indifferent among all of his maxmin strategies, and every positive type strictly prefers each of his maxmin strategies to each non maxmin strategy.*

Note that when a player has more than one maxmin strategy, no equilibrium of $\mathcal{IG}$ is strict since all maxmin strategies are then best replies. But the indeterminacy caused by this indifference is inconsequential because any mixture of maxmin strategies is itself a maxmin strategy. Consequently, neither player is forced to employ any particular randomization over his maxmin strategies in order to maintain equilibrium.

Theorem 4.1 leads to a purely classical interpretation of equilibria of two-person zero-sum games, because each player deliberately employs a maxmin strategy (which often involves randomization) and each is certain that the other will do so. We now explore the interpretation our model yields for equilibria of general two-person games.

5. General Two-Person Games

Our objective, as above, is to interpret any equilibrium of G through a nearby equilibrium of $\mathcal{IG}$. To do so requires the game G to be sufficiently robust. The following assumptions make this precise.

5.1. Genericity

Recall from Section 3 that $G = (u_i, X_i)_{i=1,2}$ is a finite two-person game with mixed strategy sets M_i, and that $v_i(m_i)$ is i's payoff in G when he chooses m_i and his opponent plays a best reply to m_i (breaking ties in i's favor if necessary).

For each $x_j \in X_j$, let $C_i(x_j)$ denote those elements of M_i against which x_j is a best reply for j. Consequently, each $C_i(x_j)$ is a convex polyhedron and so possesses finitely many extreme points. Let $\{m_{i1}, ..., m_{iK_i}\}$ denote the union over x_j of the extreme points contained in all the $C_i(x_j)$.

We shall require the following genericity assumptions:

A.1. Every equilibrium of G is regular.[13]

A.2. For each player i, $v_i(m_{i1}), ..., v_i(m_{iK_i})$ are distinct.

Both A.1 and A.2 are satisfied for all but perhaps a closed subset of games, G, having Lebesgue measure zero (in payoff space for any fixed finite number of pure strategies). The proof of this is standard in the case of A.1 (van Damme (1991, Chapter 2.6, Theorem 2.6.1, p. 42)) and can be found in Appendix B for A.2.

An equilibrium of $\mathcal{IG} = (U_1, U_2, M_1, M_2, F_1, F_2)$ is *essentially strict* if F_i-almost every type of each player i has a unique best choice in M_i. The role of genericity assumption A.2 is to ensure essential strictness, as the following result shows.

Proposition 5.1. *If G satisfies genericity assumption A.2 and each F_i is atomless, then every equilibrium of $\mathcal{IG}$ is essentially strict and almost every type of each player i employs some mixed strategy in $\{m_{i1}, ..., m_{iK_i}\}$.*

The proof can be found in Appendix B. Consequently, for generic games, the problem of indifference does not arise in our incomplete information game. This is important because our player types in general employ non-degenerate mixed strategies. Essential strictness ensures that when non-degenerate mixed strategies are employed, this is not to make the other player indifferent. Rather, they are employed because it is strictly optimal to do so (because concealment happens to be beneficial). We now provide the main result of this section which establishes that our incomplete information game can approximate all equilibria of a generic game G.

5.2. The Main Approximation Theorem

Theorem 5.2. *If G satisfies genericity assumptions A.1 and A.2, then for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all atomless F_1, F_2 satisfying $F_i(\delta) \geq 1 - \delta$ and every equilibrium m^* of G, $\mathcal{IG}$ has an essentially strict equilibrium whose induced distribution on the joint pure strategies in G is within ε of m^*.*

[13] For the definition of "regular equilibrium," see e.g., van Damme (1991, Chapter 2.5, Definition 2.5.1, p. 39).

Remark. According to this theorem, for any sufficiently nearby game $\mathcal{IG}$, every equilibrium of G can be approximated by some equilibrium distribution of $\mathcal{IG}$. A standard upper hemicontinuity argument establishes the converse, namely that all equilibrium distributions of nearby games $\mathcal{IG}$ must be close to some equilibrium of G.

The proof is given in Appendix B. The idea is to exploit the fact that player i's behavior in $\mathcal{IG}$ depends only upon the distribution m_j in M_j induced by j's strategy in $\mathcal{IG}$. Moreover, because by A.2 the $v_i(m_{ik})$ are distinct, all but finitely many types of player i have a unique best reply against any such distribution m_j and this best reply is one of the m_{ik}. Letting $g_i(m_j)$ denote the F_i-average over i's best replies as his type varies, it is not difficult to show that g_i is continuous. Moreover, if the mass of F_i is sufficiently concentrated near 0, then $g_i(m_j)$ is very close to a best reply in G against m_j. Consequently, $g = g_1 \times g_2$ is close to the product of the players' best reply correspondences for G. Because any regular equilibrium of G is an "essential" fixed point of G's best-reply correspondence and g is continuous, powerful results from algebraic topology allow us to conclude that g must have a fixed point near any such equilibrium of G. But, by construction, fixed points of g are the distributions on M of equilibria of $\mathcal{IG}$. The desired conclusion follows.

The theorem establishes that our incomplete information perturbation, $\mathcal{IG}$, can rationalize any equilibrium of a generic game G through a nearby equilibrium of $\mathcal{IG}$ in which the players have *strict* incentives to play their part. While this result is reminiscent of Harsanyi (1973), we have already seen that such an equilibrium of $\mathcal{IG}$ sometimes involves a positive measure of a player's types using *non-degenerate* mixed strategies.

5.3. The Interpretation

Let m^* be an equilibrium of a two-person game G that satisfies A.1 and A.2. Suppose that $\mathcal{IG}^n$ converges to G,[14] that σ^n is an equilibrium of $\mathcal{IG}^n$ for every n, and that the induced distributions, $\bar{\sigma}^n$, converge to m^*. By 5.1, almost every type of player i strictly optimizes in σ^n by employing one of the mixed strategies $\{m_{i1}, ..., m_{iK_i}\}$, so that σ_i^n entails some fraction of i's types, μ_{ik}^n, say, employing m_{ik} for each k. Hence, the other player is certain that player i will employ one of the m_{ik}, but is uncertain about which of the m_{ik} player i will employ. His conjecture, or belief, is that player i will employ m_{ik} with probability μ_{ik}^n. Finally,

[14] That is, the cdf's F_i^n of $\mathcal{IG}^n$ converge to mass points at zero as $n \to \infty$.

because the distribution of σ^n converges to m^*, we must have that for each player i,

$$m_i^* = \mu_{i1}^* m_{i1} + ... + \mu_{iK_i}^* m_{iK_i},$$

where μ_{ik}^* is the limiting fraction of i's types employing m_{ik}.[15] This decomposition of m_i^* therefore leads us to the following interpretation.

Each player i's equilibrium mixture, m_i^*, can be expressed as a convex combination of the mixed strategies $\{m_{i1}, ..., m_{iK_i}\}$. Each mixed strategy given positive weight in the convex combination represents a strategy that i might deliberately employ, while the weight on that mixed strategy represents the opponent's belief that i will employ it.

We have already seen that strictly mixed equilibria in zero-sum games have degenerate decompositions in which all of the weight is placed on the equilibrium mixed strategy. Consequently, such equilibria can always be interpreted from the purely classical point of view where the players deliberately randomize because concealment is beneficial.

Under what conditions is concealment beneficial in the nonzero-sum case? Equivalently, when does the above decomposition place positive weight on at least one non-degenerate mixed strategy? In such cases our interpretation of a mixed equilibrium will involve the classical view. Alternatively, under what conditions will the players instead wish to reveal their pure choices? Equivalently, when does the above decomposition give positive weight only to pure strategies. In such cases, our interpretation is similar to the Bayesian view. (See, for example, The Battle of the Sexes, in Section 2). These questions are taken up next.

6. When To Conceal, When To Reveal

In $\mathcal{IG}$, when a player of a given type strictly prefers to employ a non-degenerate mixed strategy from G, it is because that type strictly prefers concealing the pure choices in the support of that mixed strategy. When this occurs and the equilibrium of $\mathcal{IG}$ is near an equilibrium of G, our interpretation of G's equilibrium will involve (perhaps only partially) the classical view that randomization is deliberate. This motivates the following definition.

[15] Assume without loss that $\mu_{ik}^n \to \mu_{ik}^*$.

Definition 6.1. *An equilibrium m of G is strongly concealing for player i if there exists $\eta > 0$ such that for all sufficiently small $\varepsilon > 0$ and all atomless distributions F_1, F_2 satisfying $F_i(\varepsilon) \geq 1 - \varepsilon$, every equilibrium of $\mathcal{IG} \equiv (U_1, U_2; M_1, M_2, F_1, F_2)$ whose distribution on $X_1 \times X_2$ is within ε of m has the property that the F_i-measure of player i's types employing non-degenerate mixed strategies from G is at least η.*[16]

Thus, an equilibrium m of G is strongly concealing if for all nearby atomless incomplete information games $\mathcal{IG}$, a positive fraction of types must strictly mix in all equilibria near m.[17]

Theorem 4.1 implies that a mixed equilibrium of a zero-sum game is strongly concealing for i if and only if i has no pure maxmin strategy. Consequently, the fifty-fifty equilibrium of Matching Pennies is strongly concealing for both players.

As we shall see, the unique mixed equilibrium of the nonzero-sum game Modified Rock-Scissors-Paper from Section 2 is also strongly concealing for both players because, like the Matching Pennies' equilibrium, its support contains a "cyclic best reply sequence."

Formally, a *best reply sequence* in G is a finite sequence $x^1, x^2, ..., x^n$ of joint pure strategies such that in each step, say from x^k to x^{k+1}, one player's strategy is unchanged and the other player's strategy in x^{k+1} is a best reply to the opponent's strategy in x^k. A best reply sequence is *cyclic* if at least two of its elements are distinct and the first and last are identical.

The proofs of the following results can be found in Appendix B.

Theorem 6.2. *Suppose that m^* is an equilibrium of G. If the support of m^* contains a cyclic best reply sequence along which best replies are unique, then m^* is strongly concealing for both players.*

For generic games, players have unique best replies against pure strategies and so along best reply sequences. This leads to the following corollary.

[16] Reny and Robson (2002) also define an equilibrium to be merely *concealing* if these conditions hold for some, as opposed to all, atomless distributions F_i. They point out that some games possess equilibria that are concealing but not strongly concealing. We shall not discuss this weaker concept here.

[17] Theorem 5.2 ensures that under A.1 and A.2 no m can be strongly concealing simply because the particular F_i admit no equilibria of $\mathcal{IG}$ near m.

Corollary 6.3. *Generically, if a completely mixed equilibrium is not strongly concealing for either player, then beginning from any joint pure strategy, alternately best replying to one another eventually leads the players to a pure strategy equilibrium.*

Theorem 6.2 is driven in part by the fact that when best replies are unique, a best reply sequence can cycle only if, somewhere along it, some player's payoff strictly falls when the other player switches to a best reply.

On the other hand, suppose that player i's payoff falls nowhere along any best reply sequence. This means that beginning from any joint pure strategy, player i is, generically, made *better off* when player j switches to a best reply against i's strategy. Simply put, player i benefits when j finds out i's pure strategy choice. In such cases one would expect that concealment is harmful, i.e. that player i would prefer to reveal his choice. Our final result shows that this is indeed the case. Note that this result applies, in particular, to The Battle of the Sexes as well as to a whole class of coordination games. In all such games then, our interpretation of their equilibria involves only the Bayesian view. No player deliberately randomizes because randomization is actually harmful.

Theorem 6.4. *Suppose G satisfies genericity assumption A.2. If player i's payoff is weakly increasing along every best reply sequence in G, then in every equilibrium of $\mathcal{IG}$, all but perhaps finitely many types of player i strictly optimize by employing a pure strategy. In particular then, no equilibrium of G is strongly concealing for i.*

7. The Significance of Unused Strategies

We now demonstrate that whether an equilibrium is strongly concealing or not can depend on the payoffs to unused strategies. The reason for this is that unused strategies may be best replies when the opponent's mixed strategy is found out.

Consider, for example, the game of Figure 7.1. The 2×2 matrix in the top-left corner is Matching Pennies and the fifty-fifty Matching Pennies equilibrium remains a regular equilibrium of this game. However, although the fifty-fifty mixture is strongly concealing in Matching Pennies, without the strategies U and D, it is not strongly concealing here, when they are present.

The reason that the fifty-fifty equilibrium is not strongly concealing here is that each player knows that if he uses the pure strategy H or T and his opponent

	H	T	U	D
H	1,-1	-1, 1	2, 2	-3,-3
T	-1, 1	1,-1	-3,-3	2, 2
U	2, 2	-3,-3	-3,-3	-3,-3
D	-3,-3	2, 2	-3,-3	-3,-3

Figure 7.1: The Role of Unused Strategies

finds this out, the opponent will choose either U or D, giving the player his highest possible payoff of 2. Indeed, any nondegenerate mixture over H and T is strictly worse for a player than one of the pure strategies H or T.

To see this, consult Figure 7.2, where the solid line is player 1's payoff when player 2 finds out that 1 employs the mixed strategy: H with probability p and T with probability $1 - p$. (The labels H, T, U and D refer to player 2's best reply.) The dotted line gives 1's payoff, conditional on being found out, from the lottery in which the pure strategies H and T are chosen with probability p and $1 - p$, respectively. Player 1's payoff from any such lottery is constant and equal to 2. Since every nondegenerate mixture gives a payoff strictly less than 2, and the lottery and the mixture are equivalent if player 1 is not found out, the lottery is strictly better than the mixture for any positive type of player 1. Hence, no positive type will employ any such mixture, and the fifty-fifty equilibrium is not strongly concealing.

Another way to see that the fifty-fifty equilibrium is not strongly concealing is to note that both players' payoffs in Figure 7.1 are strictly increasing along every best reply sequence. Because A.2 holds generically, we can perturb the game slightly so that A.2 holds and then appeal to Theorem 6.4.

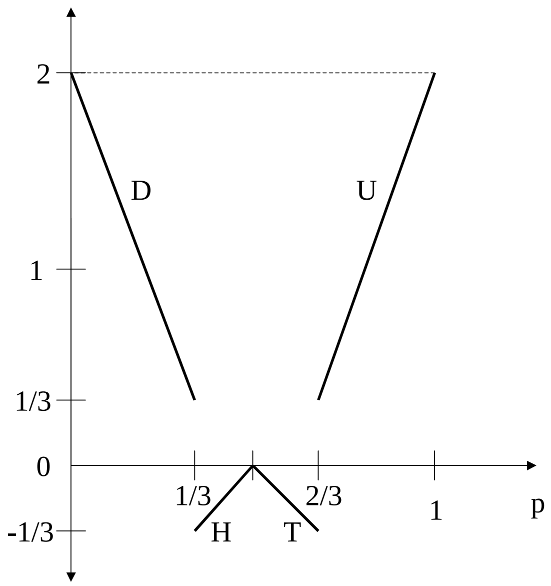

Figure 7.2: Player 1's payoff when observed by player 2

8. $\mathcal{IG}$ as the Reduced Form of a Dynamic Game

Finally, we address a key issue with a static model. This issue was described by von Neumann and Morgenstern as follows (VNM, 17.3, pp.146-8):

> On the one hand we have always insisted that our theory is a static one and that we analyze the course of one play and not that of a sequence of successive plays. But on the other hand we have placed considerations concerning the danger of one's strategy being found out into an absolutely central position. How can the strategy of a player—particularly one who plays a random mixture of several different strategies—be found out if not by repeated observation!

Although von Neumann and Morgenstern went on to argue that a dynamic model was nevertheless unnecessary, their argument is not entirely convincing. It is therefore worth pointing out that the static game $\mathcal{IG}$ from Section 3 is consistent with a fully dynamic interpretation in which no player's mixed strategy choice is ever directly revealed to the opponent. Rather, each player's mixed strategy choice is deduced by an opponent only after many observations of the realizations of the player's mixed strategy. We now sketch a simple dynamic model leading to this interpretation of $\mathcal{IG}$.

Suppose that the two-person nonzero-sum game, G, is repeatedly played by randomly matching, in each period, players from two large populations, so that there is no possibility of two particular players meeting more than once. Within each population there are two "varieties" of players. Variety I players, the focus of attention, do not observe the history of an opponent and must pay a small positive cost to implement any strategy that is other than zero-recall. Variety II players are not subject to such a cost and observe their opponent's history (i.e., the opponent's past *pure* actions) before play. Each player's type is fixed once and for all, and a player's type is the probability that he is matched with a variety II opponent in any given period. Hence, sending the type distributions to mass points at zero is equivalent to sending the fraction of variety II players in each population to zero. When there are no variety II players, the dynamic game is simply an infinite repetition of G between players who meet at most once and observe only their own histories. Both variety I and variety II players have expected liminf of the mean payoffs.

It can be shown that, if the fraction of variety II players in each population is close enough to zero, then for any equilibrium, (σ_1, σ_2), of the static game $\mathcal{IG}$, there is an equilibrium of the dynamic game in which a variety I player of type t_i employs $\sigma_i(\cdot|t_i)$ in each period, regardless of the history. Further, it can be shown that almost every type of variety I player strictly prefers this zero-recall strategy to any other strategy available in the dynamic game, whether finite-recall or not.[18]

Hence the static game $\mathcal{IG}$ has the following dynamic interpretation. A player's type, t_i, is not the probability that the opponent finds out his mixed strategy. Rather, it is the probability that the opponent observes the *realizations* of his mixed strategy choices in all *previous* periods. Thus, a player's mixed strategy is deduced by an opponent through repeated observations of the player's past actions. The reduced form model, $\mathcal{IG}$, is a parsimonious representation of this.

[18] The presence of a complexity cost is for simplicity. Similar conclusions can be shown to hold without such a complexity cost (see Reny and Robson (2002)).

Appendices

A. Compatibility of Beliefs

The game $\mathcal{IG}$ can be interpreted as part of the following extensive form game, where the players know that they will be playing G, but do not necessarily know whether their mixed strategy choices are made simultaneously.

- Nature begins by choosing each t_i, $i = 1, 2$, independently according to H_i on $[0, 1]$.
- Each player i is privately informed of t_i and Nature then determines whether the game is simultaneous according to the following event partition.
- With probability $(1 - t_1)(1 - t_2)$ neither player receives any additional information before simultaneously choosing a mixed strategy.
- With probability $t_i(1-t_j)$ player i receives no additional information but in fact makes his mixed strategy choice in M_i first, before player j, who is then informed of i's mixed strategy choice prior to choosing a mixed strategy in M_j.
- With probability $t_1 t_2$ it is common knowledge that the two players choose their mixed strategies simultaneously.
- After the players choose their mixed strategies, G is played with those strategies and the game ends.

Note first that in the extensive form, each player knows he may find out the other player's mixed strategy. Of course, in these subgames, he simply best replies to the revealed mixed strategy of the opponent. Second, note that when it is common knowledge that the players choose their strategies simultaneously, the resulting game is simply G, and so any equilibrium of G can be specified in this event.

Thus the remaining decision faced by a player i in the above extensive form occurs when he receives no additional information prior to making his mixed strategy choice. In this case, player i assigns probability t_i to the event that the opponent finds out his strategy, just as in $\mathcal{IG}$.

Further, when i receives no additional information, he must update his beliefs concerning j's type. According to Bayes' rule, i's updated beliefs about j's type are given by the distribution

$$F_j(t_j) = \frac{\int_0^{t_j}(1-t)dH_j(t)}{\int_0^1(1-t)dH_j(t)}.$$

These distributions provide the F_1 and F_2 given in the definition of $\mathcal{IG}$ and can be shown to yield the appropriate expected payoffs. In particular, the posterior for the opponent's type is independent of own type.

Thus, $\mathcal{IG}$ is the part of this extensive form game in which each player has not found out the opponent's mixed strategy but believes it is possible that the opponent will find out his.

B. Proofs

Proof of Theorem 4.1. The "if" part of the first statement is straightforward. Hence, we proceed with the "only if" part.

Even though G is a zero-sum game, $\mathcal{IG}$ will typically not be. However, $\mathcal{IG}$ is best reply equivalent to the zero-sum game of incomplete information, $\mathcal{IG}^0$, that results when each player i's payoff function is replaced by

$$u_i(m_1, m_2) + \frac{t_i}{1-t_i}v_i(m_i) - \frac{t_j}{1-t_j}v_j(m_j).$$[19]

The two games of incomplete information therefore have the same sets of equilibria. Throughout the remainder of the proof, the term "maxmin strategy" will refer to a maxmin strategy in the zero sum game G (not the zero-sum game $\mathcal{IG}^0$).

$\mathcal{IG}^0$ clearly has an equilibrium in which every type of each player chooses a maxmin strategy, giving $\mathcal{IG}^0$ a value of v^0, say.[20] Moreover, because beginning from such an equilibrium player 1's payoff rises above v^0 when a positive measure of player 2's types choose a non-maxmin strategy (owing to the term $-v_2(m_2)$ appearing in 1's payoff and because $F_2(0) = 0$), every equilibrium must involve

[19] This particularly simple argument requires each $\int \frac{t}{1-t}dF_i(t)$ to be finite. A similar proof, which involves a separate argument for types near unity, delivers the result even when one or both integrals are infinite.

[20] If the value of the zero-sum game G is v, then $v^0 = v\{1 + \int \frac{t}{1-t}dF_1(t) - \int \frac{t}{1-t}dF_2(t)\}$.

almost every type of player 2 employing, with probability one, a maxmin strategy. A similar argument applies to player 1. This proves of the "only if" part.

So, in $\mathcal{IG}$, F_j-a.e. type of player j employs one of his maxmin strategies. Consequently, player i can obtain at most his value whether or not he is found out and so is indifferent among all of his maxmin strategies. Furthermore, by employing a non maxmin strategy player i's payoff cannot be above his value if he is not found out and his payoff will be strictly below his value if he is found out. Therefore, every positive type strictly prefers every maxmin strategy to every non maxmin strategy. ■

Proof of Genericity of A.2. We wish to show that for fixed finite sets of pure strategies X_1 and X_2, and for all but a closed and Lebesgue measure zero set of pairs of the players' payoff matrices, for each $i = 1, 2$ the values $v_i(m_{i1}), ..., v_i(m_{iK_i})$ are distinct.

Let $n_i = |X_i|$ and let $\mathcal{U}_2$ denote the set of $n_1 \times n_2$ payoff matrices for player 2 in which every submatrix with at least two entries: (i) has full rank after adding a single row of 1's, and (ii) if square, is non singular.

The set of $n_1 \times n_2$ payoff matrices $\mathcal{U}_1$ for player 1 is defined analogously except that "row" is replaced by "column" in (i) above. The usage of "row" and "column" in the following paragraph assumes that $i = 1$ and $j = 2$. In the analogous alternative case, interchange "row" and "column" throughout the paragraph.

Viewing $\mathcal{U}_j$ as a subset of $\mathbb{R}^{n_1 n_2}$, $\mathcal{U}_j$ is open and its complement has Lebesgue measure zero. For any payoff matrix $u_j \in \mathcal{U}_j$, we may construct for each x_j in X_j the convex polyhedral set $C_i(x_j) \subseteq M_i$—which we now write $C_i(x_j; u_j)$ to make explicit the dependence upon u_j. Let $E_i(u_j)$ be the finite union over $x_j \in X_j$ of the finite sets of extreme points of $C_i(x_j; u_j)$. An implication of conditions (i) and (ii) in the definition of $\mathcal{U}_j$ is: $(*)$ if a sequence $u_j^n \in \mathcal{U}_j$ converges to $u_j^0 \in \mathcal{U}_j$, and for every n, m_{i1}^n and m_{i2}^n are distinct elements of $E_i(u_j^n)$ converging to m_{i1}^0 and m_{i2}^0 respectively, then m_{i1}^0 and m_{i2}^0 are distinct elements of $E_i(u_j^0)$. To see this we shall first show that for every $u_j \in \mathcal{U}_j$, $m_i \in E_i(u_j)$ if and only if the submatrix of u_j whose rows are determined by i's pure strategies in the support of m_i and whose columns are determined by j's pure u_j-best replies against m_i, is square.[21] So, suppose first that for some $x_j \in X_j$, m_i is extreme in $C_i(x_j; u_j)$. If the submatrix has fewer rows than columns, then because m_i makes j indifferent between the columns, the submatrix will not have full rank after the addition of a row of 1's, in violation of (i). But if there are fewer columns than rows, then in addition

[21] See also Shapley (1974, Assumption 2.2).

to m_i, there are many linear combinations of the rows, with weights summing to unity, that are proportional to a row of 1's. If $z(x_i)$ denotes the weight on each row x_i in one such solution, z, distinct from m_i, then for $|\alpha| > 0$ small enough $(1-\alpha)m_i + \alpha z$ is in $C_i(x_j; u_j)$ contradicting the fact that m_i is extreme.[22] Hence, the submatrix must be square. Conversely, suppose the submatrix is square. Choose any $x_j \in X_j$ that is u_j-best against m_i. Consequently, m_i is in $C_i(x_j; u_j)$ and we shall show that m_i is actually extreme in $C_i(x_j; u_j)$. This is obviously the case if the submatrix is 1×1, so suppose that it is 2×2 or larger and that m_i is a strict convex combination of distinct elements, m_i', in $C_i(x_j; u_j)$. Each x_j' that is u_j-best against m_i must also be best against each of the m_i', otherwise such an x_j' would not be as good as x_j against m_i. But by (ii) the non singularity of the submatrix implies that, among strategies in M_i—like the m_i'—whose supports are contained in m_i's, m_i is the only one against which each such x_j' is best for j. Thus each $m_i' = m_i$ and we conclude that m_i is extreme in $C_i(x_j; u_j)$. Returning to $(*)$, let us show that $m_{ik}^0 \in E_i(u_j^0)$, $k = 1, 2$. For each $k = 1, 2$, assume without loss that the rows and columns of the submatrix determined by m_{ik}^n and u_j^n are fixed. By the above characterization of the extreme points this submatrix is square and it suffices to show that the submatrix determined by m_{ik}^0 and u_j^0 is square. But the set of rows of the latter submatrix is a subset of those along the sequence because $m_{ik}^n \to m_{ik}^0$, while its set of columns is a superset of those along the sequence because limits of j's best replies remain best replies at the limit. Consequently, the limit matrix has at least as many columns as rows. But it cannot have strictly fewer rows, and so must be square, because m_{ik}^0 makes j indifferent between the columns, and the submatrix would then not have full rank after the addition of a row of 1's, contradicting (i). It remains to show that m_{i1}^0 and m_{i2}^0 are distinct. We have just seen that, for $k = 1$ and 2, the rows and columns determined by m_{ik}^n and u_j^n are the same as those determined by m_{ik}^0 and u_j^0. Thus it suffices to show that the rows and columns determined by m_{i1}^n and u_j^n are not identical to those determined by m_{i2}^n and u_j^n. But this follows immediately from the fact that if they were identical, then the common submatrix they determine is 1x1 or non singular, by (ii), either of which would imply that $m_{i1}^n = m_{i2}^n$, a contradiction. This establishes $(*)$.

For each of player j's payoff matrices $u_j \in \mathcal{U}_j$ define a set of player i's matrices $\mathcal{U}_i(u_j) = \{u_i \in \mathbb{R}^{n_1 n_2} : \sum_{x_i \in X_i} m_{i1}(x_i) u_i(x_i, x_{j1}) \neq \sum_{x_i \in X_i} m_{i2}(x_i) u_i(x_i, x_{j2})$, for

[22] Because $(1-\alpha)m_i + \alpha z$ makes j indifferent between the columns, continuity implies that the columns, and so x_j in particular, are best replies to $(1-\alpha)m_i + \alpha z$ for $|\alpha| > 0$ small enough. Hence, $(1-\alpha)m_i + \alpha z \in C_i(x_j; u_j)$.

all $x_{j1}, x_{j2} \in X_j$ and all $m_{i1} \neq m_{i2}$ s.t. m_{ik} is extreme in $C_i(x_{jk}; u_j)$ for $k = 1, 2\}$. Because X_j is finite and each $C_i(x_j; u_j)$ has finitely many extreme points, $\mathcal{U}_i(u_j)$ is an open subset of $\mathbb{R}^{n_1 n_2}$ whose complement has Lebesgue measure zero.

Let $\mathcal{U}(i) = \{(u_1, u_2) \in \mathbb{R}^{2n_1n_2} : \ u_j \in \mathcal{U}_j \text{ and } u_i \in \mathcal{U}_i(u_j)\}$. Note that if $(u_1, u_2) \in \mathcal{U}(1) \cap \mathcal{U}(2)$ then for $i = 1$ and 2, $v_i(m_{ik}) \neq v_i(m_{ik'})$ for all distinct $m_{ik}, m_{ik'}$ in $E_i(u_j)$, as desired. It therefore suffices to show that each $\mathcal{U}(i)$ is open with a Lebesgue measure zero complement. To see that $\mathcal{U}(i)$ is open, suppose that $(u_1^n, u_2^n) \to (u_1^0, u_2^0) \in \mathcal{U}(i)$. Because $\mathcal{U}_j$ is open, u_j^n is eventually in $\mathcal{U}_j$, and $(*)$ implies that u_i^n is eventually in $\mathcal{U}_i(u_j^n)$. Hence, (u_1^n, u_2^n) is eventually in $\mathcal{U}(i)$. To see that the complement of $\mathcal{U}(i)$ has Lebesgue measure zero in $\mathbb{R}^{2n_1n_2}$, note that for every $u_j \in \mathcal{U}_j$, the complement of the section $\mathcal{U}_i(u_j)$ has Lebesgue measure zero in $\mathbb{R}^{n_1n_2}$. Applying Fubini's theorem gives the desired result. ■

Proof of Proposition 5.1. The proof relies on two facts. First, for every $m_i \in M_i$, there is a lottery μ_i on $\{m_{i1}, ..., m_{iK_i}\}$ that is at least as good for i as m_i regardless of i's type and regardless of j's strategy. To see this, choose x_j so that $m_i \in C_i(x_j)$ and $v_i(m_i) = u_i(m_i, x_j)$. Clearly, such an x_j exists. Because $m_i \in C_i(x_j)$, m_i is a convex combination of the extreme points of $C_i(x_j)$. We may view the weights in this convex combination as defining a lottery, μ_i, on $\{m_{i1}, ..., m_{iK_i}\}$. Hence, we obtain, for every $t_i \in T_i$ and every strategy σ_j for player j in $\mathcal{IG}$,

$$\begin{aligned}(1-t_i)u_i(m_i, \bar{\sigma}_j) + t_i v_i(m_i) &= (1-t_i)u_i(m_i, \bar{\sigma}_j) + t_i u_i(m_i, x_j) \\ &= \sum_k \mu_{ik} \left[(1-t_i)u_i(m_{ik}, \bar{\sigma}_j) + t_i u_i(m_{ik}, x_j)\right] \\ &\leq \sum_k \mu_{ik} \left[(1-t_i)u_i(m_{ik}, \bar{\sigma}_j) + t_i v_i(m_{ik})\right],\end{aligned}$$

as desired, where the inequality follows because μ_{ik} is positive only when x_j is u_j-best for j against m_{ik} and because, by definition, $v_i(m_{ik}) \geq u_i(m_{ik}, x_j)$ for all such x_j. Note that a consequence of the above inequality is that a player's type has a unique best reply against the opponent's strategy if and only if he has a unique best reply among $\{m_{i1}, ..., m_{iK_i}\}$.

Second, for a fixed mixture $m_i \in M_i$ and a fixed distribution, $\bar{\sigma}_j$, over X_j induced by the opponent's strategy in $\mathcal{IG}$, player i's payoff, $(1-t_i)u_i(m_i, \bar{\sigma}_j) + t_i v_i(m_i)$, is linear in his type t_i. Consequently, because $v_i(\cdot)$ takes on distinct values for distinct extreme points m_{ik}, at most one type can be indifferent between any two of the extreme points.

Together, the two facts imply that at most finitely many types can have multiple best replies among all the extreme points and hence also among all the m_i in M_i. The result then follows because F_i is atomless. ■

The proof of Theorem 5.2 relies on an intuitive corollary of powerful results from algebraic topology.

Corollary B.1. *Suppose U is a bounded, open set in $\mathbb{R}^k$ and f, $g : cl(U) \to \mathbb{R}^k$ are continuous.*[23] *Further, suppose that f is continuously differentiable on U, that x_0 is the only fixed point of f in U, and that $|I - Df(x_0)| \neq 0$. If, for every $t \in [0, 1]$, the function $(1-t)f + tg$ has no fixed point on the boundary of U, then g has a fixed point in U.*

Proof of Corollary B.1. Since x_0 is the unique fixed point of f in U, and $|I - Df(x_0)| \neq 0$, it follows that 0 is a regular value of $c(x) = x - f(x)$. Hence, by Dold (1972, IV-5.13.4, p. 71), $\deg_0 c = \text{sgn}|I - Df(x_0)| = \pm 1$. If $d(x) = x - g(x)$, then by hypothesis, for every $t \in [0, 1]$, $(1-t)c+td$ has no zero on the boundary of U. Consequently, by Dold (1972, IV-5.13.3, p. 71 and IV-5.4, p. 67), $\deg_0 d = \deg_0 c = \pm 1$ and d has a zero in U. Hence, g has a fixed point in U. ■

Loosely, Corollary B.1 states that if x_0 is the only fixed point of f in some neighborhood, and f is not tangent to the forty-five-degree line, then continuous shifts of f will also have a fixed point in the neighborhood, so long as no fixed point escapes through the neighborhood's boundary.

Proof of Theorem 5.2.[24] Because, by A.1, every equilibrium of G is regular, G has finitely many isolated equilibria. Consequently, it suffices to establish the result for a single equilibrium, m^*, of G. Let $n_i = |X_i|$, and for every $m_i \in M_i$, extend $u_i(m_i, \cdot)$ linearly to all of $\mathbb{R}^{n_j}$. Because, by A.2, the $v_i(m_{ik})$ are distinct for each player i, for every $z_j \in \mathbb{R}^{n_j}$ there is a unique solution, $b_i(z_j|t_i) \in M_i$, to $\max_{m_i \in M_i}(1 - t_i)u_i(m_i, z_j) + t_i v_i(m_i)$ for all but perhaps finitely many $t_i \in [0, 1]$. Moreover, by the argument given in the proof of Proposition 5.1, the unique maximizer must be one of the m_{ik}. Define $g_i(z_j) = \int_0^1 b_i(z_j|t_i)dF_i(t_i)$. Because F_i is atomless and sufficiently small changes in z_j do not affect the unique best reply of an arbitrarily large fraction of i's types, $g_i : \mathbb{R}^{n_j} \to M_i$ is continuous. Also, note that if $\hat{m}$ is a fixed point of $g = g_1 \times g_2 : \mathbb{R}^{n_1+n_2} \to M$, then $\hat{m} \in M$

[23] $cl(U)$ denotes the closure of U.

[24] We owe a substantial debt to Hari Govindan who greatly simplified our original proof by providing detailed suggestions upon which the following proof is based.

and for each player i, $\hat{m}_i = \int_0^1 b_i(\hat{m}_j|t_i)dF_i(t_i)$, so that $(b_1(\hat{m}_2|\cdot), b_2(\hat{m}_1|\cdot))$ is an equilibrium of $\mathcal{IG}$ whose induced distribution on M is $\hat{m}$. Thus, given $\varepsilon > 0$, it suffices to show that for all δ small enough, g has a fixed point within ε of m^* whenever $F_i(\delta) \geq 1 - \delta$ for $i = 1, 2$. Henceforth we shall write g^δ to make explicit the dependence of g upon δ.

Because m^* is regular and there are just two players, the number of pure strategies in the support of each player's mixed strategy is the same, l say. So, assume, without loss, that the support of m_i^* is $\{x_{i1}, ..., x_{il}\}$. Define the continuously differentiable function $f_i : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_i}$ by

$$f_i(z_1, z_2) = z_i + \begin{pmatrix} 1 - \sum_k z_{ik} \\ z_{i2}(u_i(x_{i2}, z_j) - u_i(x_{i1}, z_j)) \\ . \\ . \\ z_{in_i}(u_i(x_{in_i}, z_j) - u_i(x_{i1}, z_j)) \end{pmatrix}. \tag{B.1}$$

Let U be an open ball in $\mathbb{R}^{n_1+n_2}$ containing m^*such that (i) every $z \in U$ is within ε of m^* and m^* is the only equilibrium of G in $cl(U)$, (ii) $z \in cl(U)$ implies $z_{ik} > 0$ for every $k \leq l$ and $i = 1, 2$, (iii) $z \in cl(U)$ implies $u_i(x_{ik}, z_j) - u_i(x_{i1}, z_j) < 0$ for every $k > l$ and $i = 1, 2$. Property (i) can be satisfied because, by regularity, m^* is isolated. Property (ii) can be satisfied because $m_{ik} > 0$ for every $k \leq l$, and property (iii) can be satisfied because x_{i1} is in the support of m_i^* and, by regularity, m^* is quasi-strict.

Remark 1. Letting $f = f_1 \times f_2$, we see that if $\hat{z} \in cl(U)$ is a fixed point of f, then $\hat{z}_{ik}(u_i(x_{ik}, \hat{z}_j) - u_i(x_{i1}, \hat{z}_j)) = 0$ for every $k > 1$, so that by property (ii) of U, $u_i(x_{ik}, \hat{z}_j) - u_i(x_{i1}, \hat{z}_j) = 0$ for every $k \leq l$, and by property (iii) of U, $\hat{z}_{ik} = 0$ for every $k > l$. Also, $\hat{z}$ fixed implies $1 - \sum_k \hat{z}_{ik} = 0$ so that, by property (ii) of U and $\hat{z}_{ik} = 0$ all $k > l$, $\hat{z} \in M$. Consequently, $\hat{z}$ is an equilibrium of G, which means, by property (i) of U, that $\hat{z} = m^*$. Hence, m^*, a fixed point of f in U, is the *only* fixed point of f in $cl(U)$.

Remark 2. Because m^* is regular, $|I - Df(m^*)| \neq 0$, by definition. (See van Damme (1991, p.39).)

Let ∂U denote the boundary of U. We claim that there exists $\bar{\delta} > 0$ small enough such that:

$$\forall \delta < \bar{\delta} \text{ and } \forall t \in [0, 1], \ (1 - t)f + tg^\delta \text{ has no fixed point in } \partial U. \tag{B.2}$$

Suppose not. Then, because ∂U is compact, there exists $z^\delta \to \hat{z}$, $t^\delta \to \hat{t}$, and $g^\delta(z^\delta) \to \hat{m} \in M$ as $\delta \to 0$ such that for every δ, $(1-t^\delta)f(z^\delta)+t^\delta g^\delta(z^\delta) = z^\delta \in \partial U$. Consequently,

$$(1-\hat{t})f(\hat{z}) + \hat{t}\hat{m} = \hat{z} \in \partial U. \tag{B.3}$$

Furthermore, $\hat{t} > 0$ because otherwise $\hat{z}$ would be a fixed point of f, implying, by Remark 1, that $m^* = \hat{z} \in \partial U$, a contradiction.

Because, for every $\delta > 0$, $g_i^\delta(z_j^\delta)$ is the F_i-average over t_i of maximizers of $(1-t_i)u_i(m_i, z_j^\delta) + t_i v_i(m_i)$, and $F_i(\delta) \geq 1-\delta$, $\hat{m}_i$ is a maximizer of $(1-t_i)u_i(m_i, \hat{z}_j) + t_i v_i(m_i)$ when $t_i = 0$. Hence,

$$\hat{m}_i \text{ solves } \max_{m_i \in M_i} u_i(m_i, \hat{z}_j). \tag{B.4}$$

So, because, by property (iii) of U, $u_i(x_{ik}, \hat{z}_j) - u_i(x_{i1}, \hat{z}_j) < 0$ for every $k > l$, we must have $\hat{m}_{ik} = 0$ for all $k > l$. Consequently, (B.3), (B.1), and $\hat{t} > 0$ together imply $\hat{z}_{ik} = 0$ for all $k > l$.

We'll now show that $f_{ik}(\hat{z}) = \hat{z}_{ik}$, for every $1 < k \leq l$. If $f_{ik}(\hat{z}) < \hat{z}_{ik}$ for some $1 < k \leq l$, then property (ii) of U and (B.1) imply $u_i(x_{ik}, \hat{z}_j) - u_i(x_{i1}, \hat{z}_j) < 0$ and so by (B.4) $\hat{m}_{ik} = 0 < \hat{z}_{ik}$. But this contradicts (B.3). Consequently, for every $1 < k \leq l$, $f_{ik}(\hat{z}) \geq \hat{z}_{ik}$ and so by (B.3), and because $\hat{t} > 0$, $\hat{m}_{ik} \leq \hat{z}_{ik}$. On the other hand, if $f_{ik}(\hat{z}) > \hat{z}_{ik}$ for some $1 < k \leq l$, then property (ii) of U and (B.1) imply $u_i(x_{ik}, \hat{z}_j) - u_i(x_{i1}, \hat{z}_j) > 0$ and so $\hat{m}_{i1} = 0$. By (B.3), this implies $(1-\hat{t})(1 - \sum_k \hat{z}_{ik}) + \hat{t}(-\hat{z}_{i1}) = 0$, and because $\hat{z}_{i1} > 0$ by property (ii) of U, we must then have $0 < \hat{t} < 1$ and $1 - \sum_k \hat{z}_{ik} > 0$. But this contradicts $1 = \sum_k \hat{m}_{ik} = \sum_{1<k\leq l} \hat{m}_{ik} \leq \sum_{1<k\leq l} \hat{z}_{ik} < \sum_{k \leq l} \hat{z}_{ik} = \sum_k \hat{z}_{ik}$. Hence, $f_{ik}(\hat{z}) = \hat{z}_{ik}$ for every $1 < k \leq l$, so that by (B.3) and the result of the previous paragraph, $\hat{m}_{ik} = \hat{z}_{ik}$ for all $k > 1$.

Finally, (B.3) implies $(1-\hat{t})(1 - \sum_k \hat{z}_{ik}) + \hat{t}(\hat{m}_{i1} - \hat{z}_{i1}) = 0$. But because $\hat{m}_{ik} = \hat{z}_{ik}$ for all $k > 1$ and $\sum_k \hat{m}_{ik} = 1$, we have $1 - \sum_k \hat{z}_{ik} = \hat{m}_{i1} - \hat{z}_{i1}$. Hence, $\hat{m}_{i1} = \hat{z}_{i1}$ and we may conclude that $\hat{z} = \hat{m}$. However, this implies, by (B.3), that $\hat{z} \in \partial U$ is a fixed point of f, contradicting Remark 1, and completing the proof of (B.2).

By (B.2) and Remarks 1 and 2, we may appeal to Corollary B.1 and conclude that for all $\delta < \bar{\delta}$, $g^\delta : \mathbb{R}^{n_1+n_2} \to M$ has a fixed point in U. ■

Proof of Theorem 6.2. Consider the point (x_i', x_j) on the cyclic best reply sequence that maximizes i's payoff when j's pure strategy is a best reply against i's. Consider also the next two points along the sequence, (x_i, x_j) and (x_i, x_j').

Because the sequence is a cycle and best replies are unique along it, $x_i \neq x_i'$ and $x_j' \neq x_j$. Because the cycle is contained in the support of m^*, $m_i^*(x_i) > 0$ and $m_i^*(x_i') > 0$.

Now, by construction, $u_i(x_i', x_j) \geq u_i(x_i, x_j')$. Also, because best replies are unique along the sequence, $u_i(x_i', x_j) < u_i(x_i, x_j)$ and we may choose $\gamma > 0$ small enough so that x_j is j's unique best reply against the mixed strategy m_i^γ giving x_i' probability $(1-\gamma)$ and x_i probability γ. Consequently,

$$\begin{aligned} v_i(m_i^\gamma) &= (1-\gamma)u_i(x_i', x_j) + \gamma u_i(x_i, x_j) \\ &> u_i(x_i', x_j) \\ &\geq u_i(x_i, x_j'). \end{aligned}$$

But $v_i(x_i') = u_i(x_i', x_j)$ and $v_i(x_i) = u_i(x_i, x_j')$ then imply that $v_i(m_i^\gamma) > v_i(x_i') \geq v_i(x_i)$. Consequently, m_i^γ is strictly better for i than each of the pure strategies x_i and x_i' when i's strategy is found out.

Suppose σ is an equilibrium of $\mathcal{IG}$. Given the equilibrium strategy σ_j of player j and the distribution, $\bar{\sigma}_j \in M_j$ it induces, suppose without loss that $\min(u_i(x_i', \bar{\sigma}_j), u_i(x_i, \bar{\sigma}_j)) = u_i(x_i', \bar{\sigma}_j)$. Then $u_i(m_i^\gamma, \bar{\sigma}_j) = (1-\gamma)u_i(x_i', \bar{\sigma}_j) + \gamma u_i(x_i, \bar{\sigma}_j) \geq u_i(x_i', \bar{\sigma}_j)$. Consequently, m_i^γ is at least as good as x_i' when i's strategy is not found out. Altogether, this means that m_i^γ is strictly better than x_i' for every positive type of player i against σ_j.

Consequently, if the distribution $\bar{\sigma}_i$ is close enough to m_i^*, then because the fraction of types employing x_i' is zero and $m_i^*(x_i') > 0$, a positive and bounded away from zero measure of types must employ non-degenerate mixed strategies. ■

Proof of Theorem 6.4. Suppose that σ is an equilibrium of $\mathcal{IG}$. As can be seen from the proof of Proposition 5.1, A.2 implies that i's best reply to σ_j, $\sigma_i(t_i)$, is unique for all but perhaps finitely many t_i. It therefore suffices to show that when t_i's best reply is unique, it is pure.

So, let m_i be t_i's unique best reply against σ_j. . Suppose that, upon finding out m_i, a best reply for player j which breaks ties in i's favor is x_j. Consider the lottery, μ_i, in $\Delta(M_i)$ giving probability $m_i(x_i)$ to each pure strategy x_i. If j does not find out i's strategy choice, this lottery yields player i the same payoff as the mixed strategy m_i. If j finds out i's strategy choice, the lottery yields i an expected payoff of $\sum_{x_i \in X_i} m_i(x_i) v_i(x_i)$, because j finds out the outcome of the lottery. This payoff must be at least as large as $v_i(m_i) = \sum_{x_i \in X_i} m_i(x_i) u_i(x_i, x_j)$, since if player j has a best reply to x_i that differs from x_j, switching to it cannot hurt player i, by hypothesis. Hence,

$$
\begin{aligned}
& (1-t_i)u_i(m_i, \bar{\sigma}_j) + t_i v_i(m_i) \\
\leq & \sum_{x_i \in X_i} m_i(x_i)[(1-t_i)u_i(x_i, \bar{\sigma}_j) + t_i v_i(x_i)],
\end{aligned}
$$

which says that, against σ_j, t_i's payoff from employing his unique best reply m_i is no higher than his payoff from employing the lottery μ_i. Hence, one of the pure strategies in the support of the lottery must be a best reply against σ_j, which, by uniqueness, implies that m_i must be this pure strategy. ■

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