HEDONIC EQUILIBRIUM

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ABSTRACT. This paper describes hedonic equilibrium and shows how and why the concept has to be modified when characteristics of traders on both sides of the market are endogenous.

1. INTRODUCTION

In the most elementary economic transaction, a buyer has to find a seller to 'match' with in order to complete a transaction. We don't ordinarily think much about matching in this context because the buyer in this example only cares about a commodity and its price. He or she cares not at all about which seller supplies it. In many problems this isn't the case. Though it is common to assume it away, a firm cares intensely which worker it hires, not just about the wage it has to pay. Marriage markets are another example in which the characteristics of a partner mean a lot more than any transfer that is made to facilitate a marriage. A recent and very applied literature in matching has studied other important problems. For example, the deferred acceptance algorithm is either used or has been suggested as a way to match new lawyers with law firms, medical residents with hospitals (Roth (2003)), students to public schools (Abdulkadrolu and Sonmez (2003)), and students to colleges (Gale and Shapley (1962)). A recent paper by (Roth, Sonmez and Utku Unver (2005)) uses the methods of stable matching to matching kidney donors and patients.

The typical approach is to assume that the characteristics of the various partners available are fixed exogenously, then to try to find good ways to pair them. A problem that has received less attention is what to do when the characteristics of the traders involved in the market are endogenous. This is a problem that is well known in practise. For example universities use standardized testing to select applicants for admission and decry the fact that students spend too much time

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prepping for the admission test, and not enough time building skills that they can use later.

For large matching markets, like the labour or marriage market, a useful tool for thinking about this problem is *hedonic equilibrium* (Rosen (1974) or Mas-Colell (1975)). The models in this literature typically assume that characteristics on one side of the market are fixed, while the other side adapts in response. Rosen's original paper involves a continuum of firms of different (but exogenously given) qualities and a continuum of consumers of different tastes who offer different amounts of money for these. A more recent application is the paper by Bulow and Levin (2006) that studies wages in a labour market where the characteristics of workers are fixed. This paper was a response to an anti-trust suit filed against the medical residents matching program in 2002 claiming that it suppressed wages.

For the purposes of this paper, the interest in hedonic equilibrium stems from a paper by Peters and Siow (2002) that made two contributions. The first was to show that there was no need to restrict attention to problems in which only one side of the market could tailor its characteristics to the matching process as Rosen had done, and as was common in other literature. In the marriage market that they studied both sides actively engage in investment. The second was to show that there was no need to monetize these characteristics the way they are, say, in the housing market where each characteristic of a house is assigned a monetary value. Utility in the marriage market typically isn't transferable, so the idea that a certain physical characteristic has a monetary value isn't natural.

The point of this paper is to illustrate how to adapt this idea for use with problems like the residents matching problem where there are relatively large numbers of traders on both sides of the market, but where the characteristics of both sides of the market are endogenous. The adaptation is needed because there is a flaw in the hedonic argument that simply isn't apparent with one sided investment - it can't be supported as the limit of a sequence of non-cooperative equilibrium. The off equilibrium payoffs that it uses to support overall equilibrium aren't credible, even approximately (for example (Peters 2004, Felli and Roberts 2000)). We hope to show that a suitably modified version of hedonic equilibrium can be applied to such problems.

In the transferable utility case, where every characteristic can be assigned a money value, the hedonic approach has a nice mathematical structure, illustrated by Ekeland (2003). His formalism extends to problems without money. We sketch his conceptual argument here, since it isn't hard. Consider a marriage market where men have innate characteristics drawn from some distribution X, while women have characteristics in another space Y. Depending on these characteristics, men choose acquired characteristics from a space W, their education, their personality, their appearance and so on. Women do the same, choosing from the set H. Then, they arrive at the marriage market and engage in a stable match of some kind based on the fact that payoffs to a match between a man and a woman depend on the innate and acquired types of the matched pair.

Let Z be the product of W and H. Imagine an auctioneer who, instead of announcing a price for each characteristic, simply creates a surface $\{z \in Z : \omega(z) = 0\}$. Men then choose the point $z = (h, w) \in Z$ satisfying $\omega(z) \leq 0$ that they most prefer, with the understanding that if they bring their part of the characteristic w to the market, then they will match with a partner who has characteristic h. Women do the same except that they choose z such that $\omega(z) \geq 0$. The choices that men make induce a measure on the set Z, the same for women. Assume that the mass of men and women is the same, the auctioneer adjusts ω until the measures that each sides' choices induce on Z are the same. This is what market clearing means. The interpretation is not that characteristics have money prices, but that if a man brings a certain set of characteristics w to the market, then he will be able to accurately predict the quality of the partner it will attract using the relationship described by the auctioneer. Ekeland (2003) establishes the existence of an equilibrium of this kind when preferences are quasi-linear under otherwise very weak assumptions.

The complication for hedonic equilibrium occurs at the 'edges' of the market. In a labour market with assortative matching, for example, there is a worst worker and firm who will match in equilibrium. The worker brings some level of education, the firm a wage. If they make a bilaterally efficient investment wage decision, as competitive equilibrium says they must, then each would like to reduce his or her own wage or educational investment, but according to the hedonic argument, won't do so for fears of losing their partner. Yet if the worker and firm are truly the worst on the market, then there is no worse partner. Small reductions in wages and investments should then have no impact on their partner, and the hedonic solution unravels.[1]

This paper illustrates how to construct an equilibrium very similar to the hedonic equilibrium using the limits of Bayesian Nash equilibrium from finite versions of the game to determine the out of equilibrium payoffs. Apart from providing an illustration of a case where large numbers do not eliminate strategic play, the problem also nicely illustrates the main difference between competitive equilibrium and game theory - off equilibrium behavior in the competitive model is largely arbitrary, whereas the game theoretic treatment takes pains to make off equilibrium behavior look reasonable.

The paper begins with a description of the hedonic equilibrium with two-sided endogeneity, articulated more or less as it was in Peters and Siow (2002). We illustrate the difficulty with hedonic equilibrium, then illustrate how the solution concept can be modified to get around the problem. We finish by mentioning some of the implications for problems that are typically understood using hedonic ideas.

2. Fundamentals

The market consists of m firms and n workers respectively with $n > m$ and $n = \tau m$. Each firm has a privately known characteristic x. If is commonly believed that these are independently drawn from a distribution F on a closed connected interval $X = [\underline{x}, \overline{x}] \subset \mathbb{R}^+$. This characteristic measures the value of worker investment to the firm. Firms with higher types have higher marginal value for worker human capital. Similarly, each worker has a type y that affect his or her investment cost. Again it is assumed that these are independently drawn from a distribution G on a closed connected interval $Y = \left[\underline{y}, \overline{y}\right] \subset \mathbb{R}^+$. The distributions F and G are both assumed to be differentiable, with both densities F' and G' uniformly bounded above.

When we want to think about a continuum of workers and firms, m and n will be treated as measurable sets. The measure of the set m will be 1 while the measure of the set n will be τ. The measures F and G will then represent the distribution of characteristics on m and n.

Each firm has a single job that it wants to fill with one worker. Each worker wants to fill one job. In order to match, firm i chooses a wage $w_i \in W \subset \mathbb{R}^+$. Each worker j chooses a human capital investment $h_j \in H \subset \mathbb{R}^+$. Workers and firms are then matched assortatively, with the most skilled worker (the worker with the highest h_j) being hired by the firm with the highest wage, and similarly for lower wages and investments. Ties are resolved by flipping coins.

Payoffs for firms and workers depend on their characteristic, their investment or wage, and on the investment or wage of the partner with whom they are eventually matched. The payoff of a firm who offers wage w_i and is matched with a worker of type h_j is

$$v\left(x_j\right) h_j - w_i \tag{1}$$

where $v\left(x_j\right)$is a monotonically increasing function of x_j. The corresponding payoff for a worker whose investment is h_j who finds a job at wage w_i is

$$w_i - c\left(h_j, y_j\right) \tag{2}$$

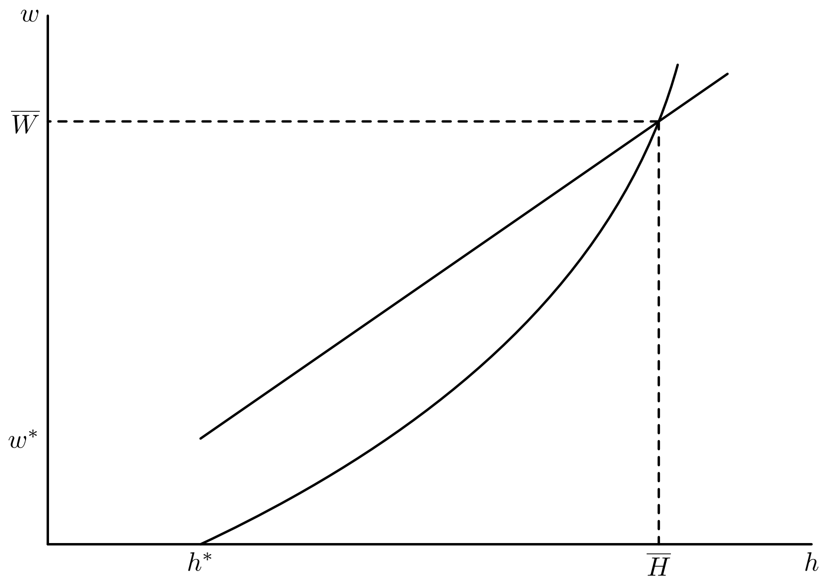
where c is a strictly convex differentiable function of h_j, and a strictly decreasing differentiable function of y_j. It is assumed that marginal cost is strictly decreasing in type y_j. Each worker must make a minimal investment h^* which we refer to as the worker's *bilateral Nash investment* since it is the investment that the worker would make in a bilateral relationship in which the quality of his partner were certain. It is also assumed there is a minimum wage $w^* > 0$ that firms must offer, and refer to this as the bilateral Nash wage of firms.

Firms always match in this model, since they are on the short side of the market. So a firm can always guarantee itself a payoff at least $v\left(x_i\right) h^* - w^*$ by offering the minimum wage and matching with the least qualified employable worker (whose investment is always at least h^*. In this sense this payoff is firm x_i's maximin payoff. Similarly, a worker has maximin payoff $-c\left(h^*, y_j\right)$ since the worst that can happen when the worker invests h^* is that she doesn't match. A pair $\left(w_i, h_j\right)$ is *rationalizable* for a worker of type y_j and a firm of type x_j if both the worker and firm attain at least their maximin payoff in a match with this wage and investment. Say that a wage w_i is rationalizable for a firm of type x_i if there is some type of worker y_j and some investment h_j such that a match between a worker of type y_j and a firm of type x_i in which investments are equal to $\left(w_i, h_j\right)$ is rationalizable. Finally, a wage is rationalizable, if it is rationalizable for some firm. We make the following assumption about the set of feasible investments and wages.

Condition 2.1. The sets H and W are bounded intervals containing all rationalizable investments and wages.

To help see the meaning of this restriction, consider the following diagram:

The diagram is drawn in the space $H \times W$ which corresponds with the space Z is the general description of hedonic equilibrium given above. Workers have indifference curves that represent the various levels of investment in human capital they would be willing to make given different wages in return. These indifference curves constitute a family that resemble the convex upward sloping curve in the diagram. The indifference curve drawn is the one that the best worker attains when she is unemployed. Firms have a similar family of indifference curves. They are linear, and the one drawn corresponds with the best firms' maximin value, attained by offering the minimum wage w^* and matching with a worker who makes the minimum investment. Wage investment pairs



that lie between these two indifference curves are rationalizable in a match between the highest worker and firm type. Indifference curves for lower type workers are steeper, while indifference curves for lower firm types are flatter. The maxi-min outcomes for lower type workers and firms are the same as they are for the highest types. So the set of wage investment pairs that are rationalizable for lower types are contained within the intersection of these two indifference curves by the single crossing property of preferences. Then the set of all rationalizable wages and investments is contained within the rectangle whose upper boundaries $\overline{H}$ and $\overline{W}$ are given by the intersection of these two curves. Condition 2.1 says that this rectangle is contained in the cross product of H and W.

3. The competitive (Hedonic) model with Two-Sided Endogenous Types.

Let $Z = W \times H$. In this section m and n are measurable sets. An allocation is a pair of measurable mappings $\alpha^f(\cdot) = \left(w^f(\cdot), h^f(\cdot)\right) : X \to Z$ and $\alpha^w(\cdot) = (w^w(\cdot), h^w(\cdot)) : Y \to Z$. Write $\alpha \equiv \left\{\alpha^f, \alpha^w\right\}$. An allocation α is feasible if the measures it induces on $Z^+ = \{(w', h') \in Z : w' > 0\}$ are the same. This restriction requires that some workers remain unmatched. In this formulation, unmatched workers are mapped to points in Z/Z^+. An allocation α is Pareto optimal if there does not exist an alternative feasible allocation for which almost all workers and firms are at least as well off, while some set of workers or firms having strictly positive measure are made strictly better off.

A feasible allocation α is a competitive equilibrium if there exists an hedonic (non-linear) functional $\omega : H \times W \to \mathbb{R}$ such that for every $x \in X$,

$$\alpha^f(x) \in \arg\max_{(h',w')} \{v(x)h' - w' : \omega(h',w') \leq 0\}$$

and for every $y \in Y$,

$$\alpha^w(y) \in \arg\max_{(h,w)} \{w' - c(h',y) : \omega(h',w') \geq 0\}$$

As preferences are quasi-linear, the functionals can be chosen to be linearly separable in w' (for example, see (Ekeland 2003)).[23] Then the constraint for firms could be written $w' \geq \omega(h)$, which gives the hedonic wage interpretation that is common in labour economics.

The existence of competitive equilibrium in the quasi-linear case is studied by (Ekeland 2003). When preferences aren't quasi-linear, but the dimension of Z is small, sufficient conditions for existence are provided in (Peters and Siow 2002) and (Han 2002). When the qualities of one side of the market are given exogenously, existence is established (without quasi-linearity but with one indivisible commodity) by (Mas-Colell 1978). Interest here is focused on a couple of the properties of competitive equilibrium. First

Proposition 3.1. *Suppose preferences are monotonic. Then every competitive equilibrium is Pareto optimal.*

The proof of this theorem is straightforward, and simply mimics the textbook argument. An easy way to compare hedonic and truncated hedonic equilibria is to use some pictures. The first figure below describes a hedonic or fully competitive equilibrium for the worker firm problem.

There are more workers than firms in this market, so not all the workers can be accommodated with jobs. Let y_0 be the worker type such that the measure of the set of workers with better types is equal to the measure of the set of firms, i.e., $(1 - G(y_0))\tau = 1$. We refer to this worker type as the *marginal* worker. *The* hedonic (competitive) wage function begins at the maximin point $(h^*, 0)$, then travels up the indifference curve of the marginal worker to the point e, where this marginal worker's indifference curve is tangent to an indifference curve of the worst type firm (given by the line segment FF in the figure). The marginal worker is that worker whose type is such that the measure of the set of workers with higher types is equal to the measure of the set of firms. Workers whose types are below that of the marginal type will make their bilateral Nash investments and remain unmatched. The indifference curve of such a worker type is drawn in the picture. The

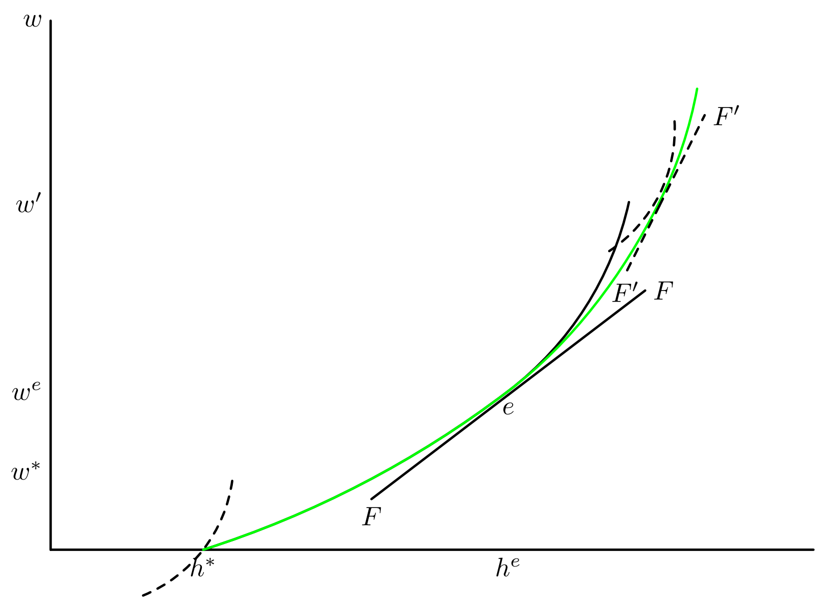


Figure 1. Hedonic Equilibrium

hedonic wage then moves up in such a way that successively higher type workers and firms indifference are tangent to one another. The dashed indifference curve $F'F'$ represents an indifference curve for a higher type firm. This indifference curve is steeper than the indifference curve through FF because the higher type firm has a higher value for human capital. This firm matches with a worker whose indifference curve is given by the dashed convex curve tangent to it. This higher type worker has a lower cost of acquiring education, so his or her indifference curve is flatter than the marginal worker's indifference curve.

The green segment of the hedonic line above the point e is sensible in that if a worker considers cutting investment below the level they are supposed to supply in equilibrium, the firm with whom they are currently matched will be able to replace them with a worker of only infinitesimally lower quality. In equilibrium this new worker will be happy to make the move since they will be matched in equilibrium with a firm paying an infinitesimally lower wage.

As noted above, this is not true for the marginal worker who is supposed to make investment h_e in equilibrium and be matched with a firm paying wage w_e. To support a hedonic equilibrium, this worker needs to believe that if he or she cuts investment, they will be matched with a lower wage firm. One way to support the hedonic outcome is to have this wage give them the same expected payoff as they enjoyed in equilibrium, so the deviation is unprofitable. However, there is no firm in the market offering a wage below w_e, so this expectation is unjustified. Furthermore, there is no way for a firm of the lowest type to replace this deviating worker, since all other workers have either invested bilateral Nash, or are already matched with firms who are paying higher wages. Similar arguments apply to firms.

4. Bayesian Equilibrium

The problem is how to modify the competitive solution to account for this behavior in the bottom of the distribution.[4]

On the workers' side of the market, the resolution is relatively straightforward. Imagine for a moment that the market has a finite, but potentially very large number of workers competing for jobs.

Workers now play a Bayesian game to determine who matches with whom. Suppose that workers use strategies that are monotonically increasing functions of their types so the match that a worker makes depends only on his rank in the realized distribution of worker types. Suppose that firms also use a common monotonically increasing strategy w. Then the equilibrium strategy for a worker of type y_i should satisfy the following necessary condition:

$$\mathbb{E}\sum_{k=n-m}^{n-1}\frac{(n-1)!}{k!\,(n-1-k)!}G\left(y_i\right)^k\left(1-G\left(y_i\right)\right)^{n-1-k}\tilde{w}_{k-(n-m)+1:m}-c\left(h\left(y_i\right),y_i\right)\geq$$

(3)

$$\mathbb{E}\sum_{k=n-m}^{n-1}\frac{(n-1)!}{k!\,(n-1-k)!}G\left(y'\right)^k\left(1-G\left(y'\right)\right)^{n-1-k}\tilde{w}_{k-(n-m)+1:m}-c\left(h\left(y'\right),y_i\right)$$

for every $y'\in Y$. In these two expressions, $\tilde{w}_{k:m}$ is the realized value of the k^{th} order statistic of firms wages. The binomial probabilities indicate how likely it is that the worker out-invests different numbers of workers on his side of the market. So the long summation is just his expected wage. The term on the right hand side gives the same calculation when the worker acts and invests as if his type were y' instead of y_i.

This condition ignores the possibility that the worker might like to act in a way that is different from every other type. This possibility is ruled out in two ways. First the function h must satisfy $h\left(\underline{y}\right)=h^*$. Then it isn't possible to invest an amount that is lower than all other types. Investing more than all other types is unprofitable since it doesn't result in any improvement in the average wage the firm earns.

It is straightforward to verify from the single crossing condition of preferences, that the condition above will be satisfied if the appropriate first order condition is satisfied for every type. This gives the differential equation

$$h'(y_i)\, c_h(h(y_i), y_j) = \mathbb{E} \sum_{k=n-m}^{n-1} \gamma_k'(y_i)\, \tilde{w}_{k-(n-m)+1:m} \tag{4}$$

where

$$\gamma_i(y_i) = \frac{(n-1)!}{k!\,(n-1-k)!} G(y_i)^k (1 - G(y_i))^{n-1-k}$$

This is a differential equation with initial value $h(\underline{y}) = h^*$. The equation satisfies the appropriate Lipschitz condition because of the fact that c_{yy} is bounded away from zero by the strict convexity assumption. By standard theorems (for example (Kreider, Kuller and Ostberg 1968), Theorem 9-7), it has a unique (and evidently monotonically increasing) solution which varies continuously (in the sense of uniform convergence) as the vector of expected wages changes.

The relation

$$y \to \left(h(y), \sum_{k=n-m}^{n-1} \frac{(n-1)!}{k!\,(n-1-k)!} G(y)^k (1 - G(y))^{n-1-k}\, \mathbb{E}\tilde{w}_{k-(n-m)+1:m} \right) \tag{5}$$

parametrically traces out something that looks like a hedonic price functional of the kind that might be estimated by a labour economist. The exact relationship between wages and investments is driven by the equilibrium of the Bayesian game instead of market clearing. Since the lowest worker type makes an equilibrium investment equal to h^*, and because the equilibrium strategy is continuous, this parametrization associates and expected wage with every investment between h^* and $h(\overline{y})$. By standard properties of the Bayesian equilibrium, the menu of investment expected wage pairs has the property that the workers' equilibrium strategy picks out each worker type's favorite element of the menu.

An analogous argument applies for firms. The common strategy that firms use in equilibrium must satisfy a condition that resembles (4), given by

$$v(x_i) \sum_{t=0}^{m-1} \phi_t(x_i)\, \mathbb{E}\tilde{h}_{t+(n-m+1):n} - w(x_i) \geq$$

$$v(x_i) \sum_{t=0}^{m-1} \phi_t(x_i')\, \mathbb{E}\tilde{h}_{t+(n-m+1):n} - w(x_i') \tag{6}$$

where the expectation refers to the expectations of the various order statistics of workers investments when workers use their equilibrium strategy. The corresponding differential equation is

$$w'(x_i) = v(x_i) \sum_{t=0}^{m-1} \phi_t'(x_i) \mathbb{E}\tilde{h}_{t+(n-m+1):n} \tag{7}$$

where

$$\phi_t(x_i) = \binom{m-1}{t} F^t(x_i)(1 - F(x_i))^{m-1-t}$$

This equation is easier to solve, since it gives the equation for the equilibrium strategy in closed form conditional on the values of the expected order statistics. Again, the boundary condition is that $w(\underline{x}) = w^*$.

Again, the relation

$$x_i \longmapsto \left(\sum_{t=0}^{m-1} \phi_t(x_i) \mathbb{E}\tilde{h}_{t+(n-m+1):n}, w(x_i) \right) \tag{8}$$

maps out a simple hedonic relationship between the wage a firm offers and the expected quality of the worker it will hire in return. As the worst firm type must always attain her equilibrium payoff by offering the wage w^*, this relationship defines a menu of wage-expected quality pairs that are available to firms when acting against the equilibrium strategies of their opponents. The menu includes returns for all wages between the minimum wage w^* and the highest wage offered by any firm type.

This remedies the difficulty with the basic hedonic solution, which as we have pointed out, assigns a payoff in an arbitrary and unrealistic way in order to support equilibrium. Here the payoffs to all investments are computed in a sensible way, consistent with equilibrium behavior. The hedonic solution, like all competitive solutions, is intended to apply to large markets where individual traders have little market power. This suggests that a good way to modify the competitive solution would be to try to compute what this solution looks like when the number of workers and firms is very large.

5. Equilibrium with Many Workers and Firms

The material in this section is taken from Peters (2007). Let h_m and w_m be Bayesian equilibrium strategies when there are m firms and $n = \tau m$ workers. Let h_∞ and w_∞ refer to the weak limits of these equilibrium strategies as m and n go to infinity together.[5] These functions describe the way workers and firms respectively behave in the limit when they face a continuum of opponents.

Similarly, let $\hat{w}_m(h')$ define the menu given by (5) and $\hat{h}_m(w')$ the menu given by (8). Then $\hat{w}_\infty$ and $\hat{h}_\infty$ define the pointwise limits of these functions. These latter two functions represent the hedonic payoffs that workers and firms respectively face in the limit when they play against a continuum of other workers and firms.

Proposition 5.1. $h_\infty(y) = h^*$ for each $y < y_0$. *Furthermore* $\hat{w}_\infty(h') - c(h', y_0) = -c(h^*, y_0)$ *for each* $h^* \leq h' \leq h_\infty(y_0)$.

The Proposition says that the marginal worker type y_0 receives an expected payoff in the limit which is equal to his or her payoff when they are unemployed. It also says that this is true for any investment the marginal worker makes between h^* and $h_\infty(y_0)$.

Informally the reason relies on the fact that traders use monotonic strategies in each finite game. So only the workers with the top m types will get jobs. As the economy becomes large, a worker will find himself in this set only if his type is above y_0. So workers with types below this simply won't invest in the limit, simply because there is no reward to doing so. To ensure that such workers don't want to deviate and offer something between h^* and $h_\infty(y_0)$, the expected payoff function $\hat{w}_\infty$ can be no steeper than the marginal workers indifference curve on this region.

On the other hand workers whose types are above y_0 are sure to find jobs. To prevent the higher type workers from cutting wage, the limit payoff functions between h^* and $h_\infty(y_0)$ can be no flatter than the marginal workers indifference curve. This is the formal argument that extends the hedonic return function outside the domain generated by equilibrium strategies.

A similar argument applies for firms. The return function faced by firms for wages between w^* and $w_\infty(\underline{x})$ has to coincide with the worst firm's indifference curve to prevent the higher types from deviating and offering the minimum wage.

The second Proposition (again from Peters (2007)) provides much of the characterization of the limit behavior of this matching market. For any type $y' > y_0$, call $x' = \pi(y')$ the hedonic partner of worker y' if the measure of the set of firms whose types are larger than x' is equal to the measure of the set of workers whose types are larger than y'. Assortative matching in the continuum would require that each worker whose type is above y_0 be matched with his or her hedonic partner.

Proposition 5.2. *For each* $y' \geq y_0$,

$$\hat{w}_\infty(h_\infty(y')) = w_\infty(\pi(y'))$$

A similar proposition applies for firms. The way this Proposition is used is as follows: begin with a candidate pair of limit strategies $w_\infty(\cdot)$ and $h_\infty(\cdot)$. We want to check whether this pair of strategies could constitute weak limits of some sequence of Bayesian equilibrium strategies. From Proposition 5.1, any such weak limit must have the property that $h_\infty(y') = h^*$ for every worker type below y_0, so lets assume this condition is satisfied.

Proposition 5.2 now shows how to construct the limit payoff function from the strategies h_∞ and w_∞. Again, using Proposition 5.1, the payoff to any investment between h^* and $h_\infty(y_0)$ is given by the graph of the marginal worker's indifference curve. Proposition 5.2 now says that the payoff function to each worker type in equilibrium must be equal to the candidate strategy of his or her hedonic partner. Since a worker can emulate the behavior of any type above y_0, this defines an implicit payoff function for every investment in the range of h_∞. Using this payoff function we can then compute a best reply for every worker type. We do exactly the same thing for firms. In order for our original candidate strategies to be the limits of equilibrium strategies from Bayesian games, they have to coincide with these best reply functions.

Formally

Proposition 5.3. *The strategy rules w_∞ and h_∞ are weak limits of Bayesian equilibrium strategies only if $h_\infty(y') = h^*$ for $y' < y_0$;*

$$h_\infty(y') = \arg\max \hat{w}_\infty(h'') - c(h'', y')$$

where $\hat{w}_\infty(h'') = \{w_\infty(\pi(y'')); h_\infty(y'') = h''\}$; and similarly for firms

$$w_\infty(x') = \arg\max v(x') \hat{h}_\infty(w'') - w''$$

where $\hat{h}_\infty(w'') = \{h_\infty(\pi^{-1}(x'')); w_\infty(x'') = w''\}$.

Proposition 5.3 only supplies a necessary condition for equilibrium. The element that is lacking in the Proposition is the starting value for the function w_∞. If we could tie this down, then a complete characterization of the limits of Bayesian equilibrium behavior would be possible. What remains in this section is a heuristic description of the location of this starting point.

If any firm offers a wage equal to w^*, it will have the lowest wage on offer with probability 1. Its partner will then be the worker with the $n - m + 1^{st}$ highest human capital investment. So the expected quality is simply the expected value of the $n - m + 1^{st}$ order statistic of worker investments. Denote the probability distribution of this order statistic by $\Pr\left\{\tilde{h}_{n-m+1:n} \leq x\right\}$. Without further comment, it seems reasonable to assume that this distribution converges weakly to $\Pr\left\{\tilde{h}_{n-m:n-1} \leq x\right\}$ as m and n go to infinity.

Now consider a worker who invests some amount $h^* < h' < h_\infty(y_0)$. If the random variable $\tilde{h}_{n-m:n-1}$ is more than h', then this investment cannot result in a match, and the worker will end up without a job. On the other hand if $\tilde{h}_{n-m:n-1}$ is less than h', the worker will find a job. Since the limit distribution of wages doesn't contain any wages below $w_\infty(\underline{x})$, and since this worker can't earn a higher wage than any worker who invests at or above $w_\infty(\underline{x})$, the expected wage conditional on finding a job should converge to $w_\infty(\underline{x})$.[6] By Proposition 5.2, this means that the worker's payoff conditional on finding a job is the same as his unconditional payoff when he invests $h_\infty(y_0)$. Formally

$$\hat{w}_\infty(h') = \Pr\left\{\tilde{h}_{n-m:n-1} \leq h'\right\}\hat{w}_\infty(\underline{x})$$

By Proposition 5.1, the payoff function $\hat{w}_\infty(h')$ must coincide with the graph of the marginal worker's indifference curve through h^*, which gives the distribution

$$\Pr\left\{\tilde{h}_{n-m:n-1} \leq h'\right\} = \frac{\hat{w}_\infty(h')}{\hat{w}_\infty(\underline{x})} = \frac{c(h', y_0) - c(h^*, y_0)}{c(h_\infty(y_0), y_0) - c(h^*, y_0)}$$

So the expected quality of the firm's partner when it offers the wage w^* should be equal to $\int_{h^*}^{h_\infty(y_0)} \frac{h'}{c(h_\infty(y_0),y_0)-c(h^*,y_0)} \frac{\partial c(h',y_0)}{\partial h'} dh'$.

In fact, this completes the characterization of the limits of Bayesian equilibrium. The limit strategies must satisfy the fixed point condition described in Proposition 5.3. In addition, the point $(h_\infty(y_0), w_\infty(\underline{x}))$ must lie on the same iso profit curve for firms as the point

$$\left(\int_{h^*}^{h_\infty(y_0)} \frac{h'}{c(h_\infty(y_0), y_0) - c(h^*, y_0)} \frac{\partial c(h', y_0)}{\partial h'} dh', w^*\right)$$

These conditions characterize a *Truncated Hedonic Equilibrium* as described in Peters (2006).

6. Truncated Hedonic Equilibrium

To relate all these limit results to hedonic equilibrium, a couple of remarks are helpful. First, point $(h_\infty(y_0), w_\infty(\underline{x}))$ is focal in all this, since the limit payoff functions look very competitive for investments and wages above these. Yet they look quite unusual below this point. So define $(h_0, w_0) \equiv (h_\infty(y_0), w_\infty(\underline{x}))$. The construction of the limit equilibrium hinges around this point. Next let $\hat{w}_\infty(\cdot) \equiv \omega^t(\cdot)$. The superscript t refers to 'truncated hedonic', and serves to distinguish

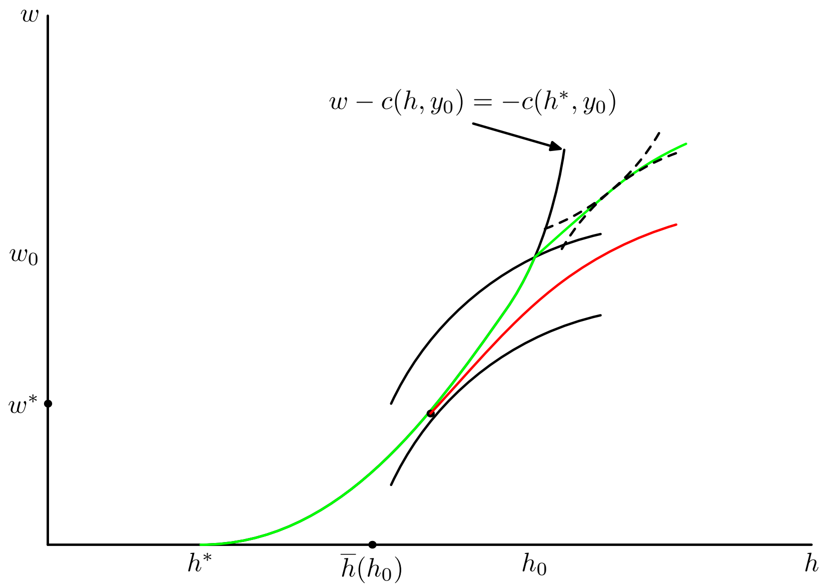


FIGURE 2. Truncated Hedonic vs Hedonic Equilibrium

this idea from the simple hedonic relationship $\omega(\cdot)$ that was defined above.

Now we can define a *Truncated Hedonic Equilibrium* as a hedonic wage relationship ω^t with initial value (h_0, w_0) that satisfies

(1) for each $h \geq h_0$ the measure of the set $\{x : \arg\max_{h'} v(x) h' - \omega^t(h') \geq h\}$ is equal to the measure of the set $\{y : \arg\max_{h'} \{\omega^t(h') - c(h', y)\} \geq h\}$;
(2) $w_0 - c(h_0, y_0) = -c(h^*, y_0)$; and
(3) $v(\underline{x}) h_0 - w_0 = v(\underline{x}) \int_{h^*}^{h_0} \frac{h'}{w_0} \frac{\partial c(h', y_0)}{\partial h'} dh' - w^*$.

The first condition is readily seen to be a reformulation of the conditions in Proposition 5.3, the second condition is the result in Proposition 5.1. The final condition is the boundary condition as discussed above.

The next figure illustrates the truncated hedonic equilibrium for one plausible specification of preferences and compares it to the pure hedonic equilibrium. In this Figure, the function $\overline{h}(h_j)$ is defined by

$$\overline{h}(h_j) = \int_{h^*}^{h_j} \frac{h'}{c(h_j, y_0) - c(h^*, y_0)} \frac{\partial c(h', y_0)}{\partial h'} dh'.$$

The truncated hedonic equilibrium is the piecewise differentiable function that starts at the point $(h^*, 0)$, follows the indifference curve of the marginal worker y_0 up to the point (h_0, w_0), then kinks and travels between the indifference curves of workers and firms. If you are viewing the paper in color, the truncated hedonic return function is the green line. Also, to make the tangencies a little clearer, the picture is drawn with concave iso-profit curves for firms. Formally the iso-profit curves for firms should be straight lines here. The distribution function for the $n - m^{th}$ order statistic of worker investments is the marginal worker's indifference curve between h^* and h_0, divided by the vertical distance w_0. Taking the expectation of the order statistic using this distribution function gives expected investment equal to $\overline{h}(h_0)$. The point $\left(\overline{h}(h_0), w^*\right)$ then determines the indifference curve for the worst firm's equilibrium payoff.

In principle, the only restriction on the location of the point $\left(\overline{h}(h_0), w^*\right)$ is that it lie to the left of h_0. It could like above or below the marginal worker's indifference curve. After all, the minimum wage is fixed exogenously in this exercise. So simply vary this wage to get the point $\left(\overline{h}(h_0), w^*\right)$ to lie either above or below the curve. This point could even lie on the worker's indifference curve. It it did, a very non-robust possibility is that the worst firm's iso-profit curve is tangent to the marginal worker's indifference curve at this point. If that is the case, then the hedonic equilibrium would be supported as the limit of some sequence of Bayesian equilibrium.

The figure deals with the case where the point $\left(\overline{h}(h_0), w^*\right)$ lies above the marginal worker's indifference curve. As the truncated hedonic return function is kinked at the point (h_0, w_0), a lot of different firm and worker types will pool at this point. It is perhaps confusing that in a large finite Bayesian game, there is no pooling at all since investment and wage strategies are both monotonically increasing. However, worker types near y_0 all make very similar investments when there are many workers and firms, and this looks like pooling in the limit.

Since Bayesian equilibrium ensures that even in the limit all firms and workers are matched assortatively, the pools at (h_0, w_0) must have equal measure. Types from both sides will be added to this pool until the marginal types added to the pool on both sides have tangent indifference curves. The remaining workers and firms are matched in the manner of a competitive equilibrium. They match with wages and investments that are bilaterally efficient.

The simple hedonic (competitive) is represented by the curve that starts at the tangency of the worst firm and marginal worker's indifference curves. The line is drawn in red if you are seeing the picture in color. One immediate consequence is that workers and firms have higher human capital investments and wages in a truncated hedonic equilibrium than they do in a competitive equilibrium. One result of this is that all firms are worse off in the truncated hedonic equilibrium than they are in the competitive equilibrium. Further, all worker types above y_0 are better off in the truncated hedonic equilibrium than in the plain competitive equilibrium.

Yet the outcome is not Pareto optimal. The workers and firms pooled together at (h_0, w_0) are basically enforcing the equilibrium. They don't want to privately reduce their wage or investment because they might lose a match, or match with a partner whose quality is lower. However, if they could negotiate any contract specifying the wage and human capital investments they like, then they would mutually prefer that lower wages be paid, and that the worker acquire less human capital. In this regard, the anti-trust suit against the residents matching program was misplaced - the joint competition in wages and investment results in wages that are actually higher than they would be in a competitive equilibrium.

The pool of workers and firms at (h_0, w_0) is the result of the following kind of externality: workers human capital investment is sunk, so if they don't find a job, this investment is lost. To avoid this, lower quality workers increase their human capital investments to reduce the unemployment probability. This creates additional unemployment risks which induce other low quality workers to invest. This negative externality is somewhat standard. On the other side of the market, this higher investment makes the low quality workers more attractive to firms, who bid up wages to compete for these better workers. The higher wages create even larger incentives for the low quality workers to avoid unemployment, and so on.

The pool of low quality workers and firms supports a fairly common property of wage distributions, in that they are skewed to the left. There are many workers who are paid low wages, then a tail of higher wage workers stretches out to the right. this skewness is a direct consequence of the pooling and is property that is very distinct from the distribution implied by a competitive equilibrium, which is given in a completely exogenous way by the distribution of types.

Finally, notice that one of the impacts of the pool at the bottom is that it generates a much larger gap in the incomes of the employed and the unemployed. As an aside, it might be useful to note that assortative matching in the marriage market based on education was the original motivation for the paper Peters and Siow (2002). This kind of assortative matching is referred to in a recent Statistics Canada report as "educational homogamy". Homogamy is blamed for increasing income inequality between families since 1970. As described above, the incentive effects of assortative matching with ex ante investment create two sorts of inequality. The one just mentioned would lead the married and unmarried to have very different education levels. This gap is larger than it would be in a competitive equilibrium. Inequality across families also increases, at least relative to what would occur with simple random matching where all families would make minimal educational investments.

7. Conclusion

Bayesian equilibria in matching problems with two sided investment support allocations that differ from straightforward hedonic equilibrium because of the way they treat payoffs to out of equilibrium investments. These equilibrium allocations support larger wages and investments than simple hedonic equilibria. In fact, for the lowest types of workers and firms, these investments and wages are both too high in the sense that any matched pair would jointly benefit by lowering both. The long side of the market benefits (relative to the competitive equilibrium).

The model is very specialized in the sense that matching is assortative. Most interesting matching problems involve much more complex characteristics for workers and firms and allow the possibility that traders on one side of the market might disagree about how to rank the traders on the other side. A very general existence theorem for hedonic equilibrium exists when preferences are quasi-linear. Absent quasi-linearity, existence theorems exist only for special cases (for example, Peters and Siow (2002)).

Truncated hedonic equilibrium has only been defined as an equilibrium for a large game with assortative matching. It isn't clear to what extent the considerations here will impact investments in more complex problems where characteristics are high dimensional and traders disagree about how to rank partners. It seems reasonable to conjecture that the problems with hedonic equilibrium will become worse with higher dimensional characteristics. The basic logic is that the worst worker who expects to be employed is considerably better for the firm with whom he matches than the next best alternative who has no investment at all because he never expected to compete for a job. With multi-dimensional characteristics, there are many market edges like this. A firm who is forced in equilibrium to match with his least desirable worker in one dimension may find that the next best alternative for him varies in many different characteristics. If this alternative is strictly worse that the worker who he is supposed to hire in equilibrium, the worker may know this and cut investment.

The simple assortative matching case nonetheless illustrates something quite unsettling about stable matching. It can have very bad incentive properties. The focus on stable matching algorithms as a good way of organizing matching problems like residents matching may be misguided. The results here suggest that the anti-trust case directed at the deferred acceptance algorithm might have missed the mark. Wages are higher than they are in a completely competitive market. However, incentive would be improved with matching procedures that could commit to unstable outcomes, especially in off equilibrium situations. For example, a fully efficient outcome could be supported simply by imposing a lower level on investment, then committing to leave participants with characteristics below this level unmatched, which would typically not be stable.

Notes

[1]In the Felli Roberts argument, firms must bid for workers ex post, so there is potentially a way for firms to penalize workers who cut investment below the efficient level. Their argument goes as follows: the worst firm has to pay the worst (employed) worker just enough to keep him from jumping to his next best alternative, which for the worst worker is unemployment. When this worker chooses his investment, the return he gets is his personal benefit when he is unemployed, not the marginal benefit of the investment to the firm. So his investment incentives are wrong. So efficient investments unravel from the bottom in their story as well.

[2]More generally, the hedonic pricing problem allows the characteristic that the firm chooses, h in this case, to be drawn from a much larger dimensional space. However, the hedonic pricing problem doesn't allow the dimension of the worker's characteristic to be higher than 1 - the worker always pays money for the different qualities of good.

[3]The 'price' of capital in this story is normalized to 1. This wouldn't work if the characteristic chosen by firms was of high dimension. The hedonic pricing equilibrium can still be defined, but the pricing function would have to a more general non-linear functional of investment.

[4]The proofs and technical details for all of the arguments that follow are given in the papers Peters (2004), Peters (2006) and Peters (2007).

[5]The weak limit of a function is equal to its pointwise limit at every point where this limiting function is continuous. The existence of a weak limit is guaranteed by the fact that the functions h_m and w_m are always increasing, and the fact that their domains are contained in a bounded interval.

[6]It is only possible (at least for me) to prove that this conditional expected wage is bounded above by $w_\infty(\underline{x})$. This is the formal difference between strict limits of Bayesian equilibrium as described in Peters (2007) and Truncated Hedonic Equilibrium as described in Peters (2006).

References

Abdulkadrolu, Atila, and Tayfun Sonmez (2003) 'School choice, a mechanism design approach.' *American Economic Review*

93(3), 729

Bulow, Jeremy, and Jonathan Levin (2006) 'Matching and price competition.' *American Economic Review* 96(3), 652–658

Ekeland, Ivar (2003) 'An optimal matching problem.' Technical Report, University of British Columbia. Unpublished report

Felli, L., and K. Roberts (2000) 'Does competition solve the holdup problem?' London School of Economics working paper

Gale, David, and Lloyd Shapley (1962) 'College admissions and the stability of marriage.' *American Mathematical Monthly* 69, 9–15

Han, Seungjin (2002) 'Menu theorems for bilateral contracting problems.' University of Toronto, working paper

Kreider, Donald L., Robert G. Kuller, and Donald R. Ostberg (1968) *Elementary Differential Equations* (Reading , Massachusetts: Addison Wesley Publishing Company)

Mas-Colell, Andreu (1975) 'A model of equilibrium with differentiated commodities.' *Journal of Mathematical Economics* 2(2), 263–295

Mas-Colell, Andreu (1978) 'A model of equilibrium with differentiated commodities.' *Journal Of Mathematical Economics* 2, 263–295

Peters, Michael (2004) 'The pre-marital investment game.' University of British Columbia Working Paper

— (2006) 'Truncated hedonic equilibrium.' University of British Columbia Working Paper

— (2007) 'Matching with competition in characteristics.' UBC working paper

Peters, Michael, and Aloysius Siow (2002) 'Competing premarital investments.' *Journal of Political Economy* 110(3), 592–609. University of Toronto working paper UT-ECIPA-PETERS-00-01

Rosen, Sherwin (1974) 'Hedonic prices and implicit markets: Product differentiation.' *Journal of Political Economy* 82(1), 34–55

Roth, Alvin E. (2003) 'The origins, history and design of the resident match.' *Journal of the American Medical Association* 289(7), 909–912

Roth, Alvin E., Tayfun Sonmez, and M. Utku Unver (2005) 'Pairwise kidney exchange.' *Journal of Economic Theory* 125(2), 151–188. available at http://ideas.repec.org/a/eee/jetheo/v125y2005i2p151-188.html