Bilateral Matching and Bargaining with Private Information

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Abstract

We explore the role of private information in bilateral matching and bargaining. Our model is a replica of Mortensen and Wright (2002), but with private information. A simple necessary and sufficient condition on the parameters of the model for existence of equilibrium with entry is obtained. As in Mortensen and Wright (2002), we find that equilibrium is unique and has the property that every meeting results in trade when the discount rate is sufficiently small. There are also equilibria in which not every meeting results in trade. All equilibria converge to perfect competition as the frictions of search costs and discounting are removed. We find that private information may deter entry. Because of matching externalities, this entry-deterring effect of private information may be welfare-enhancing.

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JEL Classification Numbers: C73, C78, D83.

1 Introduction

Can making information private increase efficiency of a dynamic matching and bargaining market? When a bargaining situation is taken as a stand-alone game, economists generally believe that private information reduces efficiency (following the classic result of Myerson and Satterthwaite (1983)). This is because privately informed traders can demand the terms of trade that are better than those they are willing to accept, which can result in no trade even when trade would be mutually profitable.

Many bargaining situations are not, however, stand-alone games but are imbedded in markets. Think of a buyer of a house who is currently bargaining with the seller. Should they not agree on the price, it is likely that each of them will pursue other options, with values that are endogenously determined by demand and supply conditions of the market.

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Or think about the labor market where workers search for jobs and bargain with employers, with their outside options once again determined by demand and supply.[1]

In dynamic matching and bargaining models, private information can have a role that goes beyond the pure inefficiency effect in bargaining. In the environments of these models, the cost of searching for a trading partner may be important. Traders who are not very optimistic about their prospects in the market may abstain from entering altogether, creating a negative search externality on traders who are on the opposite side of the market and a positive externality on those who are on the same side. Of course, the incentives to enter are affected by the expected payoffs traders hope to obtain when they are matched and bargain. So what happens in bargaining affects the incentives to enter, which provides a channel for the private information to affect entry.

In order to explore these multiple roles of private information, we develop a dynamic matching and bargaining model that has many features found in the labor search literature. Our model is a private-information replica of the dynamic matching and bargaining model of Gale (1987), enriched with a general Pissarides (2000)-style matching function as in Mortensen and Wright (2002).

We study the steady state of a market with continuously inflowing cohorts of buyers and sellers who are randomly matched pairwise and bargain under private information. The inflowing buyers are heterogeneous in that buyers have valuations (and sellers have costs) that are drawn from some distribution and remain unchanged through their lifetime. The valuations and costs are private information.

As in most of the literature, we consider what we call the take-it-or-leave-it offer protocol in which seller proposes an offer with probability α, and the buyer makes an offer with a complementary probability.[2] There are frictions due to costly search, at the rates κ_B for buyers and κ_S for sellers, and time discounting at the rate $r \geq 0$.

We find that under private information equilibria have structure that is essentially the same as under full information, as in Mortensen and Wright (2002). We find that even with private information, there may be a full-trade equilibrium, the one in which every meeting results in a trade.[3] We derive a necessary and sufficient condition for existence of such an equilibrium. In particular, again similar to Mortensen and Wright (2002), it exists provided r is sufficiently small.

But there is also a usual possibility that not every meeting results in a trade. We show however that this cannot happen if r is small, implying that the full-trade equilibrium is the unique equilibrium of the model. The intuition for this is as follows. Recall that Myerson and Satterthwaite (1983) show that the bargaining outcome is necessarily inefficient in a static model provided that the supports of type distributions of buyers and sellers overlap,

[1] Cognizant of this, labor economists have created a large literature on two-sided search in the labor market, surveyed for example in Mortensen and Pissarides (1999) and more recently Rogerson, Shimer, and Wright (2005). By focusing on a real-world matching and bargaining market, and in order to reproduce empirical patterns, papers in this literature have incorporated realistic features of search costs and matching technology. But most if not all of this literature assumes that bargaining transpires under full information.

[2] The take-it-or-leave-it offer protocol has also been in the focus of the vast literature on two-sided search in the labor market, see for example the survey by Mortensen and Pissarides (1999). Atakan (2007) extend the results of Riley and Zeckhauser (1983) and shows that even if traders are allowed to offer general mechanisms, they can do not better than making take-it-or-leave-it offers.

[3] The notion of a full-trade equilibrium is due to Satterthwaite and Shneyerov (2007).

but can be efficient if they do not overlap. We show that, although the setting is dynamic, in the steady state it can essentially be reduced to a static setting in which the types are replaced by what we call dynamic types.[4] As the discount rate gets small, we show that the support of the distribution of dynamic types shrinks, and that the presence of search costs makes the supports non-overlapping for small r. Given this, it is plausible (and we prove that this is in fact the case) that traders make offers that are always accepted.

The uniqueness of equilibrium is important in its own sake, but it also allows us to compare welfare under private and public information. We find that, with private information and when r is small, there is less entry. Why this is so can be understood through the following logic. In both models, there are marginal entrants: the lowest-value participating buyers and the highest-cost participating sellers. These marginal entrants have different incentives to enter depending on whether information is private or public. Under public information, the traders ceteris paribus obtain positive rents when they propose, and obtain zero rents when they accept offers that must be only marginally good to them. Under private information, the proposers obtain (again ceteris paribus) smaller rents when they propose, but larger rents when they are on the responding side. But for the marginal entrants, in both models, the rents are zero when they are responding. This means that the marginal entrants enjoy larger rents under full information. These rents make it attractive for additional, less efficient traders to enter the market.

Because of entry costs and matching externalities, this additional entry may not necessarily be socially beneficial. Our finding is that social welfare may be larger if information is private if the following two conditions are satisfied. First, the elasticity of the matching function with respect to the mass of one side of traders is higher than bargaining weight of that side (α for sellers and $1-\alpha$ for buyers). For concreteness, take sellers and suppose their elasticity σ_S is higher than their bargaining weight. Then our second condition is that the share of the total surplus attributable to buyers when $r = 0$, is large. This is intuitive because sellers impose a positive externality on buyers, allowing the latter to match quicker.

Our uniqueness result has yet another implication. Convergence to perfect competition has always been in the focus of the matching and bargaining literature, although most of it until very recently has assumed full information. As a by-product of our uniqueness result, we are also able to show convergence to perfect competition in our model.

We are also able to prove a necessary and sufficient condition for existence of a nontrivial equilibrium (i.e. equilibrium with positive entry). The presence of search costs can obviously lead to market breakdown if the costs are very large, but it is also interesting to know how large these are. In our model, the arrival processes of buyers and sellers are Poisson with arrival rates ℓ_B and ℓ_S respectively. (They are functions of ζ, the equilibrium ratio of the mass of buyers to the mass of sellers in the market.) The average waiting times are $1/\ell_B$ and $1/\ell_S$, and the accumulated search cost until the next meting is equal to $K = \kappa_B/\ell_B + \kappa_S/\ell_S$. The maximal gain from trading is 1 (the type supports are assumed to be $[0, 1]$). So one can see that $K < 1$ is a necessary condition for the market to be sustainable. Remarkably, we are also able to show that this condition is sufficient. To our knowledge, this is the first general no-breakdown result for bilateral matching and

[4] The dynamic type of a buyer is the difference between his valuation and the continuation value; the dynamic type of a seller is the sum of his cost and continuation value. The notion of a dynamic type is due to Satterthwaite and Shneyerov (2007).

bargaining markets with search costs.

The structure of the paper is as follows. Section 2 introduces our model. Section 3 presents and discusses our results about existence and uniqueness of a full-trade equilibrium, and its convergence to perfect competition. Section 4 states the general existence theorem and outlines its proof. Section 5 contains welfare comparison to the full information model. Section 6 reviews the related literature and provides some directions for future research. The proofs of most results are in the Appendix.

2 The Model

The players of our model are potential buyers and potential sellers of a homogeneous, indivisible good. Each buyer has a unit demand for the good, while each seller is able to produce one unit of the good. Potential buyers are heterogeneous in their valuations (or types) v over the good. Potential sellers are also heterogeneous in their costs (or types) c of producing the good. For simplicity, we assume $v, c \in [0, 1]$. Time is continuous and infinite horizon. The details of the model are described as follows:

- Entry: Potential buyers and potential sellers are continuously born at rate b and s respectively. The type of a new-born buyer is drawn i.i.d. from the c.d.f. $F(v)$ and the type of a new-born seller is drawn i.i.d. from the c.d.f. $G(c)$. Each trader's type will not change once it is drawn. Entry (or participation, or being active) is voluntary. Each potential trader decides whether to enter the market once they are born. Those who does not enter will get zero payoff. Those who enter must incur the participation cost continuously at the rate κ_B for buyers and κ_S for sellers, until they leave the market.

- Matching: Active buyers and active sellers are randomly and continuously matched pairwise with the rate of matching given by a matching function $M(B, S)$, where B and S are the numbers of active buyers and active sellers currently in the market.

- Bargaining: Once a pair of buyer and seller is matched, they bargain: with probability $\alpha \in (0, 1)$, the seller makes a take-it-or-leave-it offer to the buyer, then the buyer chooses either to accept or reject. And with probability $1 - \alpha$ the buyer proposes and the seller responds. We call this the *take-it-or-leave-it* bargaining protocol.

- If a type v buyer and a type c seller successfully trade at a price p, then they leave the market with (current value) payoff $v - p$, and $p - c$ respectively. If the matched pair fails to trade, both traders can either stay in the market waiting for another match (and incur the participation costs) as if they were never matched, or simply exit and never come back. The instantaneous discount rate is $r \geq 0$.

We make the following assumptions on the primitives of our model.

Assumption (distributions of inflow types) The cumulative distributions $F(v)$ and $G(c)$ of inflow types have densities $f(v)$ and $g(c)$ on $(0, 1)$, bounded away from 0 and ∞: $0 < \underline{f} \leq f(v) \leq \bar{f} < \infty$, $0 < \underline{g} \leq g(c) \leq \bar{g} < \infty$.

Assumption (matching function) The matching function M is continuous on $\mathbb{R}^2_{++}$, nondecreasing in each argument, constant returns to scale (i.e. homogeneous of degree one), and satisfies $\lim_{B\to 0} M(B,S) = \lim_{S\to 0} M(B,S) = 0$.

It turns out to be more convenient to work with a normalized matching function. Let

$$\zeta \equiv \frac{B}{S}$$

be the steady-state ratio of buyers to sellers, and define

$$m(\zeta) \equiv M(\zeta, 1).$$

Since the matching technology is assumed to be constant returns to scale, it is easy to see that $m(\zeta)$ is also equal to $M(B,S)/S$, the expected probability that a seller is matched over a time period of length 1. Similarly, $m(\zeta)/\zeta$ is equal to $M(B,S)/B$, the expected probability that a buyer is matched over a time period of length 1. Note that $m(\zeta)$ and $m(\zeta)/\zeta$ are nondecreasing and nonincreasing respectively in ζ, and m is continuous on $\mathbb{R}_{++}$. In this notation, the Poisson arrival rates for buyers and sellers become

$$\begin{aligned} \ell_B(\zeta) &\equiv \frac{m(\zeta)}{\zeta}, \\ \ell_S(\zeta) &\equiv m(\zeta). \end{aligned}$$

Notice that an uninteresting no-trade equilibrium always exists in which all potential traders do not enter. In the following, we will study steady-state market equilibria in which positive trade occurs. Let us simply call them *nontrivial steady-state equilibria*.

We now proceed to the definition of a nontrivial steady-state equilibrium. It is useful to represent each trader's world as a continuous-time Markov chain, as shown in Figure 1 for buyers.A trader is born into the "inactive" state, and has to decide immediately whether to enter to the market and search a partner, or simply exit. Let $\chi_B : [0,1] \to \{0,1\}$ and $\chi_S : [0,1] \to \{0,1\}$ be the buyers' and sellers' entry-decision functions in the inactive state. For example, $\chi_B(v) = 1$ means type v buyer enters; $\chi_S(c) = 0$ means type c seller does not enter. Let $A_B \subset [0,1]$ and $A_S \subset [0,1]$ be the sets of active buyers' and sellers' types, i.e.

$$\begin{aligned} A_B &\equiv \{v \in [0,1] : \chi_B(v) = 1\}, \\ A_S &\equiv \{c \in [0,1] : \chi_S(c) = 1\}. \end{aligned}$$

Once in the "searching" state, the trader waits until a new trading opportunity arrives. This happens after a time period of random length t has elapsed. (Recall that t is exponentially distributed with mean $1/\ell_B$ for buyers and $1/\ell_S$ for sellers.) The arrival of a trading opportunity moves a trader from the searching state to the "matched" state. Once in the matched state, the trader immediately proceeds either to the proposing state (with probability α for sellers and $1-\alpha$ for buyers), or to the responding state (with the complimentary probabilities). Let $p_B(v)$ and $p_S(c)$ be the proposing strategies used by buyers and sellers respectively.[5] Similarly, let $\tilde{v}(v)$ and $\tilde{c}(c)$ be the acceptance levels, characterizing the

[5] Implicitly, every traders are assumed to use symmetric pure strategies. However, we will claim that it is essentially without loss of generality. See the remark below.

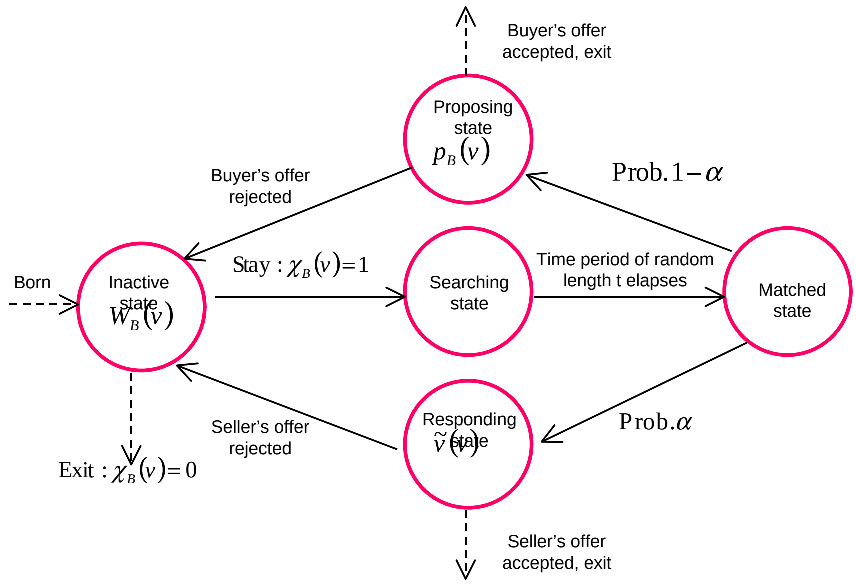

Figure 1: Markov chain of a buyer

responding policies of buyers and sellers respectively. Precisely, in a proposing state, type v buyers will propose the trading price $p_B(v)$, while in a responding state, they will accept a proposed price p if and only if $\tilde{v}(v) \geq p$. Analogous meanings apply to $p_S(c)$ and $\tilde{c}(c)$.

In the event when trading is successful, the matched pair leaves the market forever with their realized gains from trade. If trading is unsuccessful, each trader is immediately back in the inactive state of her Markov chain and the cycle repeats.

Let $\Phi(v), \Gamma(c)$ be the (endogenous) steady-state cumulative distributions of types of buyers and sellers who are active. The equilibria of our model can be defined as a collection[6]

$$E \equiv \{\chi_B, \chi_S, p_B, p_S, \tilde{v}, \tilde{c}, B, S, \Phi, \Gamma\}$$

such that:

(i) given the relevant beliefs made from E, every potential and active buyers (resp. sellers) find the entry policy given by χ_B (resp. χ_S), the proposing policy $p_B(\cdot)$ (resp. $p_S(\cdot)$) and the responding policy characterized by $\tilde{v}(\cdot)$ (resp. $\tilde{c}(\cdot)$) to be their optimal policies sequentially;

(ii) E generates B, S, Φ, Γ in steady state.

The mathematical conditions for our equilibrium are as follows. Let us consider the sequential optimality of the responding strategies first. Let $W_B(v)$ be the (steady-state) equilibrium continuation payoff of a type v buyer in her inactive state, and let $W_S(c)$ be the equilibrium continuation payoff of a type c seller in her inactive state. Pick a type v buyer.[7] If she is in her responding state with an offer p at hand, her continuation payoff

[6] This definition is similar to the one in Satterthwaite and Shneyerov (2007).

[7] This type v buyer could be either active or not. If she is not active, we are considering an off-equilibrium path.

is $\max\{v-p, W_B(v)\}$. The first element $v-p$ is the continuation payoff if she accepts the offer p, while the second element $W_B(v)$ is the continuation payoff if she rejects and hence immediately get back to the inactive state. Similar logic applies to sellers' situation. Therefore, sequential optimality in the responding states requires the acceptance levels to be equal to what we shall call *dynamic trader types*[8]

$$\tilde{v}(v) \equiv v - W_B(v), \tag{1}$$
$$\tilde{c}(c) \equiv c + W_S(c). \tag{2}$$

Turning to the sequential optimality in proposing states. Our dynamic type functions $\tilde{v}(v)$ and $\tilde{c}(c)$ allow us to characterize the proposing strategies in a simple manner. To this end, it is useful to consider the distributions of traders' dynamic types, denoted as

$$\tilde{\Phi}(x) \equiv \int_{\tilde{v}(v)\leq x} d\Phi(v), \tag{3}$$
$$\tilde{\Gamma}(x) \equiv \int_{\tilde{c}(c)\leq x} d\Gamma(c). \tag{4}$$

Consider the situation where a type v buyer is in a proposing state and suppose sellers use their equilibrium responding policy characterized by $\tilde{c}(c)$ and sellers' distribution is at the equilibrium value Γ. If the buyer propose λ (one can think λ as a one-shot deviation) and this offer is accepted, her continuation payoff will be $v-\lambda$; and if her offer is rejected, she will be back to the inactive state immediately and her continuation payoff would be $W_B(v)$. Therefore, her continuation payoff in a proposing state, conditional on proposing λ, is

$$\int_{\tilde{c}(c)\leq\lambda} (v-\lambda)d\Gamma(c) + \int_{\tilde{c}(c)>\lambda} W_B(v)d\Gamma(c),$$

which can be rewritten as

$$\tilde{\Gamma}(\lambda)[\tilde{v}(v)-\lambda] + W_B(v).$$

Only the first term, which is the "capital gain part", depends on λ. Similar logic applies to sellers' situation. It is clear that sequential optimality in the proposing states is satisfied if and only if

$$p_B(v) \in \arg\max_{\lambda} \tilde{\Gamma}(\lambda)[\tilde{v}(v)-\lambda], \tag{5}$$
$$p_S(c) \in \arg\max_{\lambda} [1-\tilde{\Phi}(\lambda)][\lambda-\tilde{c}(c)]. \tag{6}$$

It follows that the equilibrium proposing policies are determined as best-responses in the static monopoly problems where the distributions of responders' types are replaced by the distributions of the responders' dynamic types and the proposers' types are replaced by the proposers' dynamic types. As we have seen, this principle applies to the responding policies as well. In general, the bargainers behave as if they are in a one-shot game with their types replaced by their dynamic types. Intuitively, trading with current partner lead a trader to give up the opportunity of searching and trading with another partner. Our dynamic type

[8] The notion of dynamic trader types is due to Satterthwaite and Shneyerov (2007).

notions are simply adjusted with the traders' opportunity cost of further searching. This observation plays a very important role in both intuition and proofs of our results.

Turn to the matched state. Suppose that all traders always use their prescribed equilibrium strategies, $\{\chi_B, \chi_S, p_B, p_S, \tilde{v}, \tilde{c}\}$ and that the stationary distributions of active seller and buyer types are at their equilibrium values Γ and Φ. Then a type v buyer's expected bargaining surplus from the meeting is equal to

$$\Pi_B(v) \equiv (1-\alpha) \int_{\tilde{c}(c) \leq p_B(v)} [v - p_B(v)] d\Gamma(c) + \alpha \int_{p_S(c) \leq \tilde{v}(v)} [v - p_S(c)] d\Gamma(c). \tag{7}$$

Further denote

$$q_B(v) \equiv (1-\alpha) \int_{\tilde{c}(c) \leq p_B(v)} d\Gamma(c) + \alpha \int_{p_S(c) \leq \tilde{v}(v)} d\Gamma(c), \tag{8}$$

the buyer's probability of a successful trade in a given meeting. With probability $1 - q_B(v)$, the bargaining turn unsuccessful. The buyer's Markov chain then moves to the inactive state, giving a continuation payoff $W_B(v)$.

Now suppose a type v buyer chooses to enter, she has to wait and stay in the searching state until the next meeting. Since the buyer's waiting time before her next meeting is exponentially distributed with mean $1/\ell_B$,[9] the discounted value of one dollar to be received at the time of next meeting is equal to

$$R_B(\zeta) \equiv \int_{t=0}^{\infty} e^{-rt} d(1 - e^{-\ell_B(\zeta)t}) = \frac{\ell_B(\zeta)}{r + \ell_B(\zeta)}. \tag{9}$$

Similarly, the accumulated discounted participation cost over the period until next meeting is equal to

$$K_B(\zeta) \equiv \int_{t=0}^{\infty} \left(\int_0^t \kappa_B e^{-rx} dx \right) d(1 - e^{-\ell_B(\zeta)t}) = \frac{\kappa_B}{r + \ell_B(\zeta)}. \tag{10}$$

Then the searching state continuation payoff, provided that the type v buyer enters, is

$$R_B(\zeta)[\Pi_B(v) + (1 - q_B(v))W_B(v)] - K_B(\zeta).$$

Since the entry decision is made in the inactive state and the trader gets 0 if she exits, the inactive state continuation payoff, $W_B(v)$, must satisfy the following recursive equation:

$$W_B(v) = \max\{R_B(\zeta)[\Pi_B(v) + (1 - q_B(v))W_B(v)] - K_B(\zeta), 0\} \tag{11}$$

where the first maximand represents the payoff for entry, the second represents the payoff for exiting. Solve (11) for $W_B(v)$, we obtain an equivalent ratio-form formula:

$$W_B(v) = \max\left\{ \frac{\ell_B(\zeta)\Pi_B(v) - \kappa_B}{r + \ell_B(\zeta)q_B(v)}, 0 \right\}.$$

Therefore, the buyers' sequentially optimal entry policy in the inactive state is

[9] That is, the distribution function of waiting time t is $1 - \exp(-\ell_B t)$.

$$\chi_B(v) = I\left\{\ell_B(\zeta)\Pi_B(v) \geq \kappa_B\right\} \tag{12}$$

where $I(\cdot)$ is the indicator function. Note that (12) implicitly assumes that traders enter if they are indifferent between entering or not. This is only for expositional simplicity because it turns out that the set of such indifferent traders is of measure 0.

Complete parallel logic applies to the sellers' side. We can define Π_S, q_S, R_S and K_S similarly:

$$\Pi_S(c) = \alpha \int_{\tilde{v}(v) \geq p_S(c)} [p_S(c) - c]\, d\Phi(v) + (1-\alpha) \int_{p_B(v) \geq \tilde{c}(c)} [p_B(v) - c]\, d\Phi(v) \tag{13}$$

$$q_S(c) = \alpha \int_{\tilde{v}(v) \geq p_S(c)} d\Phi(v) + (1-\alpha) \int_{p_B(v) \geq \tilde{c}(c)} d\Phi(v) \tag{14}$$

$$R_S(\zeta) = \frac{\ell_S(\zeta)}{r + \ell_S(\zeta)}, \quad K_S(\zeta) = \frac{\kappa_S}{r + \ell_S(\zeta)}. \tag{15}$$

Then we have the recursive equation for W_S:

$$W_S(c) = \max\left\{R_S(\zeta)[\Pi_S(c) + (1 - q_S(c))W_S(c)] - K_S(\zeta), 0\right\}, \tag{16}$$

and the sellers' sequentially optimal entry policy in the inactive state is

$$\chi_S(c) = I\left\{\ell_S(\zeta)\Pi_S(c) \geq \kappa_S\right\}. \tag{17}$$

This completes the description of the strategic part of a nontrivial steady-state equilibrium. To complete the description of nontrivial steady-state equilibrium, we turn to the steady state equations for the distributions of active buyer and seller types Φ and Γ and active trader masses B and S. In a steady-state market equilibrium, the inflow rate of every types of traders must be equal to the corresponding outflow rate. Therefore,

$$b \int_v^1 \chi_B(x) dF(x) = B\ell_B(\zeta) \int_v^1 q_B(x) d\Phi(x) \;\; \forall v \in [0,1] \tag{18}$$

$$s \int_0^c \chi_S(x) dG(x) = S\ell_S(\zeta) \int_0^c q_S(x) d\Gamma(x) \;\; \forall c \in [0,1]. \tag{19}$$

These preparations allow us to formally define nontrivial steady-state equilibrium as follows.

Definition 1 *A collection $E \equiv \{\chi_B, \chi_S, p_B, p_S, \tilde{v}, \tilde{c}, B, S, \Phi, \Gamma\}$ is a nontrivial steady-state equilibrium if there exists a pair of equilibrium payoff functions $\{W_B, W_S\}$ such that the proposing strategies p_B and p_S, responding strategies $\tilde{v}$ and $\tilde{c}$, entry strategies χ_B and χ_S satisfy the sequential optimality conditions* (5), (6), (1), (2), (12) *and* (17)*, and the distributions of active buyer and seller types Φ and Γ and active trader masses B and S solve the steady-state equations* (18) *and* (19)*, and the payoff functions W_B and W_S solve the recursive equations* (11) *and* (16).

Remark 2 *Although we implicitly assume that traders use symmetric pure strategies, this is merely for simplicity of exposition. At a cost in notation we could define trader-specific and mixed strategies and then prove that they must be (essentially) symmetric and pure because of independence, anonymity in matching, and monotonicity (proved below) of strategies. To see this, first consider the implication of independence and anonymous matching for buyers. Even if different traders follow distinct strategies, every buyer with the same type v would still face the same market environment. (This is strictly true because we assume a continuum of traders.) Therefore, for a given value v, every buyers will have the identical continuation payoff, implying essentially identical responding and entry strategies. Moreover, every buyers have identical best-response correspondence for proposing strategy. We show below that every selection from this correspondence is nondecreasing; consequently, the best-response is pure apart from a measure zero set of values where jumps occur. These jump points are the only points where mixing can occur. But because their measure is zero, the mixing has no consequence for the maximization problems of the other traders. The same logic applies to sellers.*

Our characterization of equilibria begins with showing that the equilibrium utilities $W_B(v)$ and $W_S(c)$ are necessarily nondecreasing and nonincreasing respectively. Then, since the marginal entering types $\underline{v}$ and $\bar{c}$ must recover their participation costs, it follows that the sets of active types A_B and A_S must be intervals, $A_B = [\underline{v}, 1]$ and $A_S = [0, \bar{c}]$ (recall that we resolve the ties of the marginal types by requiring them to enter).

Lemma 3 *In any nontrivial steady-state equilibrium, $W_B(v)$ and $W_S(c)$ are absolutely continuous and convex. $W_B(v)$ is nondecreasing and $W_S(c)$ is nonincreasing. Moreover,*

$$W_B(v) = \int_{\underline{v}}^{v} \frac{\ell_B q_B(x)}{r + \ell_B q_B(x)} dx \quad \text{for all } v \in [\underline{v}, 1] \tag{20}$$

$$W_S(c) = \int_{c}^{\bar{c}} \frac{\ell_S q_S(x)}{r + \ell_S q_S(x)} dx \quad \text{for all } c \in [0, \bar{c}]\,. \tag{21}$$

Corollary 4 *(a) $A_B = [\underline{v}, 1]$ and $A_S = [0, \bar{c}]$. (b) $q_B(v)$ is nondecreasing in v, while $q_S(c)$ is nonincreasing in c.*

Next, since the derivatives $W_B'(v) \in [0, 1)$ and $W_S'(c) \in (-1, 0]$, Lemma 3 implies that the acceptance strategies $\tilde{v}$ and $\tilde{c}$ (dynamic types) must be nondecreasing.

Lemma 5 *In any nontrivial steady-state equilibrium, the acceptance strategies $\tilde{v}(v) = v - W_B(v)$ and $\tilde{c}(c) = c + W_S(c)$ are absolutely continuous and nondecreasing respectively. The slopes of the acceptance strategies are*

$$\tilde{v}'(v) = \frac{r}{r + \ell_B q_B(v)} \quad (a.e.\ v \in A_B) \tag{22}$$

$$\tilde{c}'(c) = \frac{r}{r + \ell_S q_S(v)} \quad (a.e.\ c \in A_S) \tag{23}$$

Moreover, if $r > 0$, then the acceptance strategies are strictly increasing on A_B and A_S; if $r = 0$, then $\tilde{v}(\cdot)$ and $\tilde{c}(\cdot)$ are constant on A_B and A_S.

It can also be shown (in the Appendix) that the proposing strategies p_B and p_S must be nondecreasing as well.

Lemma 6 *In any nontrivial steady-state equilibrium, the proposing policies $p_B(v)$ and $p_S(c)$ are nondecreasing on A_B and A_S respectively.*

Since the dynamic opportunity costs of trading for marginal entering types of traders are zero (i.e. $W_B(\underline{v}) = W_S(\bar{c}) = 0$), we can see that the marginal entering types are equal to the corresponding dynamic types:

$$\bar{c} = \tilde{c}(\bar{c}), \quad \underline{v} = \tilde{v}(\underline{v}).$$

The sellers' minimum acceptable price $\underline{c}$ and the buyers' maximum acceptable price $\bar{v}$ are defined by:

$$\begin{aligned} \underline{c} &\equiv \inf_c \{\tilde{c}(c) : c \in A_S\} = \tilde{c}(0) \\ \bar{v} &\equiv \sup_v \{\tilde{v}(v) : v \in A_B\} = \tilde{v}(1), \end{aligned}$$

which, taken together define what we call the *acceptance interval* $[\underline{c}, \bar{v}]$. The smallest and largest offers by buyers and sellers are

$$\begin{aligned} \underline{p}_B &\equiv \inf_v \{p_B(v) : v \in A_B\} = p_B(\underline{v}), \\ \bar{p}_B &\equiv \sup_v \{p_B(v) : v \in A_B\} = p_B(\bar{v}), \\ \underline{p}_S &\equiv \inf_c \{p_S(c) : c \in A_S\} = p_S(\underline{c}), \\ \bar{p}_S &\equiv \sup_c \{p_S(c) : c \in A_S\} = p_S(\bar{c}). \end{aligned}$$

We define the *price interval* as $[\underline{p}_B, \bar{p}_S]$.

It is not too hard to see that, in order for the trade flows to be balanced in steady state, the marginal entering types $\underline{v}$ and $\bar{c}$ must be on different sides of the Walrasian price p^*, and that, p^* must always fall within the acceptance interval, i.e. $p^* \in (\underline{c}, \bar{v})$.

The following lemma (proved in Appendix) further describes the patterns of equilibrium strategies.

Lemma 7 *In any nontrivial steady-state equilibrium, $\tilde{c}(c) < p_S(c)$ and $p_B(v) < \tilde{v}(v)$ for all $c \in [0, \bar{c}]$ and all $v \in [\underline{v}, 1]$. (They imply $\underline{p}_B < \underline{v}$ and $\bar{c} < \bar{p}_S$.) Moreover, if $r > 0$, then $\underline{c} < \underline{v} \leq \underline{p}_S \leq \bar{p}_S < \bar{v}$ and $\underline{c} < \underline{p}_B \leq \bar{p}_B \leq \bar{c} < \bar{v}$, while if $r = 0$, then $\underline{c} < \underline{v} = \underline{p}_S = \bar{p}_S = \bar{v}$ and $\underline{c} = \underline{p}_B = \bar{p}_B = \bar{c} < \bar{v}$.*

In particular, in equilibrium, the buyers' offers must be lower than their dynamic opportunity valuation, and the sellers' offers must be higher than their dynamic opportunity cost. Moreover, buyers never propose anything below the lowest acceptable price of sellers $\underline{c}$, and sellers never propose anything above the highest acceptable price of buyers $\bar{v}$. In other words, $\underline{c} \leq p_B(v) < \tilde{v}(v)$ and $\bar{v} \geq p_S(c) > \tilde{c}(c)$. Furthermore, in order for the marginal entrants to recover participation costs, we must also have $\underline{c} < \underline{v}$ and $\bar{c} < \bar{v}$. Figure 2 visualizes the proposing and responding strategies of an equilibrium.

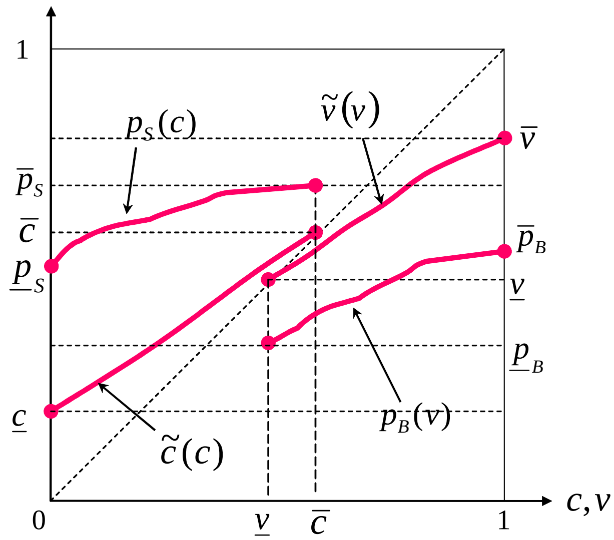

Figure 2: A non-full-trade equilibrium

3 Full-trade equilibria

The following important lemma gives participation conditions for the marginal types.

Lemma 8 *In any nontrivial steady-state equilibrium,* $\underline{p}_B = p_B(\underline{v})$*,* $\bar{p}_B = p_B(1)$*,* $\underline{p}_S = p_S(0)$*, and* $\bar{p}_S = p_S(\bar{c})$*. Moreover,*

$$\ell_B(\zeta)(1-\alpha)\tilde{\Gamma}(\underline{p}_B)(\underline{v}-\underline{p}_B) = \kappa_B \tag{24}$$

$$\ell_S(\zeta)\alpha[1-\tilde{\Phi}(\bar{p}_S)](\bar{p}_S-\bar{c}) = \kappa_S. \tag{25}$$

In the left-hand sides of equations (24) and (25) in the lemma we have marginal traders' expected profits from trading, gross of participation costs, over a short period dt, divided by the length of the period. To see the intuition behind equation (24), note that a marginal participating buyer $\underline{v}$ makes positive profit only if he meets a seller, proposes, and his offer is accepted (the combined probability is $\ell_B \cdot (1-\alpha) \cdot \tilde{\Gamma}(\underline{p}_B)$), and conditional on that, the profit is equal to the difference between his valuation and the price he proposes, $\underline{v} - \underline{p}_B$. Similar logic applies to equation (25).

There are two qualitatively different possibilities. First, it can be that at least one of the trading probabilities $\tilde{\Gamma}(\underline{p}_B)$ or $1-\tilde{\Phi}(\bar{p}_S)$ is less than 1. We call such an equilibrium a *non-full-trade equilibrium* because not every meeting results in a trade. The bargaining outcome in this class of equilibria is not ex-post efficient, in the sense that there are buyer-seller pairs with positive matching surplus (i.e. $v - c > W_B(v) + W_S(c)$ or equivalently $\tilde{v}(v) > \tilde{c}(c)$) who do not trade when they meet. If, for example, $\tilde{\Gamma}(\underline{p}_B) < 1$, then the buyers with types in a right-neighborhood of $\underline{v}$ do not trade with the sellers for whom $\tilde{c}(c) \in (\underline{p}_B, \underline{v}]$. An equilibrium of this kind is shown in Figure 2.

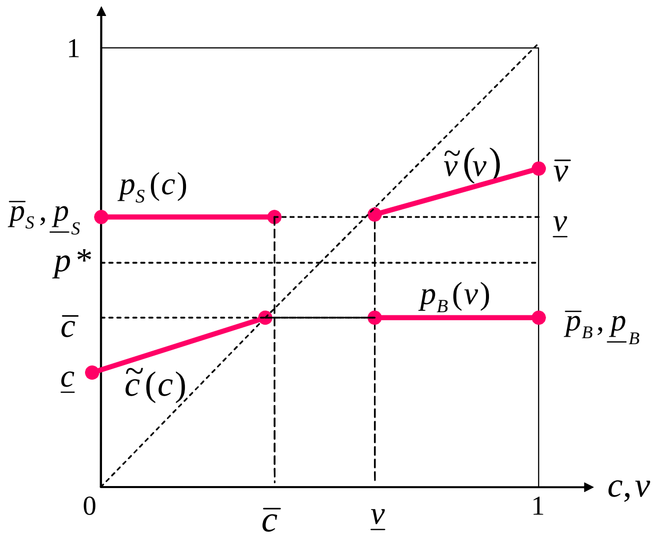

Figure 3: A full-trade equilibrium under take-it-or-leave-it offering

It may happen that the supports of the types in the market are separated, so that the marginal entrants trade with probability 1, i.e. $\tilde{\Gamma}(\underline{p}_B) = 1 - \tilde{\Phi}(\bar{p}_S) = 1$. We call such equilibria *full-trade equilibria*. Lemmas 7 and 8 imply that full-trade equilibria must have the following properties: (i) the supports for active buyers' types and active sellers' types are separate, i.e. $\underline{v} > \bar{c}$; (ii) the lowest buyer's offer $\underline{p}_B$ is exactly at the offer acceptable to all active sellers, i.e. $\underline{p}_B = \bar{c}$; and (iii) the highest seller's offer $\bar{p}_S$ is exactly at the offer acceptable to all active buyers: $\bar{p}_S = \underline{v}$. It is easy to see that the converse is also true. Thus we alternatively define full-trade equilibrium to be a nontrivial steady-state equilibrium with $\underline{p}_B = \bar{c}$ and $\bar{p}_S = \underline{v}$. An example of such an equilibrium is shown in Figure 3.

A full-trade equilibrium admits a simple characterization. Conditions (24) and (25) of Lemma 8 take the form

$$\ell_B(\zeta)(1-\alpha)\left(\underline{v} - \bar{c}\right) = \kappa_B, \tag{26}$$

$$\ell_S(\zeta)\alpha\left(\underline{v} - \bar{c}\right) = \kappa_S. \tag{27}$$

Noticing that $\ell_S(\zeta)/\ell_B(\zeta) = \zeta$, the entry equations (26) and (27) can be easily solved for ζ and $\underline{v} - \bar{c}$:

$$\zeta = \frac{1-\alpha}{\alpha}\frac{\kappa_S}{\kappa_B} \equiv z, \tag{28}$$

$$\underline{v} - \bar{c} = \frac{\kappa_B}{\ell_B(z)} + \frac{\kappa_S}{\ell_S(z)} \equiv K(z). \tag{29}$$

To complete the description of a full-trade equilibrium, note that in a steady state, the incoming flow of active buyers must equal the incoming flow of active sellers:

$$b[1 - F(\underline{v})] = sG(\bar{c}). \tag{30}$$

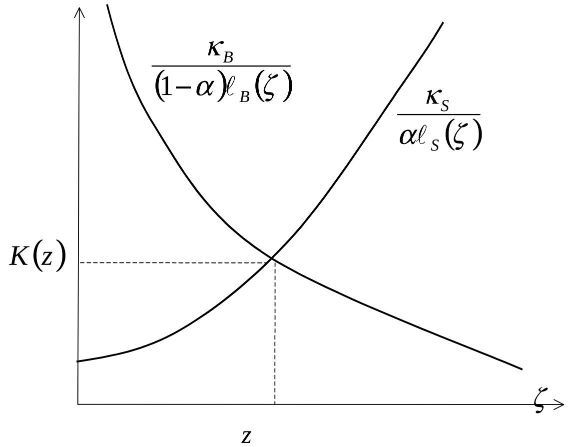

Figure 4: Interpretation of z and $K(z)$

Since $\underline{v}-\bar{c}$ are determined, $\underline{v}$ and $\bar{c}$ are uniquely pinned down by (30). Full-trade equilibrium, if exists, is uniquely characterized by equations (26), (27), and (30).[10]

It is clear from (29) that $K(z) < 1$ is a necessary and sufficient condition for existence of a solution to equations (28) – (30). The function

$$K(\zeta) = \frac{\kappa_B}{\ell_B(\zeta)} + \frac{\kappa_S}{\ell_S(\zeta)}$$

will play an important role in our analysis. It can be interpreted as the expected participation cost incurred by a buyer-seller pair (i.e. $\kappa_B/\ell_B(\zeta) + \kappa_S/\ell_S(\zeta)$) in a full-trade equilibrium until their meeting, if there is no discounting. A further insight about $K(z)$ is provided by the following lemma that will be used frequently in the proofs. The lemma shows that $K(z)$ can be interpreted either as the following maximin value or the minimax value of adjusted accumulated participation costs until the next meeting.

Lemma 9 *We have*

$$\begin{aligned} K(z) &= \max_{\zeta>0} \min \left\{ \frac{\kappa_B}{(1-\alpha)\ell_B(\zeta)}, \frac{\kappa_S}{\alpha\ell_S(\zeta)} \right\} \\ &= \min_{\zeta>0} \max \left\{ \frac{\kappa_B}{(1-\alpha)\ell_B(\zeta)}, \frac{\kappa_S}{\alpha\ell_S(\zeta)} \right\}. \end{aligned}$$

Proof. Consult Figure 4. Note that $\ell_B(\zeta)$ is a nonincreasing function, while $\ell_S(\zeta)$ is an nondecreasing function. The maximin and minimax values are realized at the intersection of the curves

$$\frac{\kappa_B}{(1-\alpha)\ell_B(\zeta)} = \frac{\kappa_S}{\alpha\ell_S(\zeta)}$$

which occurs if and only if $\zeta = z$. Q.E.D.

[10] Other endogenous variables are easily obtained. In particular, for $v \in A_B$ and $c \in A_S$, $W_B(v) = \frac{\ell_B(z)}{r+\ell_B(z)}(v-\underline{v})$, $W_S(c) = \frac{\ell_S(z)}{r+\ell_S(z)}(\bar{c}-c)$, $\Phi(v) = \frac{F(v)-F(\underline{v})}{1-F(\underline{v})}$ and $\Gamma(c) = \frac{G(c)}{G(\bar{c})}$.

But even if $K(z) < 1$, so that a solution to equations (28) – (30) exists, it may not characterize an equilibrium, since buyers may have an incentive to bid higher than $\bar{c}$, and similarly sellers may have an incentive to bid below $\underline{v}$. To rule out such deviations, we need additional conditions. Denote the virtual trader types as J_B and J_S:

$$J_B(v) = v - \frac{1 - F(v)}{f(v)}, \quad J_S(c) = c + \frac{G(c)}{g(c)}$$

Assuming that the virtual types are increasing functions, an assumption commonly made in the literature, we are able to show (in the Appendix) that the following necessary and sufficient conditions for a solution to equations (28) – (30) to characterize a full-trade equilibrium:

$$\begin{aligned} r &\leq \min\left\{\frac{\ell_B(z)(\underline{v} - \bar{c})}{\max\{\bar{c} - J_B(\underline{v}), 0\}}, \frac{\ell_S(z)(\underline{v} - \bar{c})}{\max\{J_S(\bar{c}) - \underline{v}, 0\}}\right\} \\ &\equiv r^*. \end{aligned} \tag{31}$$

(If both denominators are 0, there is no upper bound so a full-trade equilibrium exists for all $r \geq 0$.) Since the marginal types $\underline{v}$ and $\bar{c}$ in a full-trade equilibrium do not depend on the discount rate r, we can see that these conditions will be satisfied when $r \geq 0$ is sufficiently small.

We are also able to show that a full-trade equilibrium is a unique equilibrium for small $r > 0$. That is to say, there cannot be a non-full-trade equilibrium when r is small. The proof of this is based on the following lemma. This lemma proves that one important property of the full-trade equilibrium, that $K(z)$ separate the entry gap $\underline{v} - \bar{c}$ (if any) and the length of the acceptance interval $\bar{v} - \underline{c}$, carries over to all equilibria.

Lemma 10 *In any nontrivial steady-state equilibrium, we have*

$$\bar{v} - \underline{c} \geq K(z), \tag{32}$$

$$\underline{v} - \bar{c} \leq K(z). \tag{33}$$

The first inequality (32) *is strict if* $r > 0$*.*

Proof. Since $\underline{c} \leq \underline{p}_B < \underline{v} \leq \bar{v}$ and $\bar{v} \geq \bar{p}_S > \bar{c} \geq \underline{c}$, it follows from the entry conditions (24) and (25) that

$$\begin{aligned} (1-\alpha)\ell_B(\bar{v} - \underline{c}) &\geq \kappa_B, \\ \alpha\ell_S(\bar{v} - \bar{c}) &\geq \kappa_S, \end{aligned}$$

so that

$$\bar{v} - \underline{c} \geq \max\left\{\frac{\kappa_B}{(1-\alpha)\ell_B}, \frac{\kappa_S}{\alpha\ell_S}\right\} \geq K(z).$$

This proves (32). If $r > 0$, we have $\bar{v} > \underline{v}$ and $\bar{c} - \underline{c}$, which make (32) strict. (33) is proved by applying a revealed-preference argument to the same entry conditions (24) and (25). Consider the deviations in which the $\underline{v}$-buyers offer $\bar{c}$ and $\bar{c}$-sellers offer $\underline{v}$:

$$\begin{aligned} (1-\alpha)\ell_B(\underline{v} - \bar{c}) &\leq \kappa_B, \\ \alpha\ell_S(\underline{v} - \bar{c}) &\leq \kappa_S, \end{aligned}$$

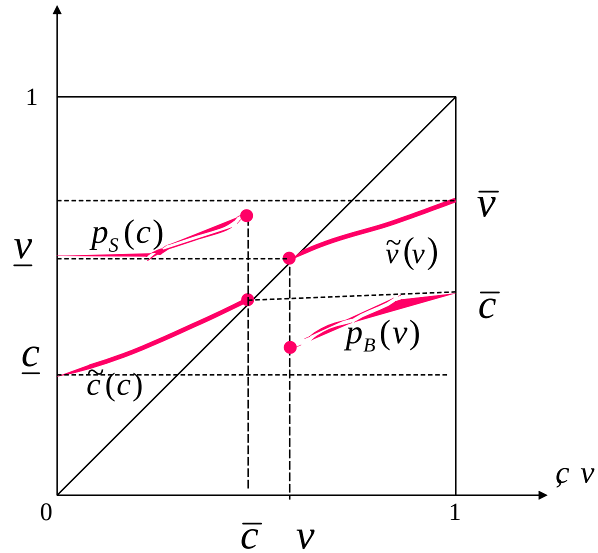

Figure 5: A full-trade equilibrium with too little entry

from which it follows that

$$\underline{v} - \bar{c} \leq \min\left\{\frac{\kappa_B}{(1-\alpha)\,\ell_B}, \frac{\kappa_S}{\alpha \ell_S}\right\} \leq K(z).$$

Q.E.D.

From Lemma 8, $\ell_B q_B(\underline{v}) \geq \kappa_B$ and $\ell_S q_S(\bar{c}) \geq \kappa_S$, and these inequalities continue to hold for all participating types because q_B (resp. q_S) is nondecreasing (resp. nonincreasing) function. Lemma 5 then implies that the slopes of acceptance strategies are bounded from above as follows:

$$\tilde{v}'(v) \leq \frac{r}{\kappa_B + r}, \qquad \tilde{c}'(c) \leq \frac{r}{\kappa_S + r}, \tag{34}$$

and therefore converge to 0 as $r \to 0$,

$$\lim_{r\to 0} \tilde{v}'(v) = \lim_{r\to 0} \tilde{c}'(c) = 0. \tag{35}$$

Lemma 10 then implies that $\bar{v} - \underline{c}$ and $\underline{v} - \bar{c}$ converge to a common limit $K(z)$.

Corollary 11 *We have*

$$\lim_{r\to 0}(\bar{v} - \underline{c}) = \lim_{r\to 0}(\underline{v} - \bar{c}) = K(z).$$

Now it is useful to introduce yet another level of equilibria classification. A non-full-trade equilibrium may either have too much entry relative to the Walrasian benchmark, $\underline{v} < \bar{c}$ (as shown in Figure 2), or too little entry, $\underline{v} > \bar{c}$ (as shown in Figure 5). Corollary 11 implies that a non-full-trade equilibrium with too much entry cannot exist when r is small.

The proof that a non-full-trade equilibrium with too little entry cannot exist is based on the following idea (the details are in the Appendix.) As $r \to 0$, it follows from (35) that the support of dynamic types narrows down to a singleton. Consequently, a marginal participating trader whose offer is in the interior of the support of the bargaining partner gains relatively little vis-a-vis proposing at the boundary of the support (i.e. seller offering $\underline{v}$ and buyer offering $\bar{c}$), but risks a substantially reduced probability of trading. We are able to show that bidding the endpoint of the support is the dominant choice, so for small r it must be that $\underline{p}_B = \bar{c}$ and $\bar{p}_S = \underline{v}$. We therefore have proven the following uniqueness result.

Proposition 12 (Existence and uniqueness of full-trade equilibrium) *If $K(z) < 1$, then $\underline{r} > 0$ exists such that for all $r \in [0, \underline{r}]$ there is a unique equilibrium and it is full-trade.*[11]

Define Walrasian price p^* as the price that clears the flows of the arriving cohorts:

$$b[1 - F(p^*)] = sG(p^*),$$

It is immediate from the characterizing equations (28) – (30) that full-trade equilibria converge to perfect competition as the frictions of time discounting and participation cost are removed.[12] Specifically, the marginal participating types converge to p^* and the acceptance interval (and hence the price interval) converges to $\{p^*\}$. Since full-trade equilibria are the only equilibria when r is small, it also follows that all equilibria of our model converge to perfect competition. We state this as a corollary.

Corollary 13 (Convergence to perfect competition) *As the frictions of participation cost and discounting are removed, all equilibria of our model converge to perfect competition, i.e.*

$$\begin{aligned} \lim_{(\kappa_B,\kappa_S)\to\mathbf{0}} \lim_{r\to 0} \underline{v} &= \lim_{(\kappa_B,\kappa_S)\to\mathbf{0}} \lim_{r\to 0} \bar{c} = p^*, \\ \lim_{(\kappa_B,\kappa_S)\to\mathbf{0}} \lim_{r\to 0} [\underline{c}, \bar{v}] &= \lim_{(\kappa_B,\kappa_S)\to\mathbf{0}} \lim_{r\to 0} [\underline{p}_B, \bar{p}_S] = \{p^*\}. \end{aligned}$$

When $r > r^*$, then a full-trade equilibrium does not exist. Note that it is possible that $r^* = \infty$ so that a full-trade equilibrium exists for all r. This may happen (as our example shows) when the search costs κ_B and κ_S are so large (but not larger than $K(z)$) that the entry gap is $\underline{v} - \bar{c}$ is sufficiently large so that both $\bar{c} - J_B(\underline{v}) < 0$ and $J_S(\bar{c}) - \underline{v} < 0$ and therefore from (31), $r^* = \infty$. On the other hand, it can also happen that $r^* < \infty$ so that a full-trade equilibrium does not exist for large r. We can easily see that this will happen if the search costs κ_B, κ_S are sufficiently small. Even stronger, we can show that $\lim_{(\kappa_B,\kappa_S)\to\mathbf{0}} r^* = 0$.

[11] Recall that for expositional simplicity we have assumed that the types are distributed on $[0,1]$. If the support were $[a_1, a_2]$, then the condition would read $K(z) < a_2 - a_1$.

[12] This interpretation of frictions is due to Mortensen and Wright (2002).

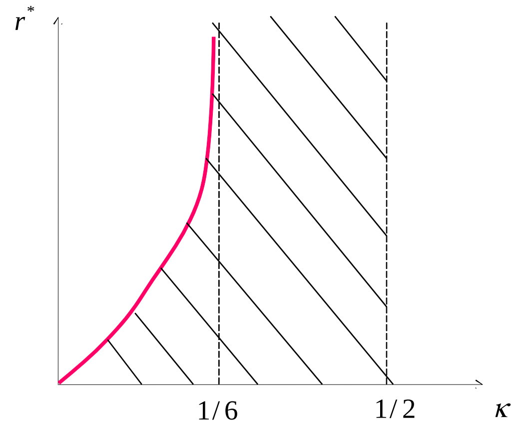

Figure 6: The values of r and κ for which a full-trade equilibrium exists in example 14 are shown by the shaded area

In a full-trade equilibrium, the entry gap $\underline{v}-\bar{c}=K(z)$, so that $\lim_{(\kappa_B,\kappa_S)\to\mathbf{0}}(\underline{v}-\bar{c})=0$. Because both $\underline{v}$ and $\bar{c}$ converge to the Walrasian price p^*, it follows that $J_B(\underline{v})\to p^*-(1-F(p^*))/f(p^*)$ and $J_S(\bar{c})\to p^*+G(p^*)/g(p^*)$. Consequently,

$$\begin{aligned}\bar{c}-J_B(\underline{v}) &\rightarrow (1-F(p^*))/f(p^*),\\ J_S(\bar{c})-\underline{v} &\rightarrow G(p^*)/g(p^*),\end{aligned}$$

and it follows from (31) that as $(\kappa_B,\kappa_S)\to\mathbf{0}$,

$$\frac{r^*}{\underline{v}-\bar{c}}\to\min\left\{\frac{\ell_B(z)f(p^*)}{1-F(p^*)},\frac{\ell_S(z)g(p^*)}{G(p^*)}\right\}.$$

Consequently, $\lim_{(\kappa_B,\kappa_S)\to\mathbf{0}}r^*=0$, so that given any $r>0$, a full-trade equilibrium does not exist when the search costs κ_B,κ_S are sufficiently small.

An example may help understand these points better.

Example 14 *Buyers and sellers are born at the same rate, which is normalized to be 1, i.e. $b=s=1$. The values and costs are uniform $[0,1]$ distributed, i.e. $F(v)=v$, $G(c)=c$. The bargaining power is evenly distributed, i.e. $\alpha=1/2$; and the search costs of buyers and sellers are also the same: $\kappa_S=\kappa_B\equiv\kappa$. The matching function is given by $M(B,S)=\min\{B,S\}$ (so that all traders who are searching at a given point in time are matched at the arrival rate 1 in this symmetric setting). One can check that the entry gap in a full-trade equilibrium is given by $\underline{v}-\bar{c}=2\kappa$ and the condition (31) takes the form $r\le 2\kappa/\max\{0.5-3\kappa,0\}$. A non-trivial equilibrium only exists if the entry gap is less than 1, i.e. $2\kappa<1$. The values of r and κ for which a full-trade equilibrium exists are shown in Figure 6.*

We collect all these findings in a theorem below, the formal proof of which is in the Appendix.

Theorem 15 *Assume that the virtual types $J_B(v)$ and $J_S(c)$ are increasing functions of their arguments. Then a full-trade equilibrium exists if and only if $K(z) < 1$, in which case there exists a unique solution to the characterizing equations $(28)-(30)$, and $r \leq r^*$ where r^* is given by (31). Moreover, an $\underline{r} > 0$ exists such that a full-trade equilibrium is the unique equilibrium of the model for $r \in [0, \underline{r}]$. If, on the other hand, $r > r^*$, then a full-trade equilibrium does not exist. In particular, given any $r > 0$, a $\bar{\kappa} > 0$ exists such that a full-trade equilibrium does not exist whenever κ_B, $\kappa_S < \bar{\kappa}$.*

4 Necessary and sufficient condition for no market breakdown

It is not hard to see that the condition $K(z) < 1$, a necessary condition for the existence of a full-trade equilibrium, is also necessary for existence of any nontrivial equilibrium of our model. Indeed, it is trivial if $r = 0$, in which case any nontrivial equilibrium is full-trade. On the other hand, if $r > 0$ and some nontrivial equilibrium exists, then Lemma 10 together with $\bar{v} - \underline{c} \leq 1$ implies the condition $K(z) < 1$.

Perhaps surprisingly, the condition $K(z) < 1$ is also sufficient for existence of a nontrivial equilibrium of our model.

Theorem 16 (No market breakdown) *A necessary and sufficient condition for existence of a nontrivial equilibrium is that $K(z) < 1$.*

Taken together with the last statement of Theorem 15, Theorem 16 implies that a non-full-trade equilibrium exists if r is sufficiently large and participation costs are sufficiently small.

Corollary 17 (Existence of a non-full-trade equilibrium) *$\bar{\kappa} > 0$, $\bar{r} > 0$ exist such that a non-full-trade equilibrium exists whenever $r > \bar{r}$ and κ_B, $\kappa_S < \bar{\kappa}$.*

This section is devoted to the main elements of the proof of Theorem 16. Our goal is to prove that there exists a tuple $(p_B, p_S, \tilde{v}, \tilde{c}, \chi_B, \chi_S, B, S, \Phi, \Gamma)$ of strategies, steady-state distributions and steady-state masses of traders, that satisfies our mathematical definition of nontrivial equilibrium of the take-it-or-leave-it offering model. However, in order to apply the fixed point theorem to do our job, it is much better to transform and reduce our space of equilibrium objects. Define $N_B : [0,1] \to \mathbb{R}_+$ and $N_S : [0,1] \to \mathbb{R}_+$ as the steady-state *unnormalized* distributions of buyers and sellers, i.e. $N_B(v) \equiv B\Phi(v)$ and $N_S(c) \equiv S\Gamma(c)$. Then we will take the tuple of payoffs and unnormalized distributions $(W_B, W_S, N_B, N_S) \equiv E$ as the primary configuration of equilibrium objects.

Indeed, our mathematical definition of a nontrivial equilibrium can be regarded as a fixed point of some mapping T that brings an initial configuration $E = (W_B, W_S, N_B, N_S)$ (from some appropriate domain) to a new configuration $E^* = (W_B^*, W_S^*, N_B^*, N_S^*)$. This mapping is defined as follows. First, we let

$$B = N_B(1),\ S = N_S(1),\ \Phi(v) = \frac{N_B(v)}{B},\ \Gamma(c) = \frac{N_S(c)}{S},\ \zeta = \frac{B}{S}. \tag{36}$$

Then determine the dynamic types $(\tilde{v}, \tilde{c})$ according to (1) and (2), and their distributions $\left(\tilde{\Phi}, \tilde{\Gamma}\right)$ according to (3) and (4). Next, we determine the best-response proposing strategies (p_B, p_S) according to (5) and (6), but whenever there are multiple best-responses, we use the maximal response for buyers and the minimal for sellers:

$$\begin{aligned} p_B(v) &= \sup\left\{\arg\max_{\lambda\in[0,1]} \tilde{\Gamma}(\lambda)[\tilde{v}(v)-\lambda]\right\} && (37) \\ p_S(c) &= \inf\left\{\arg\max_{\lambda\in[0,1]} [1-\tilde{\Phi}(\lambda)][\lambda-\tilde{c}(c)]\right\}. && (38) \end{aligned}$$

Having defined the proposing strategies, we can define and the expected profits (Π_B, Π_S) in a given meeting according to (7) and (13), as well as the probabilities of trading (q_B, q_S) according to (8) and (14). With those at hand, we can recover the resulting lifetime payoffs through their corresponding recursive equations, (11) and (16):

$$\begin{aligned} W_B^*(v) &= \max\left\{R_B(\zeta)[\Pi_B(v)+(1-q_B(v))W_B(v)]-K_B(\zeta),0\right\} && (39) \\ W_S^*(c) &= \max\left\{R_S(\zeta)[\Pi_S(c)+(1-q_S(c))W_S(c)]-K_S(\zeta),0\right\}. && (40) \end{aligned}$$

After that, we determine the best-response entry strategies as in (12) and (17), and finally determine the resultant steady-state distributions of types according to:

$$N_B^*(v) = \int_0^v \frac{\chi_B(x)b}{\ell_B(\zeta)q_B(x)}dF(x), \quad N_S^*(c) = \int_0^c \frac{\chi_S(x)s}{\ell_S(\zeta)q_S(x)}dG(x). \tag{41}$$

In the Appendix, some additional details and qualifications are provided to guarantee that this mapping is well-defined.

Our existence proof will be based on the Schauder fixed point theorem, which asserts that: if D is a nonempty compact convex subset of a Banach space and T is a continuous function from D to D, then T has a fixed point.

In order to make this theorem applicable, certain details need to be taken care of. The main difficulty is we need to make sure that as we apply the mapping T, we do not lose positive entry. To deal with this potential complication, we first prove existence of what we call an ε-equilibrium, which is an actual equilibrium in a ε-model in which positive entry always occurs because of an outside subsidy.

We modify our original model in three ways. Firstly, we add a subsidy that ensures that all buyers with type $v \geq 1-\varepsilon$ and all sellers with type $c \leq \varepsilon$ enter. In particular, every new-born trader is qualified to received a flow of subsidy for her market participation, provided that (i) her type satisfies $v \geq 1-\varepsilon$ or $c \leq \varepsilon$, and (ii) she would choose not to participate if no subsidization is available. Further, the flow rate of the subsidy for a qualified trader would be the least amount that is enough to make the trader voluntarily participate. That is, for example, a new-born buyer with type $v \geq 1-\varepsilon$ and $\ell_B(\zeta)\Pi_B(v) < \kappa_B(\zeta)$ will, conditional on entry, receive a flow amount $\kappa_B(\zeta) - \ell_B(\zeta)\Pi_B(v)$ per unit of time so that she is indifferent between entering or not. (We assume traders enter whenever indifferent.) Hence the entry conditions (12) and (17) are changed as:

$$\begin{aligned} \chi_B(v) &= I\left[\ell_B(\zeta)\Pi_B(v) \geq \kappa_B \text{ or } v \geq 1-\varepsilon\right] && (42) \\ \chi_S(c) &= I\left[\ell_S(\zeta)\Pi_S(c) \geq \kappa_S \text{ or } c \leq \varepsilon\right]. && (43) \end{aligned}$$

Because any subsidized traders are simply indifferent between entering or staying out, our equations for payoffs W_B, W_S, and bargaining strategies $\tilde{v}$, $\tilde{c}$, p_B, p_S do not need to be changed.

Although we now have a positive lower bound for the inflows of traders, we have not had a positive lower bound for the mass of traders in the market because the outflow rate (i.e. $\ell_B(\zeta)q_B(v)$ or $\ell_S(\zeta)q_S(c)$) could be potentially very large. Concerned with this, we impose the second modification, which ensures that the buyers' arrival rate $\ell_B(\zeta)$ and the sellers' arrival rate $\ell_S(\zeta)$ are bounded by $\bar{\ell}_B$ and $\bar{\ell}_S$. Specifically, given the original matching function $M(B,S)$, we replace it with a new one $\tilde{M}(B,S)$ defined as:

$$\tilde{M}(B,S) = \min\left\{M(B,S), B\bar{\ell}_B, S\bar{\ell}_S\right\}. \tag{44}$$

Notice that $\tilde{M}$ has all the properties as a matching function as long as M has. But now we make sure that

$$\ell_B(\zeta) \le \bar{\ell}_B, \;\; \ell_S(\zeta) \le \bar{\ell}_S. \tag{45}$$

While the first two modifications are added to make the mass of traders bounded from below, we also want it to be bounded from above, because our domain D need to be compact. It suffices to have a lower bound for the outflow rate ($\ell_B(\zeta)q_B(v)$ or $\ell_S(\zeta)q_S(c)$). For a type who chooses to enter without subsidization, there is naturally an upper bound for its mass because her expected trading surplus must be larger than her participation cost. More precisely, for an unsubsidized participating v-buyer, $\ell_B(\zeta)q_B(v) \ge \ell_B(\zeta)\Pi_B(v) \ge \kappa_B$. However, a subsidized buyer could have $\ell_B(\zeta)q_B(v) < \kappa_B$. Concerned with this, our third modification is, we disqualify subsidized traders in a way that ensures the outflow rates of subsidized types are at least κ_B or κ_S. In particular, the disqualification is a Poisson process, where the Poisson rate (which is contingent on type) is the least one that makes the outflow rate not lower than the lower bound κ_B or κ_S. That is, for example, a currently qualified v-buyer with $\ell_B(\zeta)q_B(v) < \kappa_B$ will be disqualified and exit immediately at a Poisson rate $\kappa_B - \ell_B(\zeta)q_B(v)$; while a currently qualified v-buyer with $\ell_B(\zeta)q_B(v) \ge \kappa_B$ will not be disqualified. Notice that for any type, either subsidized or unsubsidized, a v-buyer's gross outflow rate must be $\max\{\ell_B(\zeta)q_B(v), \kappa_B\}$. Therefore, the steady-state equations (41) are simply changed as:

$$N_B^*(v) = \int_0^v \frac{\chi_B(x)b}{\max\{\ell_B(\zeta)q_B(x), \kappa_B\}} dF(x) \tag{46}$$

$$N_S^*(c) = \int_0^c \frac{\chi_S(x)s}{\max\{\ell_S(\zeta)q_S(x), \kappa_S\}} dG(x). \tag{47}$$

It completes the descriptions of our ε-model.

In the Appendix, we show that our ε-model has at least one equilibrium, which we shall call an ε-equilibrium. Next, we prove that if $\varepsilon > 0$ is sufficiently small, then an ε-equilibrium is a true equilibrium of our model (this is Proposition 24 in the Appendix). The main idea of the proof can be illustrated graphically, see Figure 7.

First, as in Lemma 10, we show that in ε-equilibrium also, we must have $\underline{v} - \bar{c} \le K(z)$. Second, we show that the trading flows are almost balanced, the discrepancy bounded in absolute value by (a multiple of) ε. Imposing these constraints on the set of values $(\bar{c}, \underline{v})$, we obtain the set of feasible values given by the shaded area in the graph. As the

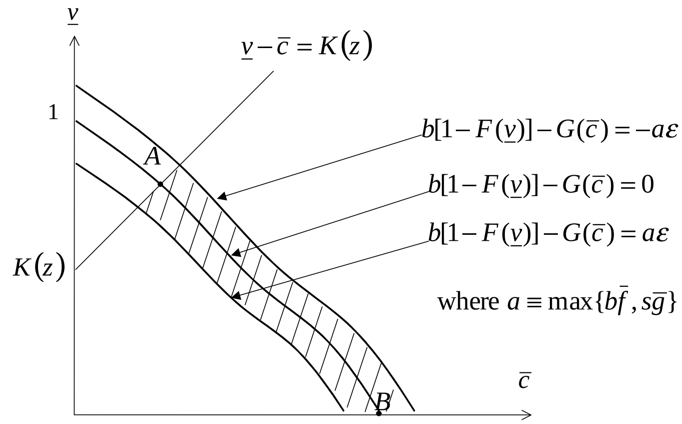

Figure 7: $\bar{c} > \varepsilon$ and $\underline{v} < 1 - \varepsilon$ for small ε

graph makes clear, the shaded area collapses to the curvilinear segment AB. Consequently, as ε gets arbitrarily small, the minimal $\bar{c}$ in the shaded area is arbitrarily close to the horizontal coordinate of point A, and the maximal feasible $\underline{v}$ is arbitrarily close to the vertical coordinate of A. It follows that for small enough $\varepsilon > 0$, the constraints $\bar{c} > \varepsilon$ and $\underline{v} < 1 - \varepsilon$ become non-binding and the ε-equilibrium becomes a true equilibrium of our model.

5 Comparison to the full information model of Mortensen and Wright (2002): private information may enhance welfare

Mortensen and Wright (2002; MW) consider a model that differs from ours only in one respect: MW assume full information, i.e. bargainers know each other's type. Consequently, proposers hold their partners to their reservation values (i.e., to their dynamic types), and the proposing strategies depend on both the trader's and his partner's type. In other words, for the buyers, the proposing strategy is $p_B(v, c) = \tilde{c}(c)$, if $\tilde{c}(c) \leq \tilde{v}(v)$, while it can be defined as any price less than $\tilde{v}(v)$ if $\tilde{c}(c) < \tilde{v}(v)$ (such a price will be rejected by the seller). Similarly, $p_S(v, c) = \tilde{v}(v)$ if $\tilde{c}(c) \leq \tilde{v}(v)$.

Even though there is no private information, not every meeting may result in a trade because it may be that $\tilde{v}(v) < \tilde{c}(c)$, so that the pair does not trade. But the same way as in our model, MW show existence of a full-trade equilibrium if r is sufficiently small (i.e., our Proposition 12 also holds assuming full information). Comparing their conditions for existence of a full-trade equilibrium, they show that, similar to our model, there is an upper bound such that a full-trade equilibrium exists if and only if the discount rate is below that bound. Unlike in our model, however, the bound is always binding (i.e. less than infinity). MW also suggest (but do not prove) that a non-full-trade equilibrium may exist.

We note that our general existence proof (Theorem 16) adapts with minor changes;

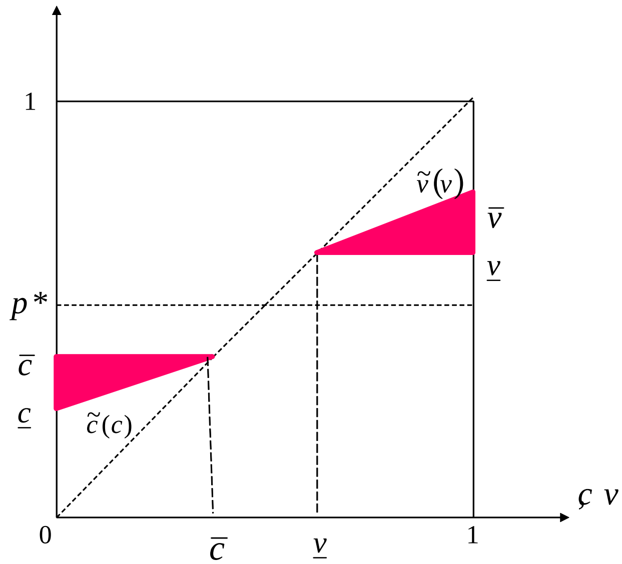

Figure 8: When types are publicly known, marginal participating types extract full rents from their partners

in particular the necessary and sufficient condition for existence of equilibrium (the no market breakdown condition) in a model with full information is the same as in our model: $K(z) < 1$.[13] The proof is even easier because we do not have to consider proposing strategies in our construction of the best-response mapping T. The only change in the definition of T is that the expected profits and trading probabilities are now

$$\Pi_B(v) = (1-\alpha)\int_{\tilde{v}(v)\geq\tilde{c}(c)}[v-\tilde{c}(c)]d\Gamma(c), \qquad q_B(v) = \int_{\tilde{v}(v)\geq\tilde{c}(c)} d\Gamma(c)$$

$$\Pi_S(c) = \alpha\int_{\tilde{v}(v)\geq\tilde{c}(c)}[\tilde{v}(v)-c]d\Phi(v), \qquad q_S(c) = \int_{\tilde{v}(v)\geq\tilde{c}(c)} d\Phi(v)$$

instead of (7), (8), (13) and (14).

The equilibrium in both models is unique for small $r \geq 0$.[14] This makes it feasible to compare the levels of social welfare under private and public information for such values of r. The marginal participating types $\underline{v}$ and $\bar{c}$ in the full-trade equilibria are equal in both models only when $r = 0$. When r increases away from 0, our marginal types do not change, while as MW show they move towards each other in their model. In other words, under complete information there is entry by extramarginal types.

To understand why this is so, note that, unlike in our model, under full information the marginal types extract full rents from the partners to whom they propose (see Figure

[13] The details of the proof are available on request.

[14] MW do not prove existence of a non-full-trade equilibrium in their model. We fill this gap by showing the changes that would be necessary to our existence proof to cover the case of public information.

8). In contrast, our marginal types are only able to extract the rents of the most inefficient partner type. As r increases away from 0, the distributions of the dynamic types become, ceteris paribus, more heterogeneous, and consequently there are more rents to be had by the marginal types. This creates incentives for the extramarginal types to enter. There are no such incentives in our model.

In the presence of matching externalities, more entry may or may not be socially desirable. Under private information, the slope of the welfare $W^p(r)$ as a function of r is

$$\frac{dW^p(0)}{dr} = -\frac{W_B^0}{\ell_B(z)} - \frac{W_S^0}{\ell_S(z)} \tag{48}$$

where

$$W_B^0 \equiv b\int_{\underline{v}}^1 (v-\underline{v})\,dF(v), \quad W_S^0 \equiv s\int_0^{\bar{c}} (\bar{c}-c)\,dG(c). \tag{49}$$

This is simply the direct effect of discounting. In particular, the effect of discounting on buyers' (resp. sellers') welfare is proportional to their expected waiting time $1/\ell_B$ (resp. $1/\ell_S$).

Under full information, on the contrary, the slope of the welfare $W^f(r)$ as a function of r, as shown in the Appendix, is

$$\frac{dW^f(0)}{dr} = -\frac{W_B^0}{\ell_B(z)} - \frac{W_S^0}{\ell_S(z)} - sG(\bar{c})\left[\kappa_B\eta_B(z) - \kappa_S\eta_S(z)\right]\zeta'(0) \tag{50}$$

where $\zeta'(0)$ is the derivative $\frac{d\zeta}{dr}$ under full information evaluated at $r=0$, and $\eta_B(\zeta)$ (resp. $\eta_S(\zeta)$) is the absolute value of the derivative of buyers' expected waiting time $1/\ell_B(\zeta)$ (resp. $1/\ell_S(\zeta)$) with respective to ζ, i.e.

$$\eta_B(\zeta) \equiv \frac{d}{d\zeta}\left[\frac{1}{\ell_B(\zeta)}\right] > 0, \quad \eta_S(\zeta) \equiv -\frac{d}{d\zeta}\left[\frac{1}{\ell_S(\zeta)}\right] > 0.$$

Other than the direct effect, the increase in r away from 0, by inducing additional entry, could increase or decrease the buyer-seller ratio ζ, which in turn affects the expected waiting time $1/\ell_B$ and $1/\ell_S$ by (in first order) $\eta_B(z)$ and $\eta_S(z)$ respectively. Thus the indirect effect on the total accumulated participation costs incurred by a cohort is the last term in (50).

In the Appendix, we also show that the difference of the two slope can be written as

$$\begin{aligned} \frac{dW^p(0)}{dr} - \frac{dW^f(0)}{dr} &= sG(\bar{c})\left[\kappa_B\eta_B(z) - \kappa_S\eta_S(z)\right]\zeta'(0) \\ &= \left[\sigma_S(z) - \alpha\right]\left(\frac{W_S^0}{\alpha\ell_S(z)} - \frac{W_B^0}{(1-\alpha)\ell_B(z)}\right) \end{aligned} \tag{51}$$

where $\sigma_S(\zeta) \equiv 1 - \zeta m'(\zeta)/m(\zeta)$ is the elasticity of the matching function with respect to the mass of sellers (i.e. $\sigma_S(\zeta) = SM_2(B,S)/M(B,S)$). It is easy to see that this slope may be either positive or negative, depending on the elasticity of the matching function as well as original welfare shares of buyers and sellers. Recall that we have shown that there is more entry for small r in the full-trade equilibrium of MW. The extramarginal sellers who enter impose a negative externality on the inframarginal sellers and a positive externality on the inframarginal buyers. A symmetric statement applies to the extramarginal buyers who enter. The positive and negative externalities completely cancel out only when the Hosios (1990) condition, i.e. $\sigma_S = \alpha$ holds, in which case we have $\frac{dW^p(0)}{dr} = \frac{dW^f(0)}{dr}$. If, for example, the elasticity σ_S is larger than sellers' bargaining weight α, and if the original share of sellers' welfare W_S^0 is large (relative to W_B^0) so that the last term in (51) is positive, then the welfare under private information is *larger* than under full information.

We formulate these findings in a proposition.

Proposition 18 *For all sufficiently small $r > 0$, the private information welfare $W^p(r)$ is either greater or smaller than the full information welfare $W^f(r)$, depending on whether the right-hand side of* (51) *is positive or negative.*

6 Related literature and concluding remarks

Most related to our paper is the recent note by Lauermann (2006b), which we believe was concurrently written. He also asks the question: can private information be welfare-enhancing in dynamic matching and bargaining models, and arrives to the same conclusion that it can, but for a very different reason and in a quite different model. There is no cost of search, and all potential traders enter. There is no discounting either. The only friction is exogenous exit rate δ.[15] The sellers have all the bargaining power, $\alpha = 1$, and all have the same cost normalized to 0. As in our model, the buyers are heterogeneous. Lauermann (2006b) considers both private and full information. He shows that, as the friction is removed ($\delta \to 0$), with private information all equilibria converge to perfect competition. In particular, the price offers converge to the sellers' cost, i.e. to 0.

In marked contrast, with full information the price offers stay bounded away from 0 even as $\delta \to 0$. The reason why this happens is the following. When sellers know buyers' valuations, they are able to extract all the surplus from the buyers, by offering the price equal to the buyer's valuation, perhaps a penny below. When a seller meets a buyer, she knows that she can guarantee at least the average buyer's valuation $\mathbb{E}(v)$ in the next meeting. Lauermann (2006b) shows that $\mathbb{E}(v)$ is bounded away from 0 as $\delta \to 0$. Since no seller will offer a price less than $(1-\delta)\mathbb{E}(v)$, it follows that the buyers with valuations below $(1-\delta)\mathbb{E}(v)$ will never trade. Since efficiency here means that sellers should trade with all buyers, this outcome is clearly less efficient with full information. Lauermann (2006b) also shows that having exogenous exit is necessary for this effect to occur. Extreme bargaining power is also necessary; it is an open question whether seller heterogeneity could be allowed.

While the entry-deterrent effect of private information has not, to our knowledge, been considered in the dynamic matching and bargaining literature, this literature is quite large and there is a number of related papers. The great majority of papers have assumed full information: Mortensen (1982), Rubinstein and Wolinsky (1985), Rubinstein and Wolinsky (1990), Gale (1986), Gale (1987) and Mortensen and Wright (2002). The first paper to look at convergence in a setting with private information is the unfinished manuscript of

[15] The model is therefore similar to Satterthwaite and Shneyerov (2005), the main difference being that Satterthwaite and Shneyerov (2005) consider multilateral meetings in which sellers run auctions among the buyers they are matched with, whereas matching is strictly bilateral in Lauermann (2006b).

Butters (1979). Other papers that have incorporated private information in some form are Wolinsky (1988), De Fraja and Sakovics (2001) and Serrano (2002).

Recently, there has been a resurgence of interest in this topic: Lauermann (2006a), Satterthwaite and Shneyerov (2007), Atakan (2007). The focus of these papers is convergence to perfect competition.[16] This is also the focus of large but less related literature on static double auctions, beginning with Chatterjee and Samuelson (1983) and Wilson (1985), followed by, among others, , Gresik and Satterthwaite (1989), Satterthwaite and Williams (1989), Satterthwaite (1989), Williams (1991), Rustichini, Satterthwaite, and Williams (1994), and more recently Satterthwaite and Williams (2002), Tatur (2005), Cripps and Swinkels (2005), Reny and Perry (2006), Fudenberg, Mobius, and Szeidl (2007). Wu (2005) studies a double auction with a small entry cost.

Given that search costs may lead to market breakdown, existence of a non-trivial equilibrium is in general not guaranteed. The literature to date has only provided existence results when search costs are small. Satterthwaite and Shneyerov (2007), for example, study a bilateral matching and bargaining model with two-sided private information in which sellers run auctions among the buyers whom they are matched with, and show existence for small search costs. Atakan (2007) has provided an important extension of Satterthwaite and Shneyerov (2007) to multiple units, and shows existence again only for small frictions.[17] We, on the other hand, derive a necessary and sufficient condition for existence (i.e. for no market breakdown) when frictions are arbitrary.

All these findings call for future research. Can our results be generalized to double auctions? Even more interesting would be to identifying the set of bargaining mechanisms for which private information may be welfare-enhancing. One difficulty would be to prove uniqueness of a full-trade equilibrium. For this, it might be fruitful to combine our approach with that of Lauermann (2006a) who develops techniques for studying general dynamic matching and bargaining markets.

[16] In addition, Hurkens and Vulkan (2006) study the role of privately observed deadlines in a matching and bargaining market.

[17] Alternatively, he is able to show existence in general, but assuming that there is costless entry in the first period.

Appendix

Proof of Lemma 3: We prove the results for buyers only. Rewrite the recursive equation for the buyers:

$$\begin{aligned}
W_B(v) &= \max\left\{R_B[\hat{\Pi}_B(v, p_B(v), \tilde{v}(v)) + (1-\hat{q}_B(p_B(v), \tilde{v}(v)))W_B(v)] - K_B, 0\right\} \\
&= \max\left\{R_B \max_{\lambda,\mu}[\hat{\Pi}_B(v, \lambda, \mu) + (1-\hat{q}_B(\lambda, \mu))W_B(v)] - K_B, 0\right\} \\
&= \max\left\{R_B \max_{\lambda,\mu}[\hat{\Pi}_B(v - W_B(v), \lambda, \mu) + W_B(v)] - K_B, 0\right\}.
\end{aligned}$$

where $\hat{\Pi}_B(v, \lambda, \mu)$ and $\hat{q}_B(\lambda, \mu)$ are conditional on proposing λ and adopting acceptance level μ:

$$\hat{\Pi}_B(v, \lambda, \mu) \equiv (1-\alpha)\int_{\tilde{c}(c)\leq\lambda}[v-\lambda]\, d\Gamma(c) + \alpha\int_{p_S(c)\leq\mu}[v - p_S(c)]\, d\Gamma(c) \tag{52}$$

$$\hat{q}_B(\lambda, \mu) \equiv (1-\alpha)\int_{\tilde{c}(c)\leq\lambda} d\Gamma(c) + \alpha\int_{p_S(c)\leq\mu} d\Gamma(c). \tag{53}$$

If $R_B = 1$ (or $r = 0$), the recursive equation indicate that whenever $W_B(v) \neq 0$, we have $\max_{\lambda,\mu}\hat{\Pi}_B(v - W_B(v), \lambda, \mu) = K_B > 0$ so that $v - W_B(v)$ must be some positive constant x. It is then easily seen that the recursive equation has a unique solution $W_B(v) = \max\{v - x, 0\}$, which is nondecreasing, continuous and convex.[18]

Now suppose $R_B < 1$ (or $r > 0$). Then the right-hand side of the recursive equation can be regarded as a contraction mapping that assigns each W_B another function on the same domain. Applying standard techniques of discounted dynamic programming, we can see that the solution W_B is unique, nondecreasing, continuous and convex.

From the continuity and monotonicity, $W_B(v)$ is absolutely continuous and hence differentiable almost everywhere. Whenever differentiable, we have

$$W_B'(v) = \chi_B(v)R_B\left\{q_B(v)[1 - W_B'(v)] + W_B'(v)\right\}.$$

Solve for $W_B'(v)$,

$$\begin{aligned}
W_B'(v) &= \frac{\chi_B(v)R_B q_B(v)}{1 - \chi_B(v)R_B[1 - q_B(v)]} \\
&= \frac{\chi_B(v)R_B q_B(v)}{1 - R_B[1 - q_B(v)]} = \chi_B(v)\frac{\ell_B q_B(v)}{r + \ell_B q_B(v)}.
\end{aligned}$$

For $v \in A_B$, the trading probability $q_B(v)$ must be strictly positive, otherwise the participation cost κ_B cannot be recovered. Thus $W_B(v)$ is strictly increasing on A_B and $A_B = [\underline{v}, 1]$.

[18] If $v - x < 0$ then $W_B(v)$ cannot be $v - x$ and hence $W_B(v) = 0$. If $v - x > 0$ then $W_B(v)$ cannot be 0 because the first maximand at $W_B(v) = 0$ is

$$\max_{\lambda,\mu}\Pi_B(v, \lambda, \mu) - K_B > \max_{\lambda,\mu}\Pi_B(x, \lambda, \mu) - K_B = 0.$$

In order to prove 20, it now suffices to show $W_B(\underline{v}) = 0$. Indeed, if $W_B(\underline{v}) \neq 0$, then either $\underline{v} = 0$ or $\underline{v} = 1$. We preclude the possibility of $\underline{v} = 1$ because we are looking at nontrivial equilibrium. $\underline{v} = 0$ is also impossible because in that case type 0 buyer cannot expect their participation cost recovered. Q.E.D.

Proof of Lemma 5: From lemma 3, $\tilde{v}(v)$ and $\tilde{c}(c)$ are absolutely continuous. Their derivatives, which exist almost everywhere on A_B and A_S, are given by

$$\tilde{v}'(v) = \frac{r}{r + \ell_B q_B(v)} \geq 0 \text{ and } \tilde{c}'(c) = \frac{r}{r + \ell_S q_S(v)} \geq 0.$$

Moreover, the above inequalities are strict if and only if $r > 0$. Q.E.D.

Proof of Lemma 7:

Step 1: $\underline{p}_S \geq \underline{v}$ and $\bar{p}_B \leq \bar{c}$.

Suppose $\underline{p}_S < \underline{v}$. Then there is some active seller with type c proposing $p_S(c) < \underline{v}$. Then her offer will be accepted with probability one and she can raise her offer without affecting this probability. We get the desired contradiction and have $\underline{p}_S \geq \underline{v}$. Similar logic with a buyer considered would show $\bar{p}_B \leq \bar{c}$.

Step 2: $\underline{c} < \underline{v}$ and $\bar{c} < \bar{v}$.

Suppose $\underline{v} \leq \underline{c}$. Then buyer with type $\underline{v}$ cannot recover the participation cost. It is because (i) when she is the proposer, she cannot get any surplus since her value $\underline{v}$ is not higher than the lowest price $\underline{c}$ acceptable by any seller; and (ii) when she is the responder, again she cannot get any surplus since, from step 1, her value $\underline{v}$ is not higher than the lowest price $\underline{p}_S$ proposed by any seller. We get the desired contradiction and have $\underline{c} < \underline{v}$. Similar logic with a $\bar{c}$ seller considered would show $\bar{c} < \bar{v}$.

Step 3: For all $c \in [0, \bar{c}]$ and all $v \in [\underline{v}, 1]$, $\tilde{c}(c) < p_S(c) \leq \bar{p}_S \leq \bar{v}$ and $\underline{c} \leq \underline{p}_B \leq p_B(v) < \tilde{v}(v)$.

Fix any $c \in [0, \bar{c}]$. From $\bar{c} < \bar{v}$ in step 2, we have $\max_\lambda \left\{ \left[1 - \tilde{\Phi}(\lambda)\right] [\lambda - \tilde{c}(c)] \right\} > 0$. (For example, the seller can propose $(\bar{c} + \bar{v})/2$.[19]) Therefore, we have $1 - \tilde{\Phi}(p_S(c)) > 0$ and $p_S(c) - \tilde{c}(c) > 0$. Notice that $1 - \tilde{\Phi}(p_S(c)) > 0$ implies $\bar{p}_S \leq \bar{v}$. It completes the proof of the first part of this step. The second part is shown by symmetric logic.

Now we have already proved $\underline{v} \leq \underline{p}_S \leq \bar{p}_S \leq \bar{v}$ and $\underline{c} \leq \underline{p}_B \leq \bar{p}_B \leq \bar{c}$. If $r = 0$, then by lemma 5, we have $\underline{v} = \bar{v}$ and $\underline{c} = \bar{c}$, and hence it proves our claims for $r = 0$ case. If $r > 0$, then again by lemma 5, then $\tilde{v}(\cdot)$ and $\tilde{c}(\cdot)$ are strictly increasing. Then $1 - \tilde{\Phi}(p_S(c)) > 0$ (resp. $\tilde{\Gamma}(p_B(v)) > 0$) implies $\bar{p}_S < \bar{v}$ (resp. $\underline{p}_B > \underline{c}$). It proves our claims for $r > 0$ case. Q.E.D.

Proof of Lemma 6: If $r = 0$, then from Lemma 7, $p_B(v)$ and $p_S(c)$ are constant on A_B and A_S respectively. The rest of this proof consider the case where $r > 0$.

Consider a buyer with type $v \geq \underline{v}$. Recall (5) and by standard argument, we have

$$\left[\tilde{\Gamma}(p_B(v_2)) - \tilde{\Gamma}(p_B(v_1))\right] \cdot [\tilde{v}(v_2) - \tilde{v}(v_1)] \geq 0.$$

From Lemma 5, if $r > 0$, then $\tilde{v}(v)$ is strictly increasing, thus $\tilde{\Gamma}(p_B(v))$ is nondecreasing in v. Now suppose $v_1 < v_2$ but $p_B(v_1) > p_B(v_2)$. Since $\tilde{\Gamma}(p_B(\cdot))$ is nondecreasing, we have

[19] Φ and Γ have positive densities on $[\underline{v}, 1]$ and $[0, \bar{c}]$ because F and G have. Then $\tilde{\Phi}$ and $\tilde{\Gamma}$ also have positive densities on $[\underline{v}, \bar{v}]$ and $[\underline{c}, \bar{c}]$ because $\tilde{v}$ and $\tilde{c}$ are continuous.

$\tilde{\Gamma}(p_B(v_1)) \leq \tilde{\Gamma}(p_B(v_2))$. On the other hand, $\tilde{\Gamma}(\cdot)$ is nondecreasing, we have $\tilde{\Gamma}(p_B(v_1)) \geq \tilde{\Gamma}(p_B(v_2))$. Therefore $\tilde{\Gamma}(p_B(v_1)) = \tilde{\Gamma}(p_B(v_2))$. Then either $\tilde{\Gamma}(p_B(v_1)) = \tilde{\Gamma}(p_B(v_2)) > 0$ or $\tilde{\Gamma}(p_B(v_1)) = \tilde{\Gamma}(p_B(v_2)) = 0$.

Suppose first that $\tilde{\Gamma}(p_B(v_1)) = \tilde{\Gamma}(p_B(v_2)) > 0$. Then type v_1 buyers could offer a lower price, namely $p_B(v_2)$, without affecting the probability of being accepted, which is positive. We get a contradiction.

Now suppose $\tilde{\Gamma}(p_B(v_1)) = \tilde{\Gamma}(p_B(v_2)) = 0$. Then $\max_\lambda \left\{\tilde{\Gamma}(\lambda)[\tilde{v}(v) - \lambda]\right\} = 0$ for $v = v_1, v_2$. But by Lamma 5 and Lamma 7, $v \geq \underline{v}$ imply $\tilde{v}(v) \geq \underline{v} > \underline{c}$. Then above objective function can be guaranteed to be strictly positive by setting $\lambda = (\underline{v} + \underline{c})/2$. Again we get a contradiction. Therefore, $p_B(v)$ are nondecreasing.

Using symmetric logic, we can prove corresponding results for the sellers' side, namely, $p_S(c)$ are nondecreasing. Q.E.D.

Proof of Lemma 8: The claims in the first sentence are straight implications of Lemma 6. For the rest, notice that by Lamma 7, $\underline{v} \leq \underline{p}_S$ and therefore the $\underline{v}$ buyer will make positive profit only when he is the proposer. His offer $\underline{p}_B$ will be accepted only if the seller's dynamic type $\tilde{c}(c) \leq \underline{p}_B$. The entry condition (12) then implies (24). Similar logic leads to (25). Q.E.D.

Proof of Proposition 12: For concreteness, focus on the sellers (a symmetric argument applies for the buyers). Since equilibrium proposing strategy is nondecreasing, it is sufficient to rule out a deviation to a higher bid $\lambda > \underline{v}$ for the sellers with type $\bar{c}$. The expected profit in a given meeting is

$$\pi_S(\bar{c}, \lambda) = (\lambda - \bar{c}) \left(1 - \tilde{\Phi}(\lambda)\right),$$

and its slope is

$$\begin{aligned} \frac{\partial \pi_S(\bar{c}, \lambda)}{\partial \lambda} &= \left(1 - \tilde{\Phi}(\lambda)\right) - (\lambda - \bar{c}) \tilde{\Phi}'(\lambda) \\ &= -\tilde{\Phi}'(\lambda) \left[\tilde{J}_B(\lambda) - \bar{c}\right] \end{aligned} \tag{54}$$

where $\tilde{J}_B(\lambda)$ is the "virtual type" that corresponds to the distribution of dynamic types $\tilde{\Phi}$,

$$\tilde{J}_B(\lambda) \equiv \lambda - \frac{1 - \tilde{\Phi}(\lambda)}{\tilde{\Phi}'(\lambda)}.$$

Given that the dynamic type $\tilde{v}(v)$ in a full-trade equilibrium is a linear function which can be calculated as

$$\tilde{v}(v) = \frac{rv + \ell_B \underline{v}}{r + \ell_B},$$

straightforward algebra shows that

$$\tilde{J}_B(\lambda) = \frac{r}{r + \ell_B} \left(J_B\left(\frac{r + \ell_B}{r}\lambda - \frac{\ell_B}{r}\underline{v}\right) + \frac{\ell_B}{r}\underline{v}\right), \tag{55}$$

where $J_B(v)$ is the virtual type function for the distribution F. Substituting (55) in the slope formula (54), we obtain

$$\frac{\partial \pi_S(\bar{c}, \lambda)}{\partial \lambda} = -\tilde{\Phi}'(\lambda) \left[\frac{r}{r + \ell_B} \left(J_B\left(\frac{r + \ell_B}{r}\lambda - \frac{\ell_B}{r}\underline{v}\right) + \frac{\ell_B}{r}\underline{v}\right) - \bar{c}\right] \tag{56}$$

Clearly, a deviation to $\lambda < \underline{v}$ is not profitable, so we only need to consider $\lambda > \underline{v}$. A necessary condition for such a deviation to be not profitable is that $\partial \pi_S(\bar{c}, \lambda) / \partial \lambda \leq 0$ at $\lambda = \underline{v}$, i.e. the expression in the brackets on the right-hand side of equation (56) is non-positive when $\lambda = \underline{v}$. This is also sufficient because of the monotonicity of J_B. This gives the inequality

$$\frac{rJ_B(\underline{v}) + \ell_B(z)\,\underline{v}}{r + \ell_B(z)} - \bar{c} \geq 0.$$

Similarly, a necessary and sufficient condition to rule out a profitable deviation by a buyer with type $\underline{v}$ is

$$\underline{v} - \frac{\ell_S(z)\,\bar{c} + rJ_S(\bar{c})}{r + \ell_S(z)} \geq 0.$$

Equivalently, we can eliminate r from both inequalities to obtain (31), the upper bound on r in text.

Proof that a non-full-trade equilibrium with too little entry cannot exist for small $r > 0$: Since offering strategies are increasing, it is sufficient to rule out the deviations by marginal participating buyers and sellers. Consider a type $\bar{c}$ seller who deviates by offering $\lambda > \underline{v}$ in all meetings. His expected payoff in a given meeting

$$\pi_S(\bar{c}, \lambda) = (\lambda - \bar{c})\left[1 - \tilde{\Phi}(\lambda)\right],$$

with the slope

$$\frac{\partial \pi_S(\bar{c}, \lambda)}{\partial \lambda} = 1 - \tilde{\Phi}(\lambda) - (\lambda - \bar{c})\,\tilde{\phi}(\lambda), \tag{57}$$

where $\tilde{\phi}$ is the density of buyers' dynamic types. This density is equal to

$$\tilde{\phi}(\lambda) = \tilde{\Phi}'(\lambda) = \frac{\phi(v(\lambda))}{\tilde{v}'(v(\lambda))}$$

where ϕ is the density of buyer types in the market and $v(\lambda)$ is the inverse of $\tilde{v}(v)$, i.e. $\tilde{v}(v(\lambda)) = \lambda$. From (34) in text, for all $v \geq \underline{v}$, $\tilde{v}'(v) \leq r/(r + \kappa_B)$, so we have

$$\tilde{\phi}(\lambda) \geq \left(1 + \frac{\kappa_B}{r}\right)\phi(v(\lambda)). \tag{58}$$

We now derive a lower bound on the endogenous density of buyers' types ϕ. From the steady-state condition, we can deduce

$$\phi(v) = \frac{bf(v)}{M(B,S)\,q_B(v)} \geq \frac{bf(v)}{M(B,S)}. \tag{59}$$

and

$$B = \int_{\underline{v}}^{1} \frac{bf(v)\,dv}{\ell_B q_B(v)} \leq \frac{b}{\kappa_B}\left[1 - F(\underline{v})\right], \tag{60}$$

where the last inequality follows from the fact that the $\underline{v}$-type buyer must recover his participation cost, $\ell_B q_B(\underline{v}) \geq \kappa_B$. Corollary 11 and the steady-state condition $b\left[1 - F(\underline{v})\right]$

$= sG(\bar{c})$ therefore implies that an $\varepsilon > 0$ exists such that $\underline{v} < 1 - \varepsilon$ for all sufficiently small $r > 0$.

From now on in this proof we are assuming that r is chosen sufficiently small. It follows from (60) and a symmetric argument for the sellers that

$$B \leq \frac{b\varepsilon\bar{f}}{\kappa_B} \equiv \bar{B}, \quad S \leq \frac{s\varepsilon\bar{g}}{\kappa_S} \equiv \bar{S}.$$

Since the matching function $M(B,S)$ is nondecreasing in each of its arguments, $M(B,S) \leq M(\bar{B},\bar{S})$. Substituting this bound into (59) we obtain the following bound on the endogenous density of buyers' types,

$$\phi(v) \geq \frac{b\underline{f}}{M(\bar{B},\bar{S})} \equiv \underline{\phi}. \tag{61}$$

Then (58) implies that $\tilde{\phi}(\lambda) \geq \left(1 + \frac{\kappa_B}{r}\right)\underline{\phi}$, and it follows that the slope of the payoff function (57) eventually becomes negative as $r \to 0$, so that the type $\bar{c}$ seller would not prefer a deviation to $\lambda > \underline{v}$. Q.E.D.

Proof of Proposition 18.

MW show that, under full information, lifetime utilities in a full-trade equilibrium are given by

$$\begin{aligned} W_B^f(v) &= \frac{(1-\alpha)\ell_B(\zeta)}{(1-\alpha)\ell_B(\zeta)+r}(v-\underline{v}), \\ W_S^f(c) &= \frac{\alpha\ell_S(\zeta)}{\alpha\ell_S(\zeta)+r}(\bar{c}-c), \end{aligned}$$

so the total welfare as a function of r is

$$\begin{aligned} W^f(r) &= \frac{(1-\alpha)\ell_B(\zeta)}{(1-\alpha)\ell_B(\zeta)+r} b\int_{\underline{v}}^1 (v-\underline{v})\,dF(v) + \frac{\alpha\ell_S(\zeta)}{\alpha\ell_S(\zeta)+r} s\int_0^{\bar{c}} (\bar{c}-c)\,dG(c) \quad (62) \\ &= \frac{(1-\alpha)\ell_B(\zeta)}{(1-\alpha)\ell_B(\zeta)+r} W_B^0 + \frac{\alpha\ell_S(\zeta)}{\alpha\ell_S(\zeta)+r} W_S^0 \end{aligned}$$

where $\underline{v}$ and $\bar{c}$ are the marginal participating types.

The indifference conditions for the marginal participating types $\underline{v}$ and $\bar{c}$ in their model are

$$(1-\alpha)\int_0^{\bar{c}} \left(\underline{v} - \frac{rc + \alpha\ell_S(\zeta)\bar{c}}{\alpha\ell_S(\zeta)+r}\right)\frac{dG(c)}{G(\bar{c})} = \frac{\kappa_B}{\ell_B(\zeta)}, \tag{63}$$

$$\alpha\int_{\underline{v}}^1 \left(\frac{rv + (1-\alpha)\ell_B(\zeta)\underline{v}}{(1-\alpha)\ell_B(\zeta)+r} - \bar{c}\right)\frac{dF(v)}{1-F(\underline{v})} = \frac{\kappa_S}{\ell_S(\zeta)}. \tag{64}$$

Note that their conditions are different from our conditions (26) and (27) because in the MW model, marginal traders make ceteris paribus a bigger profit than our marginal traders, by holding their partners to their dynamic types. In particular, while in the full-trade equilibrium of our model $\underline{v}$, $\bar{c}$ do not depend on r, MW's marginal types implicitly defined by (63) and (64) depend on r, as does the market tightness ζ. Denote these functions as $\underline{v}_*(r)$, $\bar{c}_*(r)$ and $\zeta(r)$.

Only if $r=0$, the marginal participating types are equal to those in our model and in fact, the total surpluses are also equal. Recall that our total welfare as a function of r is

$$W^p(r)=\frac{\ell_B(\zeta)}{\ell_B(\zeta)+r}W_B^0+\frac{\ell_S(\zeta)}{\ell_S(\zeta)+r}W_S^0. \tag{65}$$

Comparing (62) and (65), we see that $W^p(0)=W^f(0)$.

The slope of $W^p(r)$ at $r=0$ is much simple to compute since $\bar{c}$, $\underline{v}$ and ζ do not depend on r under private information. A direct calculation shows that

$$\frac{dW^p(0)}{dr}=-\frac{W_B^0}{\ell_B(z)}-\frac{W_S^0}{\ell_S(z)}. \tag{66}$$

Turn to the slope of $W^f(r)$ at $r=0$, note that by direct calculation,

$$\frac{dW^f(0)}{dr}=-\frac{1}{1-\alpha}\frac{W_B^0}{\ell_B(z)}-\frac{1}{\alpha}\frac{W_S^0}{\ell_S(z)}+sG(\bar{c})\left[\bar{c}_*'(0)-\underline{v}_*'(0)\right]. \tag{67}$$

To calculate $\bar{c}_*'(0)-\underline{v}_*'(0)$ that appears in the slope $dW^f(0)/dr$, we use the indifference conditions for the marginal types (63) and (64). Apply integration by parts to the left-hand side of (63) and then differentiate at $r=0$:

$$\begin{aligned}
&\frac{d}{dr}\left[(1-\alpha)\int_0^{\bar{c}_*}\left(\underline{v}_*-\frac{rc+\alpha\ell_S(\zeta)\bar{c}_*}{\alpha\ell_S(\zeta)+r}\right)\frac{dG(c)}{G(\bar{c}_*)}\right]_{r=0}\\
=\;&(1-\alpha)\cdot\frac{d}{dr}\left[\underline{v}_*-\bar{c}_*+\frac{r}{\alpha\ell_S(\zeta)+r}\int_0^{\bar{c}_*}\frac{G(c)}{G(\bar{c}_*)}dc\right]_{r=0}\\
=\;&(1-\alpha)\left[\underline{v}_*'(0)-\bar{c}_*'(0)\right]+\frac{(1-\alpha)W_S^0}{sG(\bar{c})}\cdot\frac{d}{dr}\left[\frac{r}{\alpha\ell_S(\zeta)+r}\right]_{r=0}\\
=\;&(1-\alpha)\left[\underline{v}_*'(0)-\bar{c}_*'(0)\right]+\frac{(1-\alpha)W_S^0}{\alpha\ell_S(z)\,sG(\bar{c})}.
\end{aligned}$$

Equate the derivatives of both sides of (63) and rearrange terms, we obtain

$$(1-\alpha)\left[\bar{c}_*'(0)-\underline{v}_*'(0)\right]=\frac{(1-\alpha)W_S^0}{\alpha\ell_S(z)\,sG(\bar{c})}-\kappa_B\eta_B(z)\zeta'(0) \tag{68}$$

where

$$\eta_B(z)\equiv\frac{d}{d\zeta}\left[\frac{1}{\ell_B(\zeta)}\right]_{\zeta=z}=-\frac{\ell_B'(z)}{\ell_B(z)^2}>0.$$

Similarly, by differentiating (64) and rearranging terms, we obtain

$$\alpha\left[\bar{c}_*'(0)-\underline{v}_*'(0)\right]=\frac{\alpha W_B^0}{(1-\alpha)\ell_B(z)\,b\,(1-F(\underline{v}))}+\kappa_S\eta_S(z)\zeta'(0) \tag{69}$$

where

$$\eta_S(z)\equiv-\frac{d}{d\zeta}\left[\frac{1}{\ell_S(\zeta)}\right]_{\zeta=z}=\frac{\ell_S'(z)}{\ell_S(z)^2}>0.$$

Sum (68) and (69), and insert the resulting $\bar{c}'_*(0) - \underline{v}'_*(0)$ into (67), and cancel terms, we obtain:

$$\begin{aligned}
\frac{dW^f(0)}{dr} &= -\frac{W_B^0}{\ell_B(z)} - \frac{W_S^0}{\ell_S(z)} - sG(\bar{c})\left[\kappa_B\eta_B(z) - \kappa_S\eta_S(z)\right]\zeta'(0) \\
&= \frac{dW^p(0)}{dr} - sG(\bar{c})\left[\kappa_B\eta_B(z) - \kappa_S\eta_S(z)\right]\zeta'(0). \qquad (70)
\end{aligned}$$

To compute $\zeta'(0)$, divide (68) by $1-\alpha$ and divide (69) by $-\alpha$, sum them up and rearrange terms:

$$\left[\frac{\kappa_B\eta_B(z)}{(1-\alpha)} + \frac{\kappa_S\eta_S(z)}{\alpha}\right] sG(\bar{c})\zeta'(0) = \frac{W_S^0}{\alpha\ell_S(z)} - \frac{W_B^0}{(1-\alpha)\ell_B(z)}.$$

Insert $\zeta'(0)$ into (70) and notice that

$$\frac{\kappa_B\eta_B(z)}{(1-\alpha)}\left[\frac{\kappa_B\eta_B(z)}{(1-\alpha)} + \frac{\kappa_S\eta_S(z)}{\alpha}\right]^{-1} = 1 - \frac{zm'(z)}{m(z)} = \sigma_S(z)$$

and

$$\frac{\kappa_S\eta_S(z)}{\alpha}\left[\frac{\kappa_B\eta_B(z)}{(1-\alpha)} + \frac{\kappa_S\eta_S(z)}{\alpha}\right]^{-1} = \frac{zm'(z)}{m(z)} = \sigma_B(z),$$

we obtain

$$\begin{aligned}
\frac{dW^p(0)}{dr} - \frac{dW^f(0)}{dr} &= \left[(1-\alpha)\sigma_S(z) - \alpha\sigma_B(z)\right]\left[\frac{W_S^0}{\alpha\ell_S(z)} - \frac{W_B^0}{(1-\alpha)\ell_B(z)}\right] \\
&= \left[\sigma_S(z) - \alpha\right]\left[\frac{W_S^0}{\alpha\ell_S(z)} - \frac{W_B^0}{(1-\alpha)\ell_B(z)}\right].
\end{aligned}$$

Q.E.D.

Proof of Theorem 16 (No market breakdown):

We have already seen necessity of $K(z) < 1$ in the text. To prove its sufficiency, we first introduce some definitions and lemmas.

Definition 19 *Fix $(r, \bar{\ell}_B, \bar{\ell}_S) \gg (0, \kappa_B, \kappa_S)$, and $\varepsilon \in (0, \bar{\varepsilon}]$ where*

$$\bar{\varepsilon} \equiv \min\left\{1, \frac{b}{\kappa_B}\frac{\bar{\ell}_B}{\bar{f}}, \frac{s}{\kappa_S}\frac{\bar{\ell}_S}{\bar{g}}\right\}.$$

Let $C[0,1]$ be the Banach space of real continuous bounded functions defined on $[0,1]$, endowed with the supremum norm. Then we define $D_\varepsilon \subset (C[0,1])^4$ as the set of all tuples of functions $E \equiv (W_B, W_S, N_B, N_S)$ which satisfy the following conditions:

$$0 \le W_B \le 1, \;\; 0 \le W_S \le 1$$

$$N_B(0) = N_S(0) = 0$$

$$N_B(1) \ge b[1 - F(1-\varepsilon)]/\bar{\ell}_B, \;\; N_S(1) \ge sG(\varepsilon)/\bar{\ell}_S$$

moreover W_B, W_S, N_B, N_S are Lipschitz continuous (which implies absolutely continuous and hence differentiable almost everywhere) and wherever differentiable,

$$\begin{aligned} 0 &\leq W_B'(v) \leq \bar{\ell}_B/(r+\bar{\ell}_B) \equiv \bar{R}_B < 1, \\ 0 &\leq -W_S'(c) \leq \bar{\ell}_S/(r+\bar{\ell}_S) \equiv \bar{R}_S < 1, \end{aligned}$$

$$0 \leq N_B'(v) \leq \frac{bf(v)}{\kappa_B} \leq \frac{b\bar{f}}{\kappa_B}, \;\; 0 \leq N_S'(c) \leq \frac{sg(c)}{\kappa_S} \leq \frac{s\bar{g}}{\kappa_S}.$$

(Notice that $0 \leq N_B(1) \leq b/\kappa_B$ and $0 \leq N_S(1) \leq s/\kappa_S$ are implied.) It completes the definition of the set D_ε.

Lemma 20 *D_ε is nonempty, convex and compact for any $(r, \bar{\ell}_B, \bar{\ell}_S) \gg (0, \kappa_B, \kappa_S)$ and any $\varepsilon \in (0, \bar{\varepsilon}]$.*

Proof: Obviously, D_ε is nonempty (this is where we need $\varepsilon \leq \bar{\varepsilon}$), convex and closed. Furthermore, D_ε is a uniformly bounded and equicontinuous family of functions on a compact set $[0,1]$, therefore, by Ascoli-Arzela Theorem (see e.g. Royden (1988) p.169), D_ε is compact. Q.E.D.

Definition 21 *Fix $(r, \bar{\ell}_B, \bar{\ell}_S) \gg (0, \kappa_B, \kappa_S)$ and $\varepsilon \in (0, \bar{\varepsilon}]$. Define a mapping $T_\varepsilon : D_\varepsilon \to D_\varepsilon$ by $T_\varepsilon(W_B, W_S, N_B, N_S) \equiv (W_B^*, W_S^*, N_B^*, N_S^*)$, where $W_B^*, W_S^*, N_B^*, N_S^*$ are constructed through* (36), (1), (2), (3), (4), (37), (38), (7), (13), (8), (14), (39), (40), (42), (43), (46) *and* (47), *with the matching function M underlying ℓ_B and ℓ_S replaced by $\tilde{M}$ in* (44).

Several remarks are needed to claim that our definition 21 of T_ε is legitimate, i.e. T_ε is well-defined and its range, as stated in the definition, is contained in its domain D_ε. The restrictions we impose on the domain D_ε are important to claim that. Firstly, since $N_B(1) > 0$ and $N_S(1) > 0$, the distribution variables $(B, S, \Phi, \Gamma, \zeta)$ are clearly well-defined. Second, the normalized distributions (Φ, Γ) inherit continuity from the unnormalized distributions (N_B, N_S). Third, since $r > 0$ and hence $0 \leq W_B' < 1$ and $0 \leq -W_S' < 1$, the dynamic types $\tilde{v}$ and $\tilde{c}$ are strictly increasing, which together with the continuity of (Φ, Γ) implies that the distributions of dynamic types $(\tilde{\Phi}, \tilde{\Gamma})$ are continuous.

Fourth, since $\tilde{\Gamma}$ and $\tilde{v}$ are continuous, and arg max correspondence in the definition of p_B is nonempty-valued and compact-valued. Thus p_B is well-defined; and p_B, as the supremum of the arg max correspondence, is itself a maxima. Furthermore, since the objective function of the maximization problem satisfies increasing differences in (v, λ), any selection of the arg max correspondence on the regions of types proposing serious offers is nondecreasing, and hence any other selection is essentially the same as p_B. The same logic applies to the sellers' counterpart p_S.

Fifth, W_B^* can be rewritten, by the definition of $\tilde{v}(v)$, p_B, q_B and Π_B, as:

$$W_B^*(v) = \sup \chi \cdot \left\{ R_B(\zeta)\hat{\Pi}_B(v, \lambda, \mu) + R_B(\zeta)[1 - \hat{q}_B(\lambda, \mu)] W_B(v) - K_B(\zeta) \right\}$$

where the supremum is taken over $(\lambda, \mu) \in [0,1]^2$ and $\chi \in \{0,1\}$, and where $\hat{\Pi}_B(v, \lambda, \mu)$ and $\hat{q}_B(\lambda, \mu)$ are defined in (52) and (53). It is then clear that W_B^* is continuous and nondecreasing because $\hat{\Pi}_B(\cdot, \lambda, \mu)$ and W_B are. Furthermore, since $\hat{\Pi}_B(v, \lambda, \mu) \leq \hat{q}_B(\lambda, \mu)$, we have

$$0 \leq W_B^*(v) \leq R_B(\zeta)\hat{q}_B(\lambda, \mu) + R_B(\zeta)[1 - \hat{q}_B(\lambda, \mu)] < 1$$

and since $\partial\hat{\Pi}_B(v, \lambda, \mu)/\partial v = \hat{q}_B(\lambda, \mu)$, we have,

$$0 \leq W_B^{*\prime}(v) \leq R_B(\zeta)\{q_B(v) + [1 - q_B(v)]W_B'(v)\} \leq R_B(\zeta) \leq \bar{R}_B.$$

wherever W_B^* is differentiable. Therefore W_B^* satisfies all the restrictions on it imposed by the definition of D_ε. The symmetric logic applies to the sellers' counterpart W_S^*.

Sixth, by the definition of N_B^* and χ_B, we have

$$N_B^*(1) \geq \int_{1-\varepsilon}^{1} \frac{b}{\max\{\ell_B(\zeta), \kappa_B\}} dF(x) \geq b[1 - F(1-\varepsilon)]/\bar{\ell}_B$$

and wherever differentiable,

$$0 \leq N_B^{*\prime}(v) = \frac{\chi_B(v)bf(v)}{\max\{\ell_B(\zeta)q_B(v), \kappa_B\}} \leq \frac{bf(v)}{\kappa_B}.$$

Clearly we also have $N_B^*(0) = 0$, thus N_B^* satisfies all the restrictions on it imposed by the definition of D_ε. The same logic applies to the sellers' counterpart N_S^*. We conclude that our definition 21 of T_ε is legitimate.

Lemma 22 *The mapping $T_\varepsilon : D_\varepsilon \to D_\varepsilon$ is continuous for any $(r, \bar{\ell}_B, \bar{\ell}_S) \gg (0, \kappa_B, \kappa_S)$ and any $\varepsilon \in (0, \bar{\varepsilon}]$.*

Proof: Fix any sequence $\{E_n\}_{n=1}^{\infty}$ in D_ε which is uniformly convergent to E. Let us maintain our notations used in the construction of T_ε to denote the various elements associated with the limit E, and add a subscript n to denote the various elements associated with E_n. Then we have to show that the sequence $\{E_n^*\}_{n=1}^{\infty}$ is uniformly convergent to E^*.

It is easy to see from our construction of T_ε that all the functions Φ_n, Γ_n, $\tilde{v}_n$, $\tilde{c}_n$, $\tilde{\Phi}_n$, $\tilde{\Gamma}_n$, W_{Bn}^*, W_{Sn}^*, N_{Bn}^* and N_{Sn}^* are absolutely continuous and their derivatives are uniformly bounded. Therefore, they form an equicontinuous sequence of functions with a compact domain $[0, 1]$. Hence for those functions, pointwise convergence is equivalent to uniform convergence (see e.g. Royden (1988) p.168). That is to say, once we show the pointwise convergence for one of those functions, uniform convergence for that function automatically follows.

Now obviously B_n, S_n, ζ_n, Φ_n, Γ_n, $\tilde{v}_n$ and $\tilde{c}_n$ are all convergent to their limits B, S, ζ, Φ, Γ, $\tilde{v}$ and $\tilde{c}$. Recall that $\tilde{v}_n$ is strictly increasing, absolutely continuous, and its derivative $\tilde{v}_n'$ is uniformly bounded away from zero (namely $\tilde{v}_n' \geq 1 - \bar{R}_B > 0$). These properties maintain in the limit. Then it is not hard to see that, for all $x \in [0, 1]$ and almost every $v \in [0, 1]$, we have $I[\tilde{v}_n(v) \leq x] \to I[\tilde{v}(v) \leq x]$. By a version of Lebesgue convergence theorem (see e.g. Royden (1988) p.270)[20] and (3), we have $\tilde{\Phi}_n \to \tilde{\Phi}$. The same logic shows that $\tilde{\Gamma}_n \to \tilde{\Gamma}$.

[20] Here we apply a generalized version of Lebesgue dominated convergence theorem which allows varying measure. This theorem requires setwise convergence of the measure (which is the same as pointwise convergence of c.d.f. here), pointwise convergence of the integrand, and that the integrand is dominated by an integrable function.

We next claim that $p_{Bn}(v) \to p_B(v)$ for almost all $v \in [0,1]$. In fact, $p_{Bn}(v)$ might not be convergent to $p_B(v)$. However, we will claim that the set of those v for which the non-convergence exists has zero Lebesgue measure. First notice that the objective function $\tilde{\Gamma}_n(\lambda)[\tilde{v}_n(v)-\lambda]$ in the definition of $p_{Bn}(v)$ uniformly converges as $n \to \infty$, and is continuous in λ, and the constraint set $[0,1]$ is compact, then by the Maximum Theorem, the arg max correspondence must be upper-hemicontinuous with respect to n. That is to say, any subsequential limit of $p_{Bn}(v)$ (which exists because $p_{Bn}(v) \in [0,1]$) must be maximizing the limiting objective function $\tilde{\Gamma}(\lambda)[\tilde{v}(v) - \lambda]$. Therefore, if $p_{Bn}(v)$ is not convergent to $p_B(v)$, then $\arg\max_{\lambda \in [0,1]} \left\{\tilde{\Gamma}(\lambda)[\tilde{v}(v) - \lambda]\right\}$ is not a singleton. One possibility for the above arg max being not a singleton is that $\tilde{v}(v) < \sup\{c : \tilde{\Gamma}(c) = 0\}$. It does not create problem to our concern because in that case $p_{Bn}(v) \to \sup\{c : \tilde{\Gamma}(c) = 0\} = p_B(v)$. Now suppose that $\tilde{v}(v) \geq \sup\{c : \tilde{\Gamma}(c) = 0\}$. Then by standard argument of increasing differences, we see that any selection of the above arg max correspondence must be nondecreasing in c. That is to say, if there is a c such that $\tilde{v}(v) \geq \sup\{c : \tilde{\Gamma}(c) = 0\}$ and $p_{Bn}(v)$ is not convergent to $p_B(v)$, then this v must be a discontinuous point of $p_B(\cdot)$. Besides, since p_B is nondecreasing on an interval domain, it has at most countably many discontinuous points. As a result, the set of those v for which $p_{Bn}(v)$ is not convergent to $p_B(v)$ is with measure zero with respect to Lebesgue measure. It follows that $p_{Bn}(v) \to p_B(v)$ almost everywhere with respect to the measures generated by the c.d.f.'s F, Φ and $\{\Phi_n\}_{n=1}^{\infty}$, since F, Φ and $\{\Phi_n\}_{n=1}^{\infty}$ are absolutely continuous (see e.g. Royden (1988) p.303, Problem 17). The same logic shows that $p_{Sn}(c) \to p_S(c)$ almost everywhere with respect to Lebesgue measure, and hence the measures generated by the c.d.f.'s G, Γ and $\{\Gamma_n\}_{n=1}^{\infty}$.

Now we are ready to show $W_{Bn}^* \to W_{Bn}$. Rewrite W_{Bn}^* as:

$$W_{Bn}^*(v) = \sup_{\substack{(\lambda,\mu)\in[0,1]^2 \\ \chi\in\{0,1\}}} \chi \cdot \left\{ \begin{array}{c} R_B(\zeta_n)(1-\alpha)\int I\left[\tilde{c}_n(c) \leq \lambda\right]\left[\tilde{v}_n(v) - \lambda\right] d\Gamma_n(c) \\ +R_B(\zeta_n)\alpha \int \max\left\{\tilde{v}_n(v) - p_{Sn}(c), 0\right\} d\Gamma_n(c) \\ +R_B(\zeta_n)W_{Bn}(v) - K_B(\zeta_n) \end{array} \right\}.$$

Since we have $\zeta_n \to \zeta$, $\Gamma_n \to \Gamma$, $\tilde{v}_n \to \tilde{v}$, $\tilde{c}_n \to \tilde{c}$, $W_{Bn} \to W_B$, and $p_{Sn}(c) \to p_{Sn}(c)$ for almost all c, we can see that $W_{Bn}^* \to W_{Bn}$. The same logic shows that $W_{Sn}^* \to W_S^*$.

In order to prove $N_{Bn}^* \to N_B^*$, we need to prove that q_{Bn}, Π_{Bn} and χ_{Bn} converge almost everywhere. By definition,

$$q_{Bn}(v) = (1-\alpha)\int I\left[\tilde{c}_n(c) \leq p_{Bn}(v)\right] d\Gamma_n(c) + \alpha \int I\left[p_{Sn}(c) \leq \tilde{v}_n(v)\right] d\Gamma_n(c)$$

$$\begin{aligned}\Pi_{Bn}(v) &= (1-\alpha)\int I\left[\tilde{c}_n(c) \leq p_{Bn}(v)\right]\left[v - p_{Bn}(v)\right] d\Gamma_n(c) \\ &\quad +\alpha \int I\left[p_{Sn}(c) \leq \tilde{v}_n(v)\right]\left[v - p_{Sn}(c)\right] d\Gamma_n(c).\end{aligned}$$

Pick any v such that $p_{Bn}(v) \to p_B(v)$, we have $I\left[\tilde{c}_n(c) \leq p_{Bn}(v)\right] \to I\left[\tilde{c}(c) \leq p_B(v)\right]$ for almost all c, because $\tilde{c}_n'$ is bounded away from 0. Similarly, pick any c such that $p_{Sn}(c) \to p_S(c)$, we have $I\left[p_{Sn}(c) \leq \tilde{v}_n(v)\right] \to I\left[p_S(c) \leq \tilde{v}(v)\right]$ for almost all v, because $\tilde{v}_n'$ is bounded away from 0. Thus we can see that $q_{Bn}(v) \to q_B(v)$ and $\Pi_{Bn}(v) \to \Pi_B(v)$ for almost all v.

To see that χ_{Bn} converge almost everywhere, it suffices to show

$$I\left[\ell_B(\zeta_n)\Pi_{Bn}(v) \geq \kappa_B\right] \to I\left[\ell_B(\zeta)\Pi_B(v) \geq \kappa_B\right].$$

We have already known that $\ell_B(\zeta_n)\Pi_{Bn}(v) \to \ell_B(\zeta)\Pi_B(v)$ for almost all v. Hence it suffices to show that $\ell_B(\zeta)\Pi_B(v)$ is strictly increasing around the v satisfying $\ell_B(\zeta)\Pi_B(v) = \kappa_B$. Notice that

$$\begin{aligned}
\Pi_{Bn}(v) &= \hat{\Pi}_{Bn}[v, p_{Bn}(v), \tilde{v}_n(v)] \\
&= \hat{\Pi}_{Bn}[v - W_{Bn}(v), p_{Bn}(v), \tilde{v}_n(v)] + q_{Bn}(v)W_{Bn}(v) \\
&= \sup_{(\lambda,\mu)\in[0,1]^2} \hat{\Pi}_{Bn}[v - W_{Bn}(v), \lambda, \mu] + q_{Bn}(v)W_{Bn}(v).
\end{aligned}$$

The second term of the last expression is nondecreasing since $q_{Bn}(v)$ is. The first term uniformly converges to $\sup_{(\lambda,\mu)\in[0,1]^2} \hat{\Pi}_B[v - W_B(v), \lambda, \mu]$, which is absolutely continuous, nondecreasing, and its left-hand and right-hand derivatives evaluated at the v satisfying $\ell_B(\zeta)\Pi_B(v) = \kappa_B$ must be bounded away from 0 because

$$\begin{aligned}
&\frac{d}{dv} \sup_{(\lambda,\mu)\in[0,1]^2} \hat{\Pi}_B[v - W_B(v), \lambda, \mu] \\
=\ & q_B(v)[1 - W_B'(v)] \geq \Pi_B(v) \cdot (1 - \bar{R}_B) \geq \frac{\kappa_B}{\ell_B(\zeta)}(1 - \bar{R}_B) > 0.
\end{aligned}$$

We thus conclude that $\chi_{Bn}(v) \to \chi_B(v)$ for almost all v.

Consulting the definition of $N_{Bn}^*(v)$, we see that the convergence of q_{Bn} and χ_{Bn} almost everywhere implies $N_{Bn}^*(v) \to N_B^*(v)$ for all v. As we have claimed, it implies $N_{Bn}^* \to N_B^*$ uniformly. The same logic shows $N_{Sn}^* \to N_S^*$ as well. In conclusion, the sequence $\{E_n^*\}_{n=1}^{\infty}$ is uniformly convergent to E^*. Q.E.D.

Lemma 23 *Fix any $(r, \bar{\ell}_B, \bar{\ell}_S) \gg (0, \kappa_B, \kappa_S)$ and any $\varepsilon \in (0, \bar{\varepsilon}]$. There exists a fixed point $E \in D_\varepsilon$ such that $T_\varepsilon(E) = E$. That is, our ε-model described in Section 4 has at least one equilibrium (ε-equilibrium).*

Proof: As claimed before, D_ε is a nonempty, convex and compact set in a Banach space $(C[0,1])^4$ and the mapping T_ε is continuous. Then we obtain our result by applying the Schauder Fixed Point Theorem. Q.E.D.

Proposition 24 *Fix $r > 0$. Suppose $K(z) < 1$. Then for all sufficiently small $\varepsilon > 0$, any ε-equilibrium is in fact an equilibrium of our original model with take-it-or-leave-it offering.*

Proof: Consider an ε-equilibrium. Define $\underline{v}$ as the lowest buyers' type of (either subsidized or unsubsidized) entrants and $\bar{c}$ as the highest sellers' type of (either subsidized or unsubsidized) entrants, i.e.

$$\begin{aligned}
\underline{v} &\equiv \inf\{v \in [0,1] : \chi_B(v) = 1\} \\
\bar{c} &\equiv \sup\{c \in [0,1] : \chi_S(c) = 1\}.
\end{aligned}$$

Notice that in order to claim the ε-equilibrium is a true equilibrium in our original model, it suffices to claim that, in the ε-equilibrium, $\bar{c} > \varepsilon$, $\underline{v} < 1-\varepsilon$ (i.e. no entrant is subsidized) and $\ell_B(\zeta) < \bar{\ell}_B$, $\ell_S(\zeta) < \bar{\ell}_S$ (i.e. our modification on the matching function does not have a bite).

First of all, as in Lemma 10, we want to claim that

$$\underline{v} - \bar{c} \leq K(z). \tag{71}$$

Since the assertion is trivial if $\underline{v} \leq \bar{c}$, suppose that $\underline{v} > \bar{c}$. In the ε-equilibrium, the payoff function W_B is continuous in v, thus marginal participating buyers must have zero payoff, i.e. $W_B(\underline{v}) = 0$. Therefore those marginal buyers cannot have expected profit more than their participation costs. Moreover, a marginal buyer can have profit only when she proposes, because no seller would propose less than $\underline{v}$, for the same reason as in the original model. It follows that

$$\begin{aligned} \kappa_B &\geq \ell_B(\zeta)(1-\alpha)\max\tilde{\Gamma}(\lambda)\left(\underline{v}-\lambda\right) \\ &\geq \ell_B(\zeta)(1-\alpha)\tilde{\Gamma}(\bar{c})\left(\underline{v}-\bar{c}\right) = \ell_B(\zeta)(1-\alpha)\left(\underline{v}-\bar{c}\right). \end{aligned}$$

Applying the same logic to the sellers, we have

$$\kappa_S \geq \ell_S(\zeta)\alpha\left(\underline{v}-\bar{c}\right).$$

Therefore

$$\underline{v} - \bar{c} \leq \max_{\zeta\in\mathbb{R}_{++}} \min\left\{\frac{\kappa_B}{\ell_B(\zeta)(1-\alpha)}, \frac{\kappa_S}{\ell_S(\zeta)\alpha}\right\} = K(z).$$

Second, we want to claim that, in the ε-equilibrium, the inflows of traders are approximately (although not exactly) balanced, i.e. $b[1-F(\underline{v})] \approx sG(\bar{c})$, when ε is small. By the definition of Φ and N_B^* (given in (36) and (46)) and that $N_B = N_B^*$, we have the inflow-outflow form of buyers' steady-state equation:

$$b[1-F(\underline{v})] = B\int_{\underline{v}}^{1} \max\left\{\ell_B(\zeta)q_B(v), \kappa_B\right\} d\Phi(v).$$

If no buyer is subsidized, the outflow (i.e. the right-hand side) is simply the trading outflow $B\ell_B(\zeta)\int_{\underline{v}}^1 q_B(v)d\Phi(v)$. Now consider the case in which some buyers are subsidized (which implies $\underline{v} = 1-\varepsilon$). Let $\underline{v}^0 > \underline{v}$ be the lowest type who participates without subsidization. Then the outflow is

$$\begin{aligned} & B\ell_B(\zeta)\int_{\underline{v}^0}^{1} q_B(v)d\Phi(v) + B\int_{\underline{v}}^{\underline{v}^0} \max\left\{\ell_B(\zeta)q_B(v), \kappa_B\right\} d\Phi(v) \\ = \;& B\ell_B(\zeta)\int_{\underline{v}}^{1} q_B(v)d\Phi(v) + B\int_{\underline{v}}^{\underline{v}^0} \max\left\{0, \kappa_B - \ell_B(\zeta)q_B(v)\right\} d\Phi(v). \end{aligned}$$

The first term of the last expression is the trading outflow and the second term is the disqualification outflow. The disqualification outflow is $O(\varepsilon)$:

$$\begin{aligned} B\int_{\underline{v}}^{\underline{v}^0} \max\left\{0, \kappa_B - \ell_B(\zeta)q_B(v)\right\} d\Phi(v) &\leq B\int_{1-\varepsilon}^{1} \kappa_B d\Phi(v) \\ &\leq \int_{1-\varepsilon}^{1} \kappa_B \frac{b\bar{f}}{\kappa_B} dv = b\bar{f}\varepsilon. \end{aligned}$$

Thus, in both cases,

$$0 \leq b[1 - F(\underline{v})] - B\ell_B(\zeta) \int_{\underline{v}}^{1} q_B(v) d\Phi(v) \leq b\bar{f}\varepsilon.$$

The same logic applied to the sellers' side implies

$$0 \leq sG(\bar{c}) - S\ell_S(\zeta) \int_{0}^{\bar{c}} q_S(c) d\Gamma(c) \leq s\bar{g}\varepsilon.$$

Now since the trading outflow must be balanced, i.e. $B\ell_B(\zeta) \int_{\underline{v}}^{1} q_B(v) d\Phi(v) = S\ell_S(\zeta) \int_{0}^{\bar{c}} q_S(c) d\Gamma(c)$, we have

$$|b[1 - F(\underline{v})] - sG(\bar{c})| \leq \max\left\{b\bar{f}, s\bar{g}\right\} \cdot \varepsilon. \tag{72}$$

If we let $\varepsilon \to 0$, then we have $b[1 - F(\underline{v})] - sG(\bar{c}) \to 0$ from (71), while $\underline{v} - \bar{c}$ is bounded away from 1 according to $K(z) < 1$ and (71). In the limit, we must have the strict inequalities $\bar{c} > 0$ and $\underline{v} < 1$. It follows that for all small enough $\varepsilon > 0$, we have $\bar{c} > \varepsilon$ and $\underline{v} < 1 - \varepsilon$. The shaded area in Figure 7 illustrates the feasible region of $(\bar{c}, \underline{v})$ for a small ε. In such an ε-equilibrium with small ε, no trader is subsidized. Hence the marginal entrants must be able to recover their participation costs. In particular, the entry equations (24) and (25) hold and they imply that ζ is bounded away from 0 and ∞. Thus as long as $\bar{\ell}_B$ and $\bar{\ell}_S$ are chosen to be large enough, our modification on the matching function does not have a bite. It follows that we obtain a true equilibrium in our original model. Q.E.D.

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