Uncertainty and Compound Lotteries: Calibration[1]

Yoram Halevy
University of British Columbia

Emre Ozdenoren
University of Michigan

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Abstract

The Ellsberg experiments provide an intuitive illustration that the Savage approach, which reduces subjective uncertainty to risk, is not rich enough to capture many decision makers' preferences. Recent experimental evidence suggests that decision makers reduce uncertainty to compound risk. This work presents a theoretical model of decision making in which preferences are defined on both Savage subjective acts and compound objective lotteries. Preferences are two-stage probabilistically sophisticated when the ranking of acts corresponds to a ranking of the respective compound lotteries induced by the acts through the decision maker's subjective belief. This family of preferences includes various theoretical models that have been proposed in the literature to accommodate non-neutral attitude towards ambiguity. The principle of calibration, which was used by Ramsey and de Finetti, allows an outside observer to relate preferences over acts and compound objective lotteries. If preferences abide by the calibration axioms, the evaluation of the compound lottery induced by an act through the subjective belief coincides with the evaluation of the corresponding compound objective lottery. Calibration provides the foundation that allows one to formalize and understand the tight empirical association between probabilistic sophistication and reduction of compound lotteries, for all two-stage probabilistically sophisticated preferences.

1 Introduction

One of the most important theoretical implications of the standard model of decision-making under uncertainty, is the possibility to reduce uncertainty to risk. Savage's theory of subjective expected utility distilled early ideas of Ramsey and de Finetti to provide testable behavioral axioms, which allow a modeler to derive subjective probabilities from preferences over acts. The theory implies that the information about the likelihood of events matters only to the extent that it affects the decision maker's subjective probability assessments, and does not leave any room for the decision maker's confidence in those probabilistic assessments. This result was further generalized by Machina and Schmeidler [16, 17] by relaxing the expected utility component, and focusing on the probabilistic sophistication component of behavior.

Ellsberg's [6] thought experiments challenged this conclusion: in a series of ingenious and intuitive experiments he showed that decision makers' confidence in the likelihood of events, or as it is often called the degree of ambiguity, plays a fundamental role in determining their choices. One way to summarize his results is that "there are uncertainties that are not risks" [6]. Following this line of thought several axiomatic models that generalize subjective expected utility have been proposed. They all have in common a primitive distinction between uncertainty and risk.

During the years since Ellsberg's paper, several researchers have questioned this accepted distinction between risk and uncertainty as the one underlying ambiguity aversion. Smith [26] was the first to conjecture that most decision makers who are ambiguity averse in Ellsberg's examples, would prefer a simple one-stage lottery to a compound objective lottery. If this would be the case, he wrote, we should ask "are there risks that are not risks?" [26]. Kahneman and Tversky [13] (in an early draft of their Prospect Theory contribution) conjectured that understanding of the Ellsberg paradox relies on decision makers' preferences being defined over higher-order risks. Segal [23, 24] showed that if a decision maker views the Ellsberg urn as a compound lottery, and uses non-expected utility (in which she does not multiply probabilities according to the laws of probability) then behavior consistent with ambiguity aversion may result. Rustichini [20] proposed a formal model in which preference over acts is consistent with higher-order beliefs. Robson [19] and Halevy and Feltkamp [12] provide an evolutionary and behavioral rationals for uncertainty aversion that rely on a perception of an ambiguous lottery as composed of positively correlated risks, with higher-order uncertainty.

A recent study by Halevy [11] provides support for this alternative view: it establishes a direct empirical link between decision makers' choices over objective compound lotteries and their attitudes towards ambiguity. A decision maker is said to satisfy the reduction of compound lotteries (ROCL) if her preferences depends only on the probabilities of final outcomes, which she calculates by multiplying probabilities according to the laws of probability. Halevy found that decision makers who reduce compound objective lotteries are found to be neutral to ambiguity. That is, they reduce uncertainty to risk. Furthermore, decision makers whose preferences over compound lotteries do not abide by the reduction axiom, have attitude towards ambiguity that is generally consistent with a view of the ambiguous lottery as a compound lottery. The experimental link between ROCL and attitude to ambiguity suggests that in order to understand ambiguity we need to link the DM's preferences over these two domains: compound lotteries and acts (i.e., mappings from states of nature to outcomes). This is in contrast to the standard approach in the literature which focuses only on the DM's preferences over acts to derive representations of ambiguity averse behavior.

In order to model and gain insight into the empirical link between ROCL and ambiguity aversion, we extend the domain of the DM's preferences to include the union of compound lotteries and Savage-acts. This extended domain is just sufficient for our purposes, since if we fix a set of probability measures over the state space, and consider a probability distribution over the measures in this set, each act induces a compound lottery. The induced compound lottery is constructed by first converting the act into a simple lottery with respect to each measure in the set, and then assigning to each such lottery the probability associated with the corresponding distribution.

We present a new property of preferences called Two-Stage Probabilistic Sophistication (2SPS). A decision maker's preferences satisfy 2SPS if there exists a set of probability measures over the state space (given objectively from the structure of the act), and a *fixed* subjective probability measure over the measures in this set (representing the DM's belief over the measures), such that the DM is indifferent between acts that induce the same compound lottery. That is, a DM who satisfies 2SPS reduces all uncertainty to compound risk. A DM who reduces all uncertainty to simple risk is Probabilistically Sophisticated [16], which is frequently used as a benchmark for ambiguity neutrality (Epstein [7]). 2SPS generalizes probabilistic sophistication, and allows for behavior that is non-neutral with respect to ambiguity, without restricting the utility function used to evaluate each stage of the compound lottery.

We show that 2SPS is a consequence of two new behavioral axioms called Calibration and First-Stage Probabilistic Beliefs. These axioms are very much in the spirit of Ramsey and de Finetti who used simple lotteries in order to calibrate the decision maker's subjective beliefs. Sarin and Wakker [21] used the calibration approach in order to provide foundations for subjective expected utility by calibrating Savage acts to one-stage lotteries. We use a similar methodology by calibrating Savage acts to compound lotteries. We show that if, in addition to Calibration and First-Stage Probabilistic Beliefs, preferences satisfy ROCL then they must be Probabilistically Sophisticated, and thus can not display ambiguity averse behavior. This result establishes the formal link between ROCL and ambiguity neutrality.

An equivalent statement of the result is that non-neutral attitude to ambiguity implies violation of ROCL. From a normative perspective, some may view the latter as a "mistake" and the former as "rational" choice behavior. Empirical evidence suggests that not only ROCL fails, but also that the ambiguity attitude of decision makers is closely associated to the form in which it fails. Our approach relies on this empirical evidence. Moreover, we do not consider calibrating the choices in the subjective and the objective domains as necessarily contradicting the above normative view. To the contrary, the normative foundations for ambiguity aversion may shed light on why and how decision makers violate reduction.

As a by-product, our work also provides a methodology for obtaining straightforward foundations for several representations that have been proposed in the literature. Calibration and First-Stage Probabilistic Beliefs allow to translate the structure imposed on preferences over compound lotteries to preferences over subjective acts. Using this method we provide alternative foundations for the recursive expected utility (REU) model (Klibanoff, Marinacci and Mukerji [14], Nau [18], Ergin and Gul [8], Ahn [1] and Seo [25]), and can provide similar foundations for the recursive nonexpected utility (RNEU) model. Our axioms are simple to understand and are based entirely on observable choices. They are not constrained to one functional form, and are therefore consistent with the empirical evidence found in Halevy [11] of non-uniform association between preferences over compound lotteries and attitudes to ambiguity.

In closing, we present an new example, in the spirit of Ellsberg's thought experiments, of a preference relation that is not two-stage probabilistically sophisticated. Beyond demonstrating that the property of 2SPS is refutable, it suggests that the current models of ambiguity might lack the necessary hierarchical structure, that will allow one to capture ambiguity at an arbitrary level.

2 Framework

Let $\mathcal{C}$ be an arbitrary set of consequences (prizes). $\mathcal{L}_1$ is the set of all simple objective lotteries. Elements of $\mathcal{L}_1$ are denoted by X, Y, etc. A simple lottery that yields $x_j \in \mathcal{C}$ with probability q_j for $j = 1, 2, ..., m$ is denoted by $(x_1, q_1; \ldots; x_m, q_m)$. $\mathcal{L}_2$ is the set of all compound objective lotteries. Elements of $\mathcal{L}_2$ are denoted by L_1, L_2, etc. A compound lottery that yields the simple lottery X_j with probability p_j for $j = 1, 2, ..., n$ is denoted by $(X_1, p_1; \ldots; X_n, p_n)$. Simple lotteries can be associated with two subsets of compound lotteries. The first is the set of compound lotteries that are degenerate in the first stage. We refer to this set as $\Delta \equiv \{(X, 1) : X \in \mathcal{L}_1\}$. The second is the set of compound lotteries that are degenerate in the second stage. We refer to this set as
$\Gamma \equiv \{((x_1, 1), p_1; ...; (x_n, 1), p_n) : X = (x_1, p_1; \ldots; x_n, p_n) \in \mathcal{L}_1\}$. For $X \in \mathcal{L}_1$ the elements of Δ and Γ that correspond to a given simple lottery X are denoted by δ_X and γ_X respectively.

Denote by Ω the (finite) state space, and by Σ the set of events. $\Delta(\Omega)$ is the set of probability measures on (Ω, Σ), and $\Delta(\Delta(\Omega))$ is the set of all probability measures with finite support on $\Delta(\Omega)$.

An act is a function $f : \Omega \to \mathcal{C}$, measurable with respect to Σ and assumed to have finite range. Denote by $\mathcal{F}$ the set of all acts. Given a partition $\{S_1, ..., S_m\}$ of Ω, an act that yields $x_i \in \mathcal{C}$ on event S_i is denoted by $(x_1, S_1; ...; x_m, S_m)$.

The set of all gambles is given by $\mathcal{G} = \mathcal{F} \cup \mathcal{L}_2$. Let $\succsim$ be a binary relation on $\mathcal{G}$. This is the *minimal* domain in which the decision maker can conceivably compare compound lotteries and acts. One can imagine a larger domain (as in Anscombe and Aumann's [2] original work) where preferences are defined over roulette-lotteries on horse-race acts, whose outcomes are roulette-lotteries (this domain is used by Seo [25]). Naturally, axioms in this domain are stronger. However, it includes objects that are not part of most standard choice problems. Hence, the behavioral implication of axioms there may not be transparent. We chose a minimal domain, where the axioms' behavioral

implication are easily verifiable.
We make two preliminary assumptions over the binary relation $\succsim$.

Axiom 1 *(Weak Order)* $\succsim$ *on* $\mathcal{G}$ *is complete and transitive.*

Though Weak Order is a standard axiom, note that in this framework it requires the decision maker (DM) to compare compound lotteries and acts. The following axiom requires that the DM is indifferent between a degenerate compound lottery that gives with probability one the lottery in which the outcome x is received with probability one and a degenerate act that gives x in every state.

Axiom 2 *(Degenerate Lottery/Act Equivalence) For every* $x \in \mathcal{C}$, $\delta_{(x,1)} \sim (x, \Omega)$.

Using the previous axiom sometimes we write $x \succ y$ where x and y may refer to either degenerate compound lotteries or degenerate acts. To rule out the trivial cases, assume that there exist two prizes x^*, x_* such that $x^* \succ x_*$.

3 Preferences over Compound Lotteries

This subsection presents axioms only on compound (two-stage) objective lotteries ($\mathcal{L}_2$).
The first axiom requires that there exists a function that represents the DM's preferences over compound lotteries[1].

Axiom 3 *(Representation over compound lotteries) There exists* $W : \mathcal{L}_2 \to \mathbb{R}$ *that represents* $\succsim$ *over compound lotteries.*

Generalizing monotonicity of preferences with respect to first order stochastic dominance to compound lotteries is not transparent, and is required to state our main result below. We employ a definition suggested by Segal [24] that generalizes the standard definition of first-order stochastic dominance.

[1] Representation over compound lotteries can be derived by adding appropriate continuity axiom to the weak order axiom.

Definition 1 *Let* $L_1 = (X_1, p_1; \ldots; X_n, p_n)$ *and* $L_2 = (Y_1, q_1; \ldots; Y_\ell, q_\ell)$ *be two compound lotteries.* L_1 *(strictly) dominates* L_2 *by two-stage stochastic dominance if and only if for every* $V : \mathcal{L}_1 \to \mathbb{R}$ *which is strictly increasing with respect to first-order stochastic dominance,* $\sum_{i=1}^{n} p_i V(X_i) \, (>) \geq \sum_{i=1}^{\ell} q_i V(Y_i)$.

The following axiom requires monotonicity of preferences over compound lotteries with respect to two-stage stochastic dominance.

Axiom 4 *(Two-stage Stochastic Dominance) If the compound lottery* L_1 *(strictly) dominates the compound lottery* L_2 *by two-stage stochastic dominance then* $L_1 \, (\succ) \succeq L_2$.

The distinction between two-stage and one-stage lotteries disappears if we assume that DMs care only about the ultimate probabilities of getting various prizes. The following axiom (Segal, [24]) formalizes this:

Axiom 5 *(Reduction of Compound Lotteries) Let* $X_i = \left(x_1^i, q_1^i; \ldots; x_{m_i}^i, q_{m_i}^i\right)$ *for* $i = 1, \ldots, n$ *and* $L = (X_1, p_1; \ldots; X_n, p_n)$. *Define*

$$R(L) = \left(x_1^1, p_1 q_1^1; \ldots; x_{m_1}^1, p_1 q_{m_1}^1; \ldots; x_1^n, p_n q_1^n; \ldots; x_{m_n}^n, p_n q_{m_n}^n\right).$$

Then $L \sim \delta_{R(L)} = (R(L), 1)$.

If the DM satisfies the Reduction of Compound Lotteries Axiom (ROCL), then she multiplies probabilities of final outcomes according to the laws of probability and is indifferent between any compound lottery and the resulting one stage lottery.[2] Note that ROCL is a property of preferences over compound lotteries, and by itself does not restrict preferences over Savage acts. Experimentally, however, we know that there is a strong connection between ROCL and ambiguity neutrality – a property of preferences over acts. We pursue this connection in Section 5.

[2] Strictly speaking, preferences are defined only on compound lotteries. There are two distinct subsets of compound lotteries which naturally correspond to one-stage lotteries, $\Delta, \Gamma \subset \mathcal{L}_2$. It does not matter which one of these sets is used in stating ROCL, since $L \sim \delta_{R(L)}$ for all $L \Leftrightarrow L \sim \gamma_{R(L)}$ for all L.

4 Preferences over Acts: Two-Stage Probabilistic Sophistication

In Savage's theory of Subjective Expected Utility [22] the evaluation of an act can be decomposed into two components. The DM holds beliefs over the state space and converts any act into a lottery over outcomes using this belief. This lottery is then evaluated according to expected utility. Machina and Schmeidler [16, 17] generalized Savage's theory by characterizing preferences over acts where the first component is retained (that is, the DM converts acts to lotteries using a consistent belief), but does not necessarily evaluate the resulting lotteries according to expected utility. These preferences, referred to as probabilistically sophisticated, achieve separation of beliefs from utilities without imposing the additional requirements of expected utility. Next, we present the formal definition of probabilistic sophistication.

Definition 2 *Let $\nu \in \Delta(\Omega)$. Denote by $\Phi_{\nu,f}$ the lottery induced by the act $f = (x_1, S_1; ...; X_m, S_m)$ through the measure $\nu : \Phi_{\nu,f} = (x_1, \nu(S_1); ...; x_m, \nu(S_m))$. $\succsim$ on acts is probabilistically sophisticated if there exists a probability measure $\nu \in \Delta(\Omega)$ and a functional $\widetilde{V} : \Delta(\mathcal{X}) \to \mathbb{R}$ increasing with respect to first-order stochastic dominance such that*

$$f \succsim g \Leftrightarrow \widetilde{V}(\Phi_{\nu,f}) \geq \widetilde{V}(\Phi_{\nu,g})$$

where $\Psi_{\nu,f}, \Psi_{\nu,g} \in \Delta(\mathcal{X})$ are probability distributions induced from f and g through the measure ν.

Probabilistically sophisticated preferences defined above rule out aversion to uncertainty (or ambiguity) as observed in Ellsberg type experiments[3]. Another perspective on probabilistic sophistication is that it reduces uncertainty to one-stage risk, ruling out ambiguity aversion (Epstein, [7]). Note that Machina and Schmeidler [16] do not include in their domain objective lotteries. Hence, it is conceivable that the decision maker uses $\widetilde{V}(\cdot)$ to evaluate the lottery induced by an act, but a different $V(\cdot)$ to evaluate objective

[3] Monotonicity with respect to first-order stochastic dominance is required for the typical behavior in Ellsberg to contradict the existence of probabilistically sophisticated belief. There exist definitions of probabilistic sophisticated preferences that do not require this monotonicity (Grant [10], Chew and Sagi [4]), but since our interest stems from studying ambiguity averse decision makers, and monotonicity is a normatively mild assumption, we retain it throughout.

lotteries. This possible distinction disappears in Machina and Schmeidler's later work [17], which derives probabilistic sophisticated preferences in an Anscombe-Aumann domain that includes objective lotteries, as $\widetilde{V}(\cdot)$ coincides with $V(\cdot)$.[4]

A more general class of preferences that reduce uncertainty to compound risk can accommodate non-neutral attitudes towards ambiguity in a natural way.

Definition 3 *Let $\mu \in \Delta(\Delta(\Omega))$, that is - there are $\nu_1, ..., \nu_n \in \Delta(\Omega)$ such that $\mu(\nu_j) > 0$ and $\sum_{j=1}^{n} \mu(\nu_j) = 1$. Denote by $\Psi_{\mu,f}$ the compound lottery induced by the act $f = (x_1, S_1; ...; X_m, S_m)$ through $\mu : \Psi_{\mu,f} = (X_1, \mu(\nu_1); ...; X_n; \mu(\nu_n))$ where $X_j = (x_1, \nu_j(S_1); ...; x_m, \nu_j(S_m))$. $\succsim$ on acts satisfies Two-Stage Probabilistic Sophistication (2SPS) if there exists a probability measure $\mu \in \Delta(\Delta(\Omega))$ and a functional $\widetilde{W} : \Delta(\Delta(\mathcal{X})) \to \mathbb{R}$ increasing with respect to two-stage stochastic dominance such that*

$$f \succsim g \Leftrightarrow \widetilde{W}(\Psi_{\mu,f}) \geq \widetilde{W}(\Psi_{\mu,g})$$

$\Psi_{\mu,f}, \Psi_{\mu,g} \in \Delta(\Delta(\mathcal{X}))$ are compound lotteries induced from the acts f and g through μ.

A decision maker's preferences satisfy 2SPS if every Savage-act induces a compound lottery through μ, such that the ranking of acts corresponds to the ranking of the compound lotteries induced by those acts using a utility function $\widetilde{W}(\cdot)$. Furthermore, although formally not part of our definition of 2SPS, in actual choice problems a modeler would consider only the first-stage lotteries that are consistent with the objective description of the state space.

If preferences over acts are represented by either the Recursive Expected Utility (REU) or Recursive Non-Expected Utility (RNEU) models then they are 2SPS. To clarify this point consider first REU.[5] In this model

$$\widetilde{W}(\Psi_{\mu,f}) = E_\mu \phi\left(E_{\nu_j}\left(u^2(f)\right)\right) = \sum_{j=1}^{n} u^1\left(CE\left(\sum_{i=1}^{m} u^2(f(s_i))\,\nu_j(s_i)\right)\right)\mu(\nu_j) \qquad (1)$$

[4] It is our impression that even in their 1992 paper, Machina and Schmeidler [16] intended that the utility function used to evaluate the lottery induced by an act coincides with the utility function used to evaluate objective lotteries. Otherwise, they would not have motivated their representation as capable to accommodate Allais type behavior (which is in the domain of objective lotteries).

[5] The REU functional form has been axiomatized by Klibanoff *et al* [14], Ergin and Gul [8], Nau [18] and Seo [25]. In each case, the domain of preferences includes simple acts but includes other choice objects (that varies across different representations.)

where $u^r : \mathcal{C} \to \mathbb{R}$ $(r = 1, 2)$ is increasing with respect to preferences over outcomes, $\phi = u^1 \circ (u^2)^{-1}$ and CE denotes the certainty equivalent with respect to u^2.

In the Recursive Non-Expected Utility model (Segal [23]) $\widetilde{W}(\Psi_{\mu,f})$ can be written as follows. Let $\Upsilon_{\mu,f}$ be a lottery that gives $CE\left(V\left(\Phi_{\nu_j,f}\right)\right)$ with probability $\mu(\nu_j)$ where CE denotes the certainty equivalent with respect to V and $V : \mathcal{L}_1 \to \mathbb{R}$ is increasing with respect to first-order stochastic dominance. Then,

$$\widetilde{W}(\Psi_{\mu,f}) = V(\Upsilon_{\mu,f}) \tag{2}$$

Preferences that can be represented by (1) and (2) are both increasing with respect to two-stage stochastic dominance and thus they are both 2SPS.[6]

To describe the Maxmin Expected Utility (MEU) model (Gilboa and Schmeidler [9], Casadesus-Masanell, Klibanoff and Ozdenoren [3]) consider

$$\widetilde{W}(\Psi_{\mu,f}) = \min_{\nu \in co(\mathrm{supp}(\mu))} \sum_{i=1}^{m} u(f(s_i))\,\nu(s_i) \tag{3}$$

However, this representation is not strictly increasing with respect to two-stage stochastic dominance. This is because if we improve (in the sense of first order stochastic dominance) a second stage lottery that does not receive the minimum expected utility, then the value of $\widetilde{W}$ does not change.

As in the definition of Probabilistic Sophistication, one can conceivably imagine that the decision maker uses $\widetilde{W}(\cdot)$ to evaluate the compound lottery induced by the acts, while using a different $W(\cdot)$ to evaluate compound lotteries. Indeed, this is the interpretation taken by previous work that maintained the traditional distinction between risk and uncertainty, even when applied to reduction of compound lotteries. For example, Klibanoff *et al*'s [14] preferred interpretation is that reduction of compound lotteries holds for $W(\cdot)$, but fails for $\widetilde{W}(\cdot)$. Our axioms below, as in Machina and Schmeidler [17], impose restrictions on the relations between the two functions.

5 Calibration

The following two axioms establish the essential connection between preferences over compound objective lotteries and acts, which is the focus of this

[6] If $u^1 \equiv u^2$ then REU reduces to SEU. If V is linear in probabilities then RNEU reduces to SEU.

study. The relation over these two domains was established experimentally in Halevy [11], and the current paper provides its theoretical foundation. The connection between the two domains of preferences, acts and compound lotteries is made through two calibration axioms. Axiom 6 (*Calibration*) makes the connection between subjective bets and compound lotteries where the second stage consists of objective bets. Axiom 7 (*First-Stage Probabilistic Beliefs*) extends the calibration to arbitrary acts.

Axiom 6 *(Calibration) There exists a strictly positive probability measure* $p = (p_1, \ldots, p_n)$ *such that for all disjoint events* S *and* S'*, there exist compound lotteries* $(X_1, p_1; \ldots; X_n, p_n)$ *and* $(X_1', p_1; \ldots; X_n', p_n)$ *with*

$$(x^*, S; x_*, S^c) \sim (X_1, p_1; \ldots; X_n, p_n),$$

$$(x^*, S'; x_*, S'^c) \sim (X_1', p_1; \ldots; X_n', p_n)$$

where $X_j = (x^*, q_j; x_*, (1 - q_j))$ *and* $X_j' = \left(x^*, q_j'; x_*, \left(1 - q_j'\right)\right)$ *for* $j \in \{1, ..., n\}$ *with* $q_j + q_j' \leq 1$ *and*

$$\left(x^*, S \cup S'; x_*, (S \cup S')^c\right) \sim (X_1'', p_1; \ldots; X_n'', p_n)$$

where $X_j'' = \left(x^*, q_j + q_j'; x_*, \left(1 - q_j - q_j'\right)\right)$ *for* $j \in \{1, ..., n\}$.

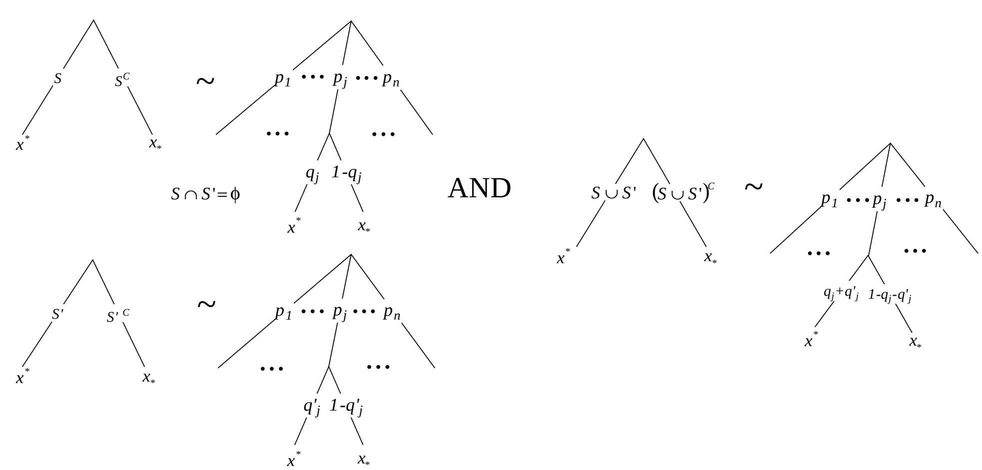

Figure 1: The Calibration Axiom

Figure 1 illustrates the Calibration Axiom that incorporates two requirements. First, the axiom guarantees the existence of a *fixed* first-stage probability measure such that the DM is indifferent between any subjective bet and some compound lottery over objective bets with these fixed first-stage probabilities. Second, the axiom requires that probabilities at the second-stage are additive. That is, the DM is indifferent between a bet on the union of two disjoint events and a compound lottery that additively aggregates the winning probabilities of the corresponding compound lotteries at the second-stage.

Axiom 7 *(First-Stage Probabilistic Beliefs) Suppose that $S_1, ..., S_m$ is a partition of Ω such that*

$$\left(x^*, S_j; x_*, S_j^c\right) \sim \left(X_j^1, p_1; \ldots; X_j^n, p_n\right)$$

where $X_j^k = \left(x^*, q_j^k; x_*, \left(1 - q_j^k\right)\right)$ for $j = 1, ..., m$ and $k = 1, ..., n$ with $\sum_{j=1}^m q_j^k \leq 1$. Then

$$(x_1, S_1; ...; x_m, S_m) \sim (X_1, p_1; \ldots; X_n, p_n)$$

where $X_k = \left(x_1, q_1^k; \ldots; x_m, q_m^k\right)$.

Figure 2 illustrates the First-Stage Probabilistic Beliefs Axiom. Fix a partition of the state space. Suppose that for each element of this partition, the decision maker is indifferent between a bet on this element and a compound lottery, where the first-stage is constant across elements in this partition, and the second stage are bets. It is intuitive to think of the second-stage probabilities of winning the bets as possible probability assessments of the corresponding event. Indeed, the axiom requires that the DM's preferences are consistent with respect to these probability assessments. Formally, consider a general act, that gives a different outcome on each event in the partition. Consider also a compound lottery, where the first-stage is identical to the first-stage above and the second-stage lotteries are obtained by assigning to each outcome the corresponding probability assessment of the event on which this outcome is attained. The axiom requires that the DM is indifferent between the act and this induced compound lottery.

Theorem 1 shows that Calibration and First-Stage Probabilistic Beliefs (together with the rest of the axioms) imply 2SPS, which captures heterogenous ambiguity attitudes. That is, if the decision maker's preferences satisfy

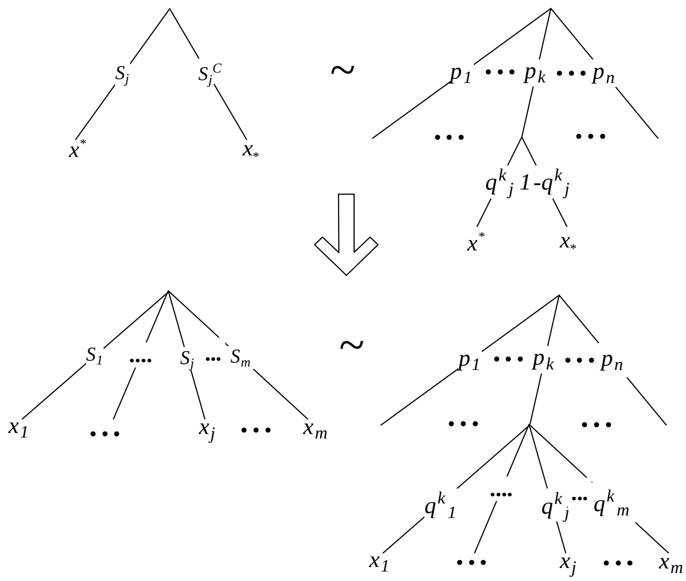

Figure 2: First-Stage Probabilistic Beliefs

our calibration axioms then for every Savage-act, there exists an indifferent compound objective lottery, such that its first-stage represents the DM's assessments of possible probability distributions over the state space (and therefore is constant across acts). Therefore, the DM is 2SPS. Moreover, the Theorem establishes theoretically the empirical link between ROCL and probabilistic sophistication, independently of a specific functional representation of preferences.

Theorem 1 *Suppose $\succsim$ satisfies 1, 2, 3, 4, 6 and 7.*

1. *Preference over acts satisfies two-stage probabilistic sophistication and $\widetilde{W}(\cdot) \equiv W(\cdot)$.*
2. *Reduction of Compound Lotteries (Axiom 5) implies probabilistic sophistication. In other words, if preferences over acts are not probabilistically sophisticated then reduction fails.*

Proof. Fix an event S in Σ. By the Calibration Axiom (Axiom 6) there exist two-stage lotteries

$$(X_1, p_1; \ldots; X_n, p_n) \text{ and } (X'_1, p_1; \ldots; X'_n, p_n)$$

where $X_j = (x^*, q_j; x_*, 1 - q_j)$ and $X'_j = \left(q'_j, x^*; x_*, 1 - q'_j\right)$ for $j \in \{1, ..., n\}$ satisfying

$$(x^*, S; x_*, S^c) \sim (X_1, p_1; \ldots; X_n, p_n),$$
$$(x^*, S^c; x_*, S) \sim (X'_1, p_1; \ldots; X'_n, p_n)$$

and

$$(x^*, \Omega) \sim (X''_1, p_1; \ldots; X''_n, p_n)$$

where $X''_j = \left(x^*, q_j + q'_j; x_*, 1 - \left(q_j + q'_j\right)\right)$ for $j \in \{1, ..., n\}$. This implies that either $q_j + q'_j = 1$ or $p_j = 0$. To see this, suppose that $p_j > 0$ and $q_j + q'_j < 1$ for some j. Then by Degenerate Lottery/Act Equivalence Axiom, $(x^*, \Omega) \sim (X, 1)$ where $X = (x^*, 1)$. Note that by Two-Stage Stochastic Dominance we know that $(X, 1) \succ (X''_1, p_1; \ldots; X''_n, p_n) \sim (x^*, \Omega)$ which is a contradiction. Moreover, by the Calibration Axiom $p_j > 0$, so it must be that $q_j + q'_j = 1$. Now let $P_j(S) = q_j$. If S and S' are disjoint events then by the Calibration Axiom (Axiom 6), $P_j(S \cup S') = P_j(S) + P_j(S')$ which proves that P_j is additive.
Define the probability measure μ in $\Delta(\Delta(\Omega))$ by $\mu(P_j) = p_j$.
Fix an act $f = (x_1, S_1; ...; x_m; S_m)$. Use axiom 6 to inductively construct

$$(x^*, S_1 \cup \cdots \cup S_{i-1}; x_*, (S_1 \cup \cdots \cup S_{i-1})^c) \sim \left(X^*_{1,i-1}, p_1; \ldots; X^*_{n,i-1}, p_n\right)$$

where

$$X^*_{j,i-1} = \left(x^*, q^*_{j,i-1}; x_*, 1 - q^*_{j,i-1}\right),$$
$$(x^*, S_i; x_*, S_i^c) \sim (X_{1,i}, p_1; \ldots; X_{n,i}, p_n)$$

where

$$X_{ji} = (x^*, P_j(S_i); x_*, 1 - P_j(S_i))$$

and

$$(x^*, S_1 \cup \cdots \cup S_i; x_*, (S_1 \cup \cdots \cup S_i)^c) \sim (X''_{1i}, p_1; \ldots; X''_{ni}, p_n)$$

where

$$X''_{ji} = \left(x^*, q^*_{j,i-1} + P_j(S_i); x_*, 1 - \left(q^*_{j,i-1} + P_j(S_i)\right)\right)$$

and $q^*_{j,i} = q^*_{j,i-1} + P_j(S_i)$. Note that

$$\begin{aligned}
& P_j(S_1 \cup \cdots \cup S_{i-1}) \\
= \; & P_j(S_1 \cup \cdots \cup S_{i-2}) + P_j(S_{i-1}) \\
= \; & q^*_{j,i-2} + P_j(S_{i-1}) \\
& \vdots \\
= \; & \sum_{k=1}^{i-1} P_j(S_k).
\end{aligned}$$

So $P_j(S_1 \cup \cdots \cup S_m) = \sum_{k=1}^{m} P_j(S_k) \leq 1$ for $j = 1, ..., n$. Now let $X_j = (x_1, P_j(S_1); ...; x_m, P_j(S_m))$, and let $L_1 = (X_1, p_1; ..; X_n, p_n)$. By Axiom 7, $(x_1, S_1; ...; x_m, S_m) \sim L$. But note L_1 is the two stage lottery induced by μ and f,i.e., $L_1 = \Psi_{\mu,f}$. Repeat the construction for an arbitrary act g, and similarly construct the two-stage lottery induced by μ and g - $\Psi_{\mu,g}$. Therefore $f \succeq g$ if and only if $\widetilde{W}(\Psi_{\mu,f}) = W(\Psi_{\mu,f}) \geq W(\Psi_{\mu,g}) = \widetilde{W}(\Psi_{\mu,g})$. Finally, $\widetilde{W}$ is increasing with respect to second-order stochastic dominance follows directly from Axiom 4, which proves part (1).

Now define the probability measure $m \in \Delta(\Omega)$ so that $m(S) = \sum_{j=1}^{n} p_j P_j(S)$. If Reduction holds then $\Psi_{\mu,f} \sim R(\Psi_{\mu,f})$. But note that $R(\Psi_{\mu,f})$ is the probability measure induced from f using m, i.e., $R(\Psi_{\mu,f}) = \Phi_{m,f}$. For any lottery X define $V(X) = W(\delta_X)$. Since reduction holds, $V(\Phi_{m,f}) = W(\delta_{\Phi_{m,f}}) = W(\Psi_{\mu,f})$. Therefore $f \succeq g$ if and only if $V(\Phi_{m,f}) \geq V(\Phi_{m,g})$. Moreover, by Theorem 5 in Segal [24] V must be increasing with respect to first-order stochastic dominance, proving part (2). ■

In Appendix A we demonstrate the usage of this framework and axioms. We start from a REU representation of preferences over compound lotteries (one may be obtained from Kreps and Porteus [15] or Segal [24]). Using the calibration and second order probabilistic belief axioms we show that preferences over Savage-acts can be represented by the recursive structure suggested recently [14, 8, 18, 25]. The advantage of this representation is that it explicitly builds on the relation between preferences over compound objective lotteries and Savage-acts and is based on behavioral testable axioms.

The second part of the Theorem establishes the theoretical link between reduction and ambiguity neutrality, independently of a specific functional form. In Anscombe and Aumann framework, Seo [25] (in an independent and concurrent study) was able to prove this link for the REU representation.

However, Halevy's findings [11] indicate that the relation between ambiguity neutrality and ROCL holds beyond the REU preferences, which account for only about one-half of the decision makers that hold a non-neutral attitude towards ambiguity.

5.1 Discussion: Relation with Empirical Findings

Theorem 1 explains two regularities observed in Halevy [11]: different decision makers may hold different first-stage beliefs (heterogeneous μ), and even for identical first-stage beliefs, the evaluation of the compound lottery induced by the act through the first-stage belief may vary (heterogeneous $\widetilde{W}(\cdot)$).

In order to demonstrate this point consider a DM who evaluates a bet on the color of a ball drawn from an urn, containing ten balls with unknown composition of red and black balls. A bet on the correct color wins $\$x$ and an incorrect bet wins $\$0. The natural second-stage is given by: $\{(\$x, k/10; \$x, 1-k/10) : k \in \{0, \ldots, 10\}\}$.

If the DM's preferences satisfy reduction of compound (objective) lotteries, then she will be indifferent among all compound objective lotteries with a symmetric first-stage measure, and the above second-stage. The second part of Theorem 1 implies that if the DM holds any symmetric first-stage subjective belief, she will be indifferent between the ambiguous Savage act described above and these symmetric compound objective lotteries (and in particular, a first-stage objective lottery that is degenerate at the second-stage lottery of $(\$x, 0.5; \$0, 0.5)$, which corresponds to a bet on the color of a ball drawn from an urn containing 5 red and 5 black balls). This holds for *any* utility function used to evaluate compound lotteries, as long as it satisfies ROCL. This is exactly Halevy's [11] first experimental result. Therefore the second part of Theorem 1 provides the theoretical foundation that is necessary to understand this observation.

If, however, the DM is not ambiguity neutral the experimental results in Halevy [11] indicate that she doesn't satisfy reduction of compound objective lotteries, and her preference ordering over Savage-acts is consistent with her preferences over compound lotteries. In other words, the DM does not make a distinction between the compound lottery induced by the act through a subjective first-stage belief, and a compound lottery in which the first-stage (appropriately calibrated) is objectively given. The first-stage probability measure in which each DM is indifferent between the compound lottery and the Savage act, varies across decision makers. In other words, the first-stage belief varies across decision makers. For example, while one DM may be indifferent between the ambiguous Savage lottery and a compound lottery where the first-stage is uniform, another may be indifferent between the same act and a compound symmetric lottery with hypergeometric first-stage (10 balls are sampled without replacement out of 20 balls, half of which are red and half black). As a result, even if two decision makers have identical preferences over compound objective lotteries they might differ in their evaluation of the ambiguous (Savage) act since it induces different compound lotteries, through the different first-stage beliefs.

Moreover, the preference ordering over compound *objective* lotteries varies across decision makers. In particular, Halevy [11] found that two utility functions can represent most decision makers' preferences (with about equal frequency). If the DM holds recursive non-expected utility (RNEU) preferences, she uses a non-expected utility function to evaluate each stage of the compound lottery (like in Segal [23]). As a result, she is indifferent between a compound lottery with a degenerate first-stage (that assigns probability 1 to a second-stage lottery that pays $\$x$ with probability 0.5) and a compound lottery with an extreme first-stage (that assigns probability 0.5 to a second-stage lottery that pays $\$x$ with probability 1 and probability 0.5 to a second-stage lottery that pays $\$x$ with probability 0) – preferences termed by Segal [24] "time neutral." If, however, the DM's preferences are represented by REU, her ranking of compound objective lotteries is monotonically decreasing in the dispersion of the first-stage probability measure.

The experimental evidence [11] indicates that (for most DMs) the ranking of the Savage act is consistent with a ranking of the compound lottery induced by the act through the first-stage belief according to RNEU or REU, which were derived in the domain of compound objective lotteries. That is, the evidence is consistent with 2SPS preferences and the fact that the function $\widetilde{W}(\cdot)$ used to evaluate the compound lottery induced by an act through the first-stage belief coincides with the function $W(\cdot)$, used to evaluate compound objective lotteries.

Although the REU representation was recently applied to decision making under ambiguity (by Klibanoff *et al* [14], Ergin and Gul [8], Nau [18]), all those applications assume that the source for the failure of reduction is the distinction between subjective (first-stage) and objective (second-stage) risks. Although formally speaking they are 2SPS according to our definition, their domain does not include compound objective lotteries. Hence, the question whether the utility function they derive, which is used to evaluate the compound lottery induced by the act through the first-stage belief, coincides with a utility function used to evaluate compound objective lotteries is a matter of interpretation that will be further discussed in Section 7.1. As noted above, the experimental finding in Halevy [11] demonstrate that there is a tight correspondence between the two. That is, the DM's ranking of a Savage act is consistent with her ranking of compound objective lotteries. As noted above, Seo [25] provides an axiomatic foundation for the REU model based on the original Anscombe-Aumann framework, in which ambiguity aversion is tied to violation of the reduction axiom in that model. Our result (Theorem 1) explains the relation between preferences over objective compound lotteries and Savage acts in a minimal domain in which both exist, without relying on a specific functional form and without excluding common preference ranking (either REU or RNEU).

6 Calibration at Work

An important implicit component in the Calibration axioms and the definition of 2SPS is that the set of second-stage probability measures is exogenous and objectively known. This set is apparent from the problem at hand. We believe that this discipline is necessary in order for the agent to be rational. Otherwise, a DM might assign positive first-stage probability to a second-stage measure that is known to be false. Under this maintained assumption it is easy to identify the first-stage probabilities directly using continuity of preferences. To illustrate this point, define continuity:

Axiom 8 *For any act f and simple lotteries $X_1, ..., X_n$, the sets* $\{(p_1, ..., p_n) \,|\, f \succsim (X_1, p_1; \ldots; X_n, p_n)\}$ *and* $\{(p_1, ..., p_n) \,|\, (X_1, p_1; \ldots; X_n, p_n) \succsim f\}$ *are both closed.*

We illustrate how to identify the first-stage probabilities with a simple example in which for each distribution there exists an event such that betting on that event is essentially betting on the distribution. Specifically, consider an urn that has two balls that are each either red or black. Suppose that the decision maker draws both balls in order from this urn and considers various bets on the color of the two balls. A natural state space for this problem is $\{BB, BR, RB, RR\}$ where IJ ($I, J \in \{B, R\}$) denotes a state in which the first ball is of color I and the second ball is of color J. The possible second-stage probabilities are also apparent from the description since either both balls are black, or both balls are red, or one ball is black and the other ball is red. These three possibilities correspond to three possible probability distributions. Denote by P_{2B} the probability measure that assigns probability one to the state BB, by P_{1B1R} the probability measure that assigns probability 1 to the event $\{BR, RB\}$, and by P_{2R} the probability measure that assigns probability 1 to the state RR. Let p_{2B}, p_{1B1R} and p_{2R} be the first-stage probabilities on the corresponding probability measures. Consider first the bet where the decision maker gets $x > 0$ dollars if both balls are red and zero otherwise. This bet corresponds to the $f(RR) = x$ and $f(BR) = f(RB) = f(BB) = 0$. By the continuity axiom there exists a compound lottery $(X_1^1, q; X_2^1, 1-q)$ where $X_1^1 = (x, 1)$ and $X_2^1 = (0, 1)$ such that $f \sim (X_1^1, q; X_2^1, 1-q)$. Thus we have identified that $p_{2R} = q$. We can identify p_{1B1R} and p_{2B} in a similar way.

7 Beyond 2SPS: A Two Urn Example

In this section we provide an example of a preference relation that is not two-stage probabilistically sophisticated. Our example has the flavor of Ellsberg's two color urn example. The first urn (Urn 1) has two balls that are each either red or black. Thus this urn may contain two black balls, two red balls, or one black and one red ball. There is no further information on how the urn is filled. The other urn (Urn 2) is filled in the following way: two balls are put in the urn where each ball is equally likely to be black or red. That is, the urn contains two black or two red balls - each with probability 1/4, and one black and one red ball with probability 1/2.

Two balls are drawn from both urns in order without replacement. We will consider several bets on the color of these balls. The state space for this problem can be written as:

$$\Omega = \{(\omega_1, \omega_2) \,|\omega_1, \omega_2 \in \{BB, BR, RB, RR\}\}$$

where ω_i specifies the colors of the balls from Urn i in the order they are drawn. For example, (BB, BR) corresponds to the state where both balls drawn from Urn 1 are black and the first ball drawn from Urn 2 is black and the second is red.

In the following, we will discuss several acts (bets), and a possible preference ordering over these acts. If this preference is 2SPS then each act corresponds to a compound lottery. As discussed above, in order to identify the corresponding compound lottery, we consider only second-stage measures that are consistent with the possible composition of the urns. The following table describes all possible compositions of the two urns and the corresponding second-stage measures over the states.

Urns Contain	Probabilities of States (with positive probability)
$(2B, 2B)$	$P_1((BB, BB)) = 1$
$(2B, 1B1R)$	$P_2((BB, BR)) = 1/2, P_2((BB, RB)) = 1/2$
$(2B, 2R)$	$P_3((BB, RR)) = 1$
$(1B1R, 2B)$	$P_4((BR, BB)) = 1/2, P_2((RB, BB)) = 1/2$
$(1B1R, 1B1R)$	$P_5((BR, BR)) = 1/4, P_5((BR, RB)) = 1/4,$ $P_5((RB, BR)) = 1/4, P_5((RB, RB)) = 1/4$
$(1B1R, 2R)$	$P_6((BR, RR)) = 1/2, P_6((RB, RR)) = 1/2$
$(2R, 2B)$	$P_7((RR, BB)) = 1$
$(2R, 1B1R)$	$P_8((RR, BR)) = 1/2, P_8((RR, RB)) = 1/2$
$(2R, 2R)$	$P_9((RR, RR)) = 1$

A two-stage probabilistically sophisticated DM assigns a probability to each of the distributions $P_1, ..., P_9$. Denote the probability assigned to distribution P_i by p_i. Let $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ be defined by

$$\begin{aligned} \alpha &= p_1 + p_2 + p_3,\ \beta = p_4 + p_5 + p_6,\ \gamma = p_7 + p_8 + p_9, \\ \alpha' &= p_1 + p_4 + p_7,\ \beta' = p_2 + p_5 + p_8,\ \gamma' = p_3 + p_6 + p_9. \end{aligned}$$

Next we consider the following four pairs of bets.

Betting on the color of the first ball drawn from Urn 1 : The DM chooses black or red. If the DM matches the color of the first ball drawn from Urn 1, she wins x dollars where $x > 0$. If the bet is incorrect, the DM receives zero. Bets on either color can be represented as a mapping from states to $\{0, x\}$ as shown below:

Bet on black from Urn 1

Urn 1\Urn 2	BB	BR	RB	RR
BB	x	x	x	x
BR	x	x	x	x
RB	0	0	0	0
RR	0	0	0	0

Bet on red from Urn 1

Urn 1\Urn 2	BB	BR	RB	RR
BB	0	0	0	0
BR	0	0	0	0
RB	x	x	x	x
RR	x	x	x	x

A second order probabilistically sophisticated DM converts these two acts into the compound lotteries as shown in Figures 3 and 4.

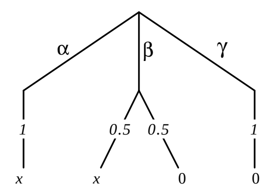

Figure 3: The compound lottery induced by a bet that the first ball drawn from Urn 1 is Black

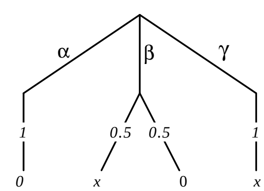

Figure 4: The compound lottery induced by a bet that the first ball drawn from Urn 1 is Red.

Suppose that the DM is indifferent between betting on red and betting on black. Two stage stochastic dominance implies that $\alpha = \gamma$.

Betting whether the balls drawn from Urn 1 **are the same or different color:** These bets can be represented as the following two acts:

Bet on the two balls from Urn 1 being of the same color

Urn 1\Urn 2	BB	BR	RB	RR
BB	x	x	x	x
BR	0	0	0	0
RB	0	0	0	0
RR	x	x	x	x

Bet on the two balls from Urn 1 being of different color

Urn 1\Urn 2	BB	BR	RB	RR
BB	0	0	0	0
BR	x	x	x	x
RB	x	x	x	x
RR	0	0	0	0

A second order probabilistically sophisticated DM converts these two acts into the compound lotteries shown in Figures 5 and 6. Suppose that the DM

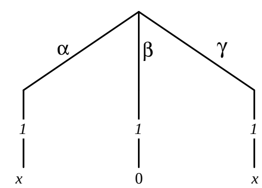

Figure 5: The compound lottery induced by a bet that both balls drawn from Urn 1 have the same color

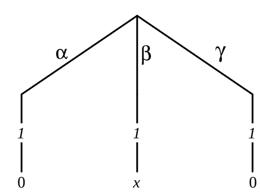

Figure 6: The compound lottery induced by a bet that the balls drawn from Urn 1 have the different color

is indifferent between these bets as well.[7] Two stage stochastic dominance

[7]It is important to note that evolutionary (Robson [19]) and behavioral (Halevy and Feltkamp [12]) rationals for ambiguity aversion rely on the species' or the decision maker's regular environment to incorporate positive correlation among different risks. This reasoning may suggest that the DM will prefer a bet that the two balls are of the same color. In other words, indifference between the bets above indicates that the DM believes that the color assignment of the first ball is uncorrelated with the color assignment of the second ball. With this in mind, we view this indifference as an empirical question, that is worth further experimental investigation. Our goal in this example, however, is to provide a sequence of plausible decisions that is inconsistent with 2SPS.

implies that $\beta = \alpha + \gamma$. Thus we must have $\alpha = \gamma = 1/4$ and $\beta = 1/2$.

Suppose also that the DM exhibits similar preferences for bets generated from Urn 2. That is, the DM is indifferent between betting on whether the first ball drawn from Urn 2 is red or black, and whether the two balls drawn from Urn 2 are of the same or different color. Similar reasoning implies that we must have $\alpha' = \gamma' = 1/4$ and $\beta' = 1/2$.

Finally, suppose that the DM is asked to choose an urn and bet on the color of the first ball drawn from the chosen urn. Suppose that the DM strictly prefers to bet on either color from the known urn (Urn 2) to betting on either color from the unknown urn (Urn 1). If her preferences were 2SPS, a bet on the color of the first ball drawn from either urn would induce the same compound lottery and she would be indifferent among all such bets. Therefore, if the hypothetical preferences described above display strict preference for a bet on the known urn to a bet on the unknown urn, they are inconsistent with 2SPS.

Next, we discuss two possible interpretations of this example.

7.1 Discussion

It is useful to compare this example to the standard two-color Ellsberg example. In that case the DM is indifferent between betting on red or black in the ambiguous urn. Probabilistic sophistication implies that the DM's subjective probability of either color is one half. However (goes the standard argument), preference for betting on the risky rather than the ambiguous urn (together with monotonicity with respect to first order stochastic dominance) implies that this belief must be smaller than one half. Hence, preferences in the Ellsberg example cannot be probabilistically sophisticated. A way to accommodate failure of probabilistic sophistication is to consider more general preferences, like those that are two-stage probabilistically sophisticated. This generalization seem to be consistent with the existing experimental evidence. Alternatively, several authors (Tversky and Wakker [27], Chew and Sagi [5] and Ergin and Gul [8]) argue that the failure of probabilistic sophistication is due to the fact that the DM perceives the two urns as two different sources of uncertainty: objective and subjective. If attention is restricted to only one source, probabilistic sophistication may be maintained on the restricted domain. For example, the DM uses $V(\cdot)$ to evaluate one-stage objective lotteries, and $\widetilde{V}(\cdot)$ to evaluate the lottery induced by the act through her belief. Machina and Schmeidler [16] are silent on this issue, although in their later work [17] the utility functions over the objective and subjective domains coincide, which rules out source preference.

The thought experiment presented above and the failure of 2SPS admits two similar interpretations. To accommodate strict preference for betting on Urn 2 one could consider higher orders of compound lotteries (e.g. three-stage), in which the DM may be three-stage probabilistically sophisticated. The need to allow for higher orders of compound lotteries suggests that this avenue of considering higher order probabilistic sophistication might not be of a closed form. In this case, the level of probabilistic sophistication may be determined endogenously by the model considered. Alternatively, the DM may perceive the two urns as different sources of compound uncertainty: subjective and objective. The DM may be 2SPS on each domain separately, but use different utility functions to evaluate compound objective lotteries and compound lotteries induced by an act through the first-stage belief (i.e. $W(\cdot)$ may be different from $\widetilde{W}(\cdot)$). For example, the DM's preferences on the subjective domain may be represented by REU, and she may be expected utility (and satisfy reduction) on the objective domain (which is Klibanoff *et al's* [14] preferred interpretation of their model). However, if one is willing to admit source preference as discussed above, Ellsberg-type preference can be accommodated without resorting to higher order beliefs. Furthermore, the empirical evidence [11] suggests that most DMs have preferences over acts that can be calibrated to their preferences over compound objective lotteries as presented in this study.

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A Recursive Expected Utility Representation

In this appendix we provide an axiomatic foundation for the recursive expected utility model. Fix a continuous function $u : \mathcal{C} \to \mathbb{R}$ and a strictly increasing continuous function $\rho : \mathbb{R} \to \mathbb{R}$ and $p_j \geq 0$ for $j = 1, ..., n$ with $\sum_{j=1}^{n} p_j = 1$. We define $V_{KP}(L) \equiv \sum_{j=1}^{n} \rho(E(u(X_j))) p_j$ for all $L = (X_1, p_1; \ldots; X_n, p_n)$.

Axiom 9 *(Kreps-Porteus)There exists a continuous function $u : \mathcal{C} \to \mathbb{R}$, a strictly increasing continuous function $\rho : \mathbb{R} \to \mathbb{R}$ and $p_j \geq 0$ for $j = 1, ..., n$ with $\sum_{j=1}^{n} p_j = 1$ such that $\succeq$ on $\mathcal{L}_2$ is represented by V_{KP}, i.e.,*

$$L_1 \succeq L_2 \text{ if and only if } V_{KP}(L_1) \geq V_{KP}(L_2).$$

This axiom can be obtained from more basic axioms for example as in Kreps and Porteus [15] or Segal [24].

Next we state the representation result:

Proposition 1 *$\succeq$ satisfies Weak Order, Calibration, Second Order Probabilistic Beliefs and Kreps-Porteus if and only there exists $p_1, ..., p_n \geq 0$ with $\sum_{i=1}^{n} p_i = 1$ and probability measures $P_1, ..., P_n : \Sigma \to [0, 1]$ such that V can be extended to the set of all acts through*

$$V_{KP}(f) = V_{KP}(\Psi_{\mu,f}) = \sum_{i=1}^{n} \rho\left(\sum_{j=1}^{m} u(x_j) P_i(S_j)\right) p_i$$

where $f = (x_1, S_1; ...; x_m, S_m)$ and the function V_{KP} represents the preference relation $\succeq$ over all gambles.

Proof. By axiom 9 there exists a continuous function $u : X \to \mathbb{R}$, a strictly increasing continuous function $\rho : \mathbb{R} \to \mathbb{R}$ and a probability measure $P : \mathcal{R} \to [0, 1]$ such that for any L^1 and L^2 in $\mathcal{L}_2$,

$$L_1 \succeq L_2 \text{ if and only if } V_{KP}(L_1) \geq V_{KP}(L_2).$$

Note that V_{KP} satisfies Two-stage Stochastic Dominance. It follows from the proof of Theorem 1 that there exists $\mu \in \Delta(\Delta(\Omega))$, and $\nu_1, ..., \nu_n \in \Delta(\Omega)$ such that $\mu(\nu_j) > 0$ and $\sum_{j=1}^{n} \mu(\nu_j) = 1$, and a functional $W : \Delta(\Delta(\mathcal{X})) \to \mathbb{R}$ increasing with respect to two-stage stochastic dominance such that

$$f \succsim g \Leftrightarrow V_{KP}(\Psi_{\mu,f}) \geq V_{KP}(\Psi_{\mu,g})$$

where $\Psi_{\mu,f}$, $\Psi_{\mu,g}$ are the compound lotteries induced by the acts f and g through μ. This concludes the proof. ■