Technical Appendix to "Screening When Not All Agents Are Strategic: Does a Monopoly Need to Exclude?"

Proof of Theorem 2.

Existence. We wish to show that there exists a 4-tuple of measurable bounded functions $\{q(\theta), t^s(\theta), g(\theta), t^\tau(\theta)\}$ solving the maximization problem (1)-(5).

Note that since the monopolist will never select to sell a quantity larger than $Q = \max\{q|u(q,1) - c(q) \geq 0\}$, consumers will never pay more than $M = u(Q,1)$. Also, without loss of generality, $t^s(\theta) \geq 0$ and $t^\tau(\theta) \geq 0$. Thus we may without loss of generality restrict the domain of maximization to functions whose range is contained in $[0,K]$ where $K = \max\{Q,M\}$.

The set of measurable functions with range $[0,K]$ coincides with $L^2(\mu)$, where μ is the measure associated with the distribution function $F(.)$. Let us endow this space of functions with *weak** topology. To be more precise, a sequence $x_n(\theta)$ converges to $x(\theta)$ in the *weak** topology iff $\int_0^1 x_n(\theta)y(\theta)f(\theta)d\theta \rightarrow \int_0^1 x(\theta)y(\theta)f(\theta)d\theta \ \forall y \in L^2(\mu)$. Setting $y(\theta) \equiv 1$ then shows that $x_n(\theta) - x(\theta) \rightarrow 0$, a.e.-$\theta$.

Since $c(\cdot)$ is continuous, and since the components of $\{q(\theta), t^s(\theta), g(\theta), t^\tau(\theta)\}$ are bounded by K, it follows from the Lebesgue dominated convergence Theorem that the objective function is a continuous functional under the *weak** topology.

By Alaoglu's Theorem (see Theorem 6.17 in Royden "Real Analysis," (1987)) a K-ball is compact in the *weak** topology. Furthermore, by Tychonoff's Theorem the product of 4 K-balls is compact in the product topology generated by the *weak** topology. Since the set $\mathcal{S}$ of all 4-tuples $\{q(\theta), t^s(\theta), g(\theta), t^\tau(\theta)\}$ whose components lie in $[0,K]$ and satisfy the constraints (2)-(5) is a closed subset of the K-ball, we conclude that $\mathcal{S}$ is compact in the product topology generated by the *weak** topology. It follows from the Weierstrass Theorem that there exists a 4-tuple of L^2 functions $\{q(\theta), t^s(\theta), g(\theta), t^\tau(\theta)\}$ in $\mathcal{S}$ that attains the maximum in (1).

Uniqueness. Since Problem (1)-(5) is equivalent to Problem (6)-(7), we will prove the uniqueness of a solution to the latter problem.

Suppose to the contrary that there exist two distinct pairs $(q_i(\theta), g_i(\theta))$ $i = 1,2$ solving (6)-(7). Fix some $\rho \in (0,1)$, and let $g_3(\theta) = \rho g_1(\theta) + (1-\rho)g_2(\theta)$, $q_3(\theta) = \rho q_1(\theta) + (1-\rho)q_2(\theta)$.

Now define $U_\tau(\theta) = \max_{\theta' \in [0,1]} u(g_3(\theta'), \theta) - u(g_3(\theta'), \theta')$, and let $\theta^*(\theta)$ be the largest corresponding maximizer. Since $u_{\theta q}(q,\theta) > 0$ and $g_3(\theta)$ is increasing, the maximand is supermodular in the choice variable, and so $\theta^*(\theta)$ is an increasing function. Furthermore, at any point θ where θ^* is continuous (which excludes at most a countable number of points), the function U_τ is differentiable with derivative $U_\tau'(\theta) = u_\theta(g_3(\theta^*(\theta)), \theta)$. At any discontinuity point of θ^* we nevertheless have $\limsup_{\theta' \downarrow \theta} \frac{U_\tau(\theta') - U_\tau(\theta)}{\theta' - \theta} \leq u_\theta(g_3(\theta^*(\theta)), \theta)$ and $\liminf_{\theta' \downarrow \theta} \frac{U_\tau(\theta') - U_\tau(\theta)}{\theta' - \theta} \geq u_\theta(g_3(\theta^*_-(\theta)), \theta)$, where θ^*_- is the left limit of θ^*.

We will show that the following modification of the quantity schedules $(\widehat{q}(\theta), \widehat{g}(\theta))$ improves the firm's expected profits. Let $\widehat{q}(\theta)$ satisfy (in a recursive fashion):

$$\widehat{q}(\theta) = \begin{cases} q_3(\theta) & \text{if } \int_0^\theta u_\theta(\widehat{q}(s), s)ds > U_\tau(\theta) \\ \max\{q_3(\theta), g_3(\theta^*(\theta))\} & \text{if } \int_0^\theta u_\theta(\widehat{q}(s), s)ds = U_\tau(\theta) \end{cases}$$

Let us show that the tuple $(\widehat{q}(\theta), g_3(\theta))$ is feasible, i.e. satisfies the constraints (8) and (7). (7) holds because $\widehat{q}(\theta)$ is constructed so that whenever $U(\theta) = \int_0^\theta u_\theta(\widehat{q}(s), s)ds = U_\tau(\theta)$ we have $U'(\theta) = u_\theta(\widehat{q}(\theta), \theta) \geq u_\theta(g_3(\theta^*(\theta)), \theta) = \limsup_{\theta' \downarrow \theta} \frac{U_\tau(\theta') - U_\tau(\theta)}{\theta' - \theta}$, where the inequality holds because $u_{\theta q}(q,\theta) > 0$ and $\widehat{q}(\theta) \geq g_3(\theta^*(\theta))$. Thus, whenever $U(\theta)$ equals $U_\tau(\theta)$, it cannot decrease below it.

To see that (8) holds note that both $q_3(\theta)$ and $\max\{q_3(\theta), g_3(\theta^*(\theta))\}$ are increasing functions. Note that $\widehat{q}(\theta)$ can fail to be increasing only if for some θ_d $U(\theta_d) = U_\tau(\theta_d)$, $U(\theta) > U_\tau(\theta)$ in a right neighborhood of θ_d, and $g_3(\theta^*(\theta_d)) > q_3(\theta_d)$. The latter inequality, the monotonicity of $g_3(\theta^*(.))$, and the continuity of $q_3(.)$ would then imply that there exists a right neighborhood of θ_d over which $g_3(\theta^*(\theta)) > q_3(\theta)$, and hence $U'(\theta) = u_\theta(q_3(\theta), \theta) < u_\theta(g_3(\theta^*_-(\theta)), \theta) \leq \liminf_{\theta' \downarrow \theta} \frac{U_\tau(\theta') - U_\tau(\theta)}{\theta' - \theta}$. This contradicts that $U(\theta) > U_\tau(\theta)$ in a right neighborhood of θ_d.

Let us now show that the objective (6) attains a strictly higher value under $(\widehat{q}(\theta), g_3(\theta))$ than under either $(q_1(\theta), g_2(\theta))$ or $(q_2(\theta), g_2(\theta))$. First, note that strict concavity of $u(q,\theta) - c(q)$ implies that $u(g_3(\theta), \theta) - c(g_3(\theta)) > \rho(u(g_1(\theta), \theta) - c(g_1(\theta))) + (1-\rho)(u(g_2(\theta), \theta) - c(g_2(\theta)))$, and the corresponding inequality applies to the second term in (6).

Second, the first term in (6) can be rewritten as $\int_0^1 \left(u(q(\theta), \theta) - c(q(\theta)) - \int_0^\theta u_\theta(q(s), s) ds \right) dF(\theta)$. Lemmas 6 and 8 imply that $\rho q_1(\theta) + (1-\rho) q_2(\theta) \leq \widehat{q}(\theta) \leq q^*(\theta)$. Since $u(q,\theta) - c(q)$ is strictly concave in q, we have $u(\widehat{q}(\theta), \theta) - c(\widehat{q}(\theta)) > \rho(u(q_1(\theta), \theta) - c(q_1(\theta))) + (1-\rho)(u(q_2(\theta), \theta) - c(q_2(\theta)))$, and the corresponding inequality applies to $\int_0^1 u(q(\theta), \theta) - c(q(\theta) d\theta$.

To complete the proof, we only need to show that for all $\theta \in [0,1]$,

$$\int_0^\theta u_\theta(\widehat{q}(s), s) ds \leq \rho \int_0^\theta u_\theta(q_1(s), s) ds + (1-\rho) \int_0^\theta u_\theta(q_2(s), s) ds \tag{43}$$

First, suppose θ is such that $\int_0^\theta u_\theta(\widehat{q}(s), s) ds = U_\tau(\theta) \equiv \int_{\theta^*(\theta))}^\theta u_\theta(g_3(\theta^*(\theta)), s) ds$. Now (3) implies that for both $i = 1, 2$ we have $\int_{\theta^*(\theta))}^\theta u_\theta(g_i(\theta^*(\theta)), s) ds \leq \int_0^\theta u_\theta(q_i(s), s) ds$. Since $u_{\theta qq}(q, \theta) \geq 0$, we also have $u_\theta(g_3(\theta), \theta) \leq \rho u_\theta(g_1(\theta), \theta) + (1-\rho) u_\theta(g_2(\theta), \theta)$, and the desired inequality holds. Second, if θ is such that $U(\theta) > U_\tau(\theta)$, then

$$\begin{aligned}
&\frac{d}{d\theta} \int_0^\theta u_\theta(\widehat{q}(s), s) ds = u_\theta(q_3(\theta), \theta) \leq \rho u_\theta(q_1(\theta), \theta) + (1-\rho) u_\theta(q_2(\theta), \theta) \\
&= \rho \frac{d}{d\theta} \int_0^\theta u_\theta(q_1(s), s) ds + (1-\rho) \frac{d}{d\theta} \int_0^\theta u_\theta(q_2(s), s) ds
\end{aligned}$$

where the inequality follows because $q_3(\theta) = \rho q_1(\theta) + (1-\rho) q_2(\theta)$ and $u_{\theta qq}(q, \theta) \geq 0$. So, (43) holds. *Q.E.D.*

Proof of Lemma 1. First, (i) is immediate, as incentive constraints in (2) imply that $q(.)$ must be nondecreasing. Next we will prove (v), (iii), (ii) and (iv) in that order. Note that our proof establishes that (iii) and (v) must hold almost everywhere on $[0,1]$. It also implies that for every solution in which (iii) and (v) fail on a set of measure zero there is an equivalent solution which: (a) differs on a set of measure zero, (b) generates the same value of (1), (c) in which (iii) and (v) hold everywhere. Focussing on equivalence classes, there is no loss of generality then to require that (iii) and (v) hold everywhere.

To establish (v), suppose that $t^\tau(\theta) < u(g(\theta), \theta)$ for some $\theta \in [0,1]$. Then the value of the second integrand in (1) can be increased pointwise by setting $t^\tau(\theta) = u(g(\theta), \theta)$ at all such θ. This modification does not violate any constraints in (2)-(4), and raises the value of (1).

Similarly, if $g(\theta) > q^*(\theta)$ for some $\theta \in [0,1]$, then the value of the second integrand in (1) can be increased pointwise by setting $g(\theta) = q^*(\theta)$ and resetting the corresponding transfer $t^\tau(\theta)$ so that $t^\tau(\theta) = u(g(\theta), \theta)$ at all such θ. This modification does not violate any constraints in (2)-(4) either.

Next, suppose that $g(\theta') > g(\theta'')$ for some $\theta', \theta'' \in [0,1]$, $\theta' < \theta''$. By part (ii) and the previous argument, $t^{\tau}(\theta) = u(g(\theta), \theta)$ and $g(\theta) \leq q^*(\theta)$ for all $\theta \in [0,1]$. So, $u(g(\theta'), \theta) - t^{\tau}(\theta') > u(g(\theta''), \theta) - t^{\tau}(\theta'')$ for all $\theta \geq \theta'$. Also, by (4), $u(q(\theta), \theta) - t^s(\theta) \geq 0 > u(g(\theta''), \theta) - t^{\tau}(\theta'')$ for all $\theta < \theta'$. Then let us modify $g(\theta'')$ raising it to $g(\theta')$ and also raise $t^{\tau}(\theta'')$ so that (5) remain binding. This modification does not violate any incentive constraints, and raises the value of the second integrand in (1) pointwise.

Next, we establish (ii). We start by showing that $\lim_{\theta \to 1} q(\theta) = q^*(1)$, which would imply that $q(1) = q^*(1)$. So, first, suppose that $q(\theta) > q^*(1)$ for some $\theta \in [0,1)$. Let $\hat{\theta} = \inf\{\theta | q(\theta) > q^*(1)\}$. Since $q(.)$ is non-decreasing, $q(\theta) > q^*(1)\ \forall \theta \in (\hat{\theta}, 1]$. Consider a modified quantity/transfer schedule $(\tilde{q}(\theta), \tilde{t}^s(\theta))$ s.t. $\tilde{q}(\theta) = q(\theta)$, $\tilde{t}^s(\theta) = t^s(\theta)$ if $\theta \in [0, \hat{\theta})$, and $\tilde{q}(\theta) = q^*(1)$, $\tilde{t}^s(\theta) = t^s(\hat{\theta}) + u(q^*(1), \hat{\theta}) - u(q(\hat{\theta}), \hat{\theta})$ if $\theta \in [\hat{\theta}, 1]$. It is easy to check that $(\tilde{q}(\theta), \tilde{t}^s(\theta))$ satisfies all incentive constraints. In particular, all constraints in (3) hold because $g(\theta) \leq q^*(1)\ \forall \theta \in [0,1]$. Also, (2) imply that $\tilde{t}^s(\theta) > t^s(\theta)$ for all $\theta \in [\hat{\theta}, 1]$. Since we also have $c(\tilde{q}(\theta)) \leq c(q(\theta))$, the firm's expected profits strictly increase as a result of this modification.

Now suppose that $\lim_{\theta \to 1} q(\theta) = \mu < q^*(1)$. Since $q(\theta)$ is nondecreasing, $q(\theta) \leq \mu\ \forall \theta \in [0,1)$. Let θ_m be well-defined by the following equality: $u(q^*(\theta_m), \theta_m) - c(q^*(\theta_m)) = u(\mu, 1) - c(\mu)$. Since $\mu < q^*(1)$, $\theta_m < 1$ and $q^*(\theta_m) > \mu$. So, $\theta_m > \theta_\mu$ where θ_μ satisfies $q^*(\theta_\mu) = \mu$. Therefore, $u(q^*(\theta_m), \theta_m) - c(q^*(\theta_m)) > u(\mu, \theta) - c(\mu) > u(q(\theta), \theta) - c(q(\theta))\ \forall \theta > \theta_m$.

Consider a modified profile $(\tilde{q}(\theta), \tilde{t}^s(\theta))$ which coincides with $(q(\theta), t^s(\theta))$ on $[0, \theta_m)$, while for $\theta \in [\theta_m, 1]$ $\tilde{q}(\theta) = q^*(\theta_m)$, $\tilde{t}^s(\theta) = t^s(\theta_m) + u(q^*(\theta_m), \theta_m) - u(q(\theta_m), \theta_m)$. The profile $(\tilde{q}(\theta), \tilde{t}^s(\theta))$ satisfies all incentive and individual rationality constraints. Moreover, the firm's expected profits strictly increase. It earns the same profits from strategic consumers with valuations in $[0, \theta_m)$, while its profits from selling to a consumer with valuation $\theta > \theta_m$ changes by:

$$u(q^*(\theta_m), \theta_m) - c(q^*(\theta_m)) - u(q(\theta), \theta) - c(q(\theta)) - \int_{\theta_m}^{\theta} u_\theta(q(s), s) ds > 0$$

The proof that it is optimal to set $q(0) = 0$ proceeds along similar lines.

To establish (iv) note that, since $q(\theta)$ is non-decreasing and bounded, it is Riemann integrable (Theorem 6.9, p.126 in Rudin, "Principles of Mathematical Analysis," (1976)) and a.e. differentiable (Theorem 3 in Royden, p. 100 in "Real Analysis," Third Edition, *Prentice Hall.*). Hence, $u(q(\theta), \theta)$ is also bounded, Riemann integrable and a.e. differentiable. Let $U(\theta) \equiv u(q(\theta), \theta) - t^s(\theta)$. Incentive constraints (2) imply that $U(\theta)$ is increasing and satisfies the following inequalities $\forall \theta, \theta' \in [0,1]$:

$$u(q(\theta'), \theta) - u(q(\theta'), \theta') \leq U(\theta) - U(\theta') \leq u(q(\theta), \theta) - u(q(\theta), \theta')$$

By the intermediate value theorem, $\exists \lambda_1, \lambda_2 \in [0,1]$ s.t. $u_\theta(q(\theta'), \lambda_1\theta + (1-\lambda_1)\theta')(\theta - \theta') \leq U(\theta) - U(\theta') \leq u_\theta(q(\theta), \lambda_2\theta + (1-\lambda_2)\theta')(\theta - \theta')$. Since, $u_\theta(q(\theta'), \theta) \leq \max_{\theta \in [0,1]} u_\theta(q^*(1), \theta) < \infty$, $U(\theta)$ is absolutely continuous. Therefore by Theorem 14, p.110 in Royden (1987) we have $U'(\theta) = u_\theta(q(\theta), \theta)$ and $U(.)$ is equal to the integral of its derivative, i.e. $U(\theta) - U(\theta') = \int_{\theta'}^{\theta} u_\theta(q(s), s) ds$. This equation implies that, as in the standard case, only downwards incentive constraints between 'adjacent' types are binding among the 'strategic' consumers whenever $q(.)$ is strictly increasing, and so $t^s(\theta) = u(q(\theta), \theta) - \int_0^{\theta} u_\theta(q(s), s) ds - U(0)$. *Q.E.D.*

Proof of Lemma 2. Since the schedule $q(.)$ is non-decreasing, by Theorems 4.29 and 4.30, p.96 in Rudin, "Principles of Mathematical Analysis," (1976), it has at most countably many points of discontinuity on $[0,1]$, and both the left-hand and the right-hand limits exist at all discontinuity points of $q(.)$.

Suppose that the optimal quantity schedule $q(.)$ is discontinuous at $x \in (0,1)$. Let $q(x-)$ and $q_+(x)$ be, respectively, the left-hand and the right-hand limits of $q(.)$ at x. Consequently, we have $q_-(x) = q_+(x) - 2\delta$ for some $\delta > 0$.

Let $G(q,\theta) = u(q,\theta) - c(q) - u_\theta(q,\theta)\frac{1-F(\theta)}{f(\theta)}$ and $\Delta(x) = q_+(x) - q_-(x)$. We will consider two different cases.

Case 1: $G(q_-(x), x) < G(q_+(x), x)$. By continuity of $G(q,\theta)$ and $f(\theta)$, $\exists\epsilon > 0$ s.t. $\forall\theta \in (x-\epsilon, x)$, $G(q(\theta)+\Delta(x),\theta) > G(q(\theta),\theta)$. Then let us replace the schedule $q(\theta)$ with modified quantity schedule $\tilde{q}(\theta)$ s.t. $\tilde{q}(\theta) = q(\theta)$ $\forall\theta \in [0, x-\epsilon) \cup (x, 1]$, $\tilde{q}(\theta) = q(\theta) + \Delta(x)$ $\forall\theta \in (x-\epsilon, x)$ and $\tilde{q}(x) = q_+(x)$. Note that $\tilde{q}(\theta)$ is increasing in θ, and all incentive constrains in (7) still hold because $\tilde{U}(\theta) \geq U(\theta)$ $\forall\theta \in [0,1]$. At the same time, the value of the objective (6) increases.

Case 2: $G(q_-(x), x) \geq G(q_+(x), x)$. By concavity of $G(q,\theta)$ in q, $G((q_-(x)+q_+(x))/2, x) > G(q_-(x), x)/2 + G(q_+(x), x)/2$. Furthermore, by continuity of $G(q,\theta)$ and $f(\theta)$, $\exists\epsilon > 0$ s.t. $\forall\theta \in (x-\epsilon, x)$, $G((q_-(x)+q_+(x))/2, \theta)f(\theta) + G((q_-(x)+q_+(x))/2, \theta+\epsilon)f(\theta+\epsilon) > G(q(\theta),\theta)f(\theta) + G(q(\theta+\epsilon), \theta+\epsilon)f(\theta+\epsilon)$.

So, let $\tilde{q}(\theta) = q(\theta)$ $\forall\theta \in [0, x-\epsilon] \cup [x+\epsilon, 1]$ and $\tilde{q}(\theta) = (q_-(x)+q_+(x))/2$ $\forall\theta \in (x-\epsilon, x+\epsilon)$. Note that $\tilde{q}(\theta)$ is increasing in θ. If $u_\theta((q_-(x)+q_+(x))/2, x)) > u_\theta(q_-(x), x)/2 + u_\theta(q_+(x), x)/2$, then ϵ can be chosen small enough that $\forall\theta \in (x-\epsilon, x)$, $u_\theta(q_-(x)+q_+(x))/2, \theta)f(\theta) + u_\theta(q_-(x)+q_+(x))/2, \theta+\epsilon)f(\theta+\epsilon) > u_\theta(q(\theta),\theta)f(\theta) + u_\theta(q(\theta+\epsilon), \theta+\epsilon)f(\theta+\epsilon)$. So under the quantity schedule $\tilde{q}(\theta)$, $\tilde{U}(\theta) \equiv \int_0^\theta u_\theta(\tilde{q}(s), s)ds \geq U(\theta) \equiv \int_0^\theta u_\theta(q(s), s)ds$ $\forall\theta \in [0,1]$. The value of (6) changes by:

$$\int_{x-\epsilon}^{x+\epsilon} \left(G((q_-(x)+q_+(x))/2, \theta) - G(q(\theta),\theta)\right) f(\theta)d\theta > 0$$

If $u_\theta((q_-(x)+q_+(x))/2, x)) \leq u_\theta(q_-(x), x)/2 + u_\theta(q_+(x), x)/2$, then it is possible that $\Delta U(x+\epsilon) = \int_{x-\epsilon}^{x+\epsilon} u_\theta(\tilde{q}(s), s) - u_\theta(q(\theta), s)ds < 0$. In this case, $\forall\theta \in (x-\epsilon, x+\epsilon)$ set $\tilde{q}(\theta) = \tilde{q}$ s.t. $\int_{x-\epsilon}^{x+\epsilon} u_\theta(\tilde{q}, s) - u_\theta(q(\theta), s)ds = 0$. Note that $\tilde{q} > (q_-(x)+q_+(x))/2$. If ϵ is sufficiently small, then the value of the Problem (6) changes approximately by:

$$\int_{x-\epsilon}^{x+\epsilon} \left(u(\tilde{g},\theta) - c(\tilde{g}) - u(q(\theta),\theta) + c(q(\theta))\right) f(\theta)d\theta > 0$$

The inequality holds by concavity of $u(\theta, q) - c(q)$ and the fact that $\tilde{q} > (q_-(x)+q_+(x))/2$.

Proof of Lemma 3. Suppose that sequence $\{\theta_n\}_{n=1}^{n=\infty}$ is s.t. $\lim_{n\to\infty} u(g(\theta_n), \theta_2) - u(g(\theta_n), \theta_n) = U_\tau(\theta_2)$ and $\lim_{n\to\infty} g(\theta_n) = \bar{g}_2$. (Such a sequence exists because $g(\theta) \in [0, q^*(1)]$ $\forall\theta \in [0,1]$, and any sequence in a compact set has a converging subsequence.) Then $\bar{g}_2 > g$. For suppose not, i.e. $\bar{g}_2 \leq g$. Then

$$U_\tau(\theta_2) - U_\tau(\theta_1) \leq \lim_{n\to\infty} u(g(\theta_n), \theta_2) - u(g(\theta_n), \theta_1) = u(\bar{g}_2, \theta_2) - u(\bar{g}_2, \theta_1) < u(g, \theta_2) - u(g, \theta_1)$$

Contradiction.

Next, consider a sequence $\{\theta_m\}_{m=1}^{m=\infty}$ s.t. $\lim_{m\to\infty} u(g(\theta_m), \theta_3) - u(g(\theta_m), \theta_m) = U_\tau(\theta_3)$ and $\lim_{m\to\infty} g(\theta_m) = \bar{g}_3$. Then $\bar{g}_3 > g$. Again, suppose otherwise i.e. $\bar{g}_3 \leq g$. We have:

$$U_\tau(\theta_3) = \lim_{m\to\infty} u(g(\theta_m), \theta_3) - u(g(\theta_m), \theta_m) \geq \lim_{n\to\infty} u(g(\theta_n), \theta_3) - u(g(\theta_n), \theta_n)$$

Since $\bar{g}_3 < \bar{g}_2$ by assumption, it follows that $\exists N, M$ s.t. $\forall n \geq N$ and $m \geq M$, $g(\theta_m) < g(\theta_n)$, and so $u(g(\theta_m), \theta_2) - u(g(\theta_n), \theta_2) > u(g(\theta_m), \theta_3) - u(g(\theta_n), \theta_2)$.

But then $\lim_{m\to\infty} u(g(\theta_m),\theta_2) - u(g(\theta_m),\theta_m) > \lim_{n\to\infty} u(g(\theta_n),\theta_2) - u(g(\theta_n),\theta_n) = \sup_{\theta'\in[0,1]} u(g(\theta_2),\theta') - u(g(\theta'),\theta')$. Contradiction. Finally, we have:

$$U_\tau(\theta_4) - U_\tau(\theta_3) \geq \lim_{m\to\infty} u(g(\theta_m),\theta_4) - u(g(\theta_m),\theta_3) = u(\bar{g}_3,\theta_4) - u(\bar{g}_3,\theta_1) > u(g,\theta_4) - u(g,\theta_3)$$

Q.E.D.

Proof of Lemma 4. Suppose that in the optimal mechanism $U(0) = \underline{u} > 0$. Consider set $Z \subset \Theta$ s.t. $\theta \in Z$ iff

$$\underline{u} + \int_0^\theta u_\theta(q(x),x)dx = \sup_{\theta'\in[0,1]} u(g(\theta'),\theta) - u(g(\theta'),\theta') \tag{44}$$

The set Z is non-empty, because otherwise the firm could reduce $U(0)$ and hence increase its expected profits without violating any of the incentive constraints in (7). Let $\hat{\theta}$ be the minimal element of Z. $\hat{\theta}$ exists because both the left-hand side and the right-hand side of (44) are continuous in θ.

Define $U_\tau(\theta) \equiv \sup_{\theta'\in[0,1]} u(g(\theta'),\theta) - u(g(\theta'),\theta')$. Note that $U_\tau(\theta)$ is continuous and strictly increasing in θ. Since $g(\theta) \leq q^*(1)\ \forall\theta$, $|U_\tau(\theta_1) - U_\tau(\theta_2)| \leq |\theta_1 - \theta_2| \max_{\theta\in[0,1]} u(\theta, q^*(1))$, $U_\tau(\theta)$ is absolutely continuous. Hence, it is almost everywhere differentiable and has finite left-hand and right-hand derivatives for all $\theta \in [0,1]$.

Now, let us demonstrate that the firm can strictly increase its expected profits by offering a modified quantity schedule $\tilde{q}(\theta)$ and setting $U(0) = 0$. To define $\tilde{q}(\theta)$, let $g_-(\theta)$ denote the left-hand limit of $g(.)$ at θ (such limit exists $\forall\theta \in [0,1]$ since $g(.)$ is increasing and bounded), and let $\theta^m = \min\{\hat{\theta}, \sup\{\theta | g_-(\theta) \leq q(\hat{\theta})\}\}$. Then for $\theta \in [0,\theta^m]$ define:

$$V(\theta) = \int_0^\theta u_\theta(\max\{g(s),q(s)\},s)ds + \int_\theta^{\hat{\theta}} u_\theta(\max\{g_-(\theta),q(s)\},s)ds$$

We will show that there exists $\theta_0 \in [0,\theta^m]$ s.t. $V(\theta_0) = \underline{u} + \int_0^{\hat{\theta}} u_\theta(q(s),s)ds$. Note that $V(\theta)$ is continuous in θ_0 and $V(0) = \int_0^{\hat{\theta}} u_\theta(q(s),s)ds < \underline{u} + \int_0^{\hat{\theta}} u_\theta(q(s),s)ds$.

Next, let us establish that $V(\theta^m) \geq U_\tau(\hat{\theta})$. Note that $V(\theta^m) = \int_0^{\theta^m} u_\theta(\max\{g(s),q(s)\},s)ds + u(q(\hat{\theta}),\hat{\theta}) - u(q(\hat{\theta}),\theta^m)$. Since $g(.)$ is nondecreasing, $\int_0^\theta u_\theta(\max\{g(s),q(s)\},s)ds \geq U_\tau(\theta)\ \forall\theta \in [0,\theta^m]$.

Since $U'(\hat{\theta}) = u_\theta(q(\hat{\theta}),\hat{\theta})$ and $\hat{\theta}$ satisfies (44), there exists $\hat{\delta} > 0$ s.t. $U_\tau(\hat{\theta}+\delta) - U_\tau(\hat{\theta}) \leq u(q(\hat{\theta}),\hat{\theta}+\delta) - u(q(\hat{\theta}),\hat{\theta})$ for all $\delta \leq \hat{\delta}$. Otherwise, $\exists\delta$ small enough that $U_\tau(\hat{\theta}+\delta) > U(\theta)$. By Lemma 3, this implies that $U_\tau(\hat{\theta}) - U_\tau(\theta^m) \leq u(q(\hat{\theta}),\hat{\theta}) - u(q(\hat{\theta}),\theta^m)$, and so $V(\theta^m) \geq U_\tau(\hat{\theta})$. Hence, by continuity of $V(\theta)$, $\exists\theta_0 \in [0,\theta^m]$ s.t. $V(\theta_0) = U_\tau(\hat{\theta})$.

Set $\tilde{q}(\theta) = \max\{g(\theta),q(\theta)\}\ \forall\theta \in [0,\theta_0)$, $\tilde{q}(\theta) = \max\{g_-(\theta_0),q(\theta)\}\ \forall\theta \in [\theta_0,\hat{\theta}]$, and $\tilde{q}(\theta) = q(\theta)\ \forall\theta \in [\hat{\theta},1]$. Let $\tilde{U}(\theta) = \int_0^\theta u(q(s),s)ds$. Then, clearly, $\tilde{U}(\theta) \geq U_\tau(\theta)\ \forall\theta \in [0,\theta_0]$. Also, $\tilde{U}(\theta) = U(\theta) \geq U_\tau(\theta)\ \forall\theta \in [\theta_e,1]$ where $\theta_e = \sup\{\theta | q(\theta) < g_-(\theta_0)\}$. Note that $\theta_e \leq \hat{\theta}$.

Suppose that $\exists\theta_l \in (\theta_0,\theta_e)$ s.t. $\tilde{U}(\theta_l) < U_\tau(\theta_l)$. Then we have $\tilde{U}(\theta_l) - \tilde{U}(\theta_0) < U_\tau(\theta_l) - U_\tau(\theta_0)$. Note that $q(\theta) = g_-(\theta_0)$ and so $\tilde{U}'(\theta) = u_\theta(g_-(\theta_0),\theta)\ \forall\theta \in [\theta_0,\theta_e]$. So, by Lemma 3 $\tilde{U}(\theta_e) - \tilde{U}(\theta_l) < U_\tau(\theta_e) - U_\tau(\theta_l)$, i.e. $U_\tau(\theta_e) > \tilde{U}(\theta_e)$. Contradiction.

When the firm implements quantity schedule $\tilde{q}(\theta)$ rather than $q(\theta)$, sets $U(0) = 0$, and does not modify $g(.)$, the change in the firm's expected profits is equal to:

$$\underline{u} + \int_0^1 (u(\tilde{q}(\theta),\theta) - u(q(\theta),\theta))f(\theta)d\theta - \int_0^1 (u_\theta(\tilde{q}(\theta),\theta) - u_\theta(q(\theta,\theta))(1-F(\theta))d\theta =$$

$$\int_0^1 (u(\tilde{q}(\theta),\theta) - u(q(\theta),\theta))f(\theta)d\theta + \int_0^1 (u_\theta(\tilde{q}(\theta),\theta) - u_\theta(q(\theta,\theta))F(\theta)d\theta > 0$$

The equality follows from the fact that $\tilde{U}(1) = U(1)$. The inequality follows because both terms in the expression on the second line are positive. The second term is positive because $\tilde{q}(\theta) \geq q(\theta)$ $\forall\theta \in [0,1]$. The first term is positive, because it is also true that $\tilde{q}(\theta) \leq g(\theta) \leq q^*(1)$ whenever $\tilde{q}(\theta) > q(\theta)$. So, since $u(q,\theta)$ is quasiconcave in q, $u(\tilde{q}(\theta),\theta) - u(q(\theta),\theta) > 0$. Q.E.D.

Proof of Lemma 6. (i) Suppose that $\exists\theta_a \in (0,1]$ s.t. $q(\theta_a) < q^{sb}(\theta_a)$. Let $\theta_b = \sup\{\theta | q(\theta) < q^{sb}(\theta)\}$. By continuity of $q(.)$, $\theta_b > \theta_a$. Then the firm can strictly increase its profits by offering a modified quantity schedule $q_n(.)$ s.t. $q_n(\theta) = q(\theta)$ $\forall\theta \in [0,\theta_a) \cup (\theta_b, 1]$ and $q_n(\theta) = q^{sb}(\theta)$ $\forall\theta \in [\theta_a, \theta_b]$ and adjusting the transfers to preserve the incentive compatibility. Inspecting (6) and (7), one can see that this modification does not violate any incentive constraints and leads to an increase in the value of the objective function (6).

(ii) If $\exists\theta_m$ s.t. $q(\theta) > q^*(\theta)$, $\forall\theta \in (\theta_m, 1)$, and either $q(\theta_m) \leq q^*(\theta)$ or $\theta_m = 0$, then the firm can increase its expected profits by replacing the quantity schedule $q(.)$ with $q_a(.)$ s.t. $q_a(\theta) = q(\theta)$ $\forall\theta \in [0,\theta_m)$ and $q_a(\theta) = q^*(\theta)$ $\forall\theta \in [\theta_m, 1]$.

Since $g(\theta) \leq q^*(\theta)$, inspection of (7) reveals that after this modification all incentive constraints in (7) continue to hold. Inspecting (6) one can also see that as a result of this modification the value of the first integral goes up, while the second remains unchanged.

It remains to consider the following case: there exist θ_1, θ_2 s.t. $q(\theta) \leq q^*(\theta)$ $\forall\theta \in [\theta_2, 1]$, $q(\theta) > q^*(\theta)$ $\forall\theta \in (\theta_1, \theta_2)$, and at least one is true: $q(\theta_1) = q^*(\theta_1)$ or $\theta_1 = 0$. Note that by continuity $q(\theta_2) = q^*(\theta_2)$.

Let us construct a modified quantity schedule $q_a(.)$ in the following way: $q_a(\theta) = q(\theta)$ $\forall\theta \in [0,\theta_1]$, $q_a(\theta) = q^*(\theta)$ $\forall\theta \in (\theta_1, \theta_2]$. Further, if $\exists\hat{\theta} \in (\theta_2, 1)$ s.t.

$$\int_{\theta_1}^{\hat{\theta}} u_\theta(q^*(s), s)ds + \int_{\hat{\theta}}^1 u_\theta(\max\{q^*(\hat{\theta}), q(s)\}, s)ds = \int_{\theta_1}^1 u_\theta(q(s), s)ds \tag{45}$$

then set $q_a(\theta) = q^*(\theta)$ $\forall\theta \in (\theta_2, \hat{\theta}]$ and $q_a(\theta) = \max\{q^*(\hat{\theta}), q(\theta)\}$ $\forall\theta \in (\hat{\theta}, 1]$. Otherwise, if the left-hand side of (45) is strictly smaller than its right-hand side $\forall\theta \in [\theta_2, 1]$, then set $q_a(\theta) = q^*(\theta)$ $\forall\theta \in (\theta_2, 1]$.

Leaving the quantity schedule $g(.)$ unmodified, we need to establish two claims. **Claim 1:** all incentive constraints (7) still hold after this modification. **Claim 2:** this modification leads to an increase in the firm's expected profits given by (6).

To establish **Claim 1**, consider $\theta_3 = \sup\{\theta | \theta \geq \hat{\theta}, q(\theta) < q^*(\hat{\theta})\}$. Then $\forall\theta \in [0,\theta_1) \cup [\theta_3, 1]$, $U_a(\theta) \equiv \int_0^\theta u_\theta(q_a(s), s)ds = U(\theta) \equiv \int_0^\theta u_\theta(q(s), s)ds$. So, (7) holds $\forall\theta \in [0,\theta_1) \cup [\theta_3, 1]$ under $q_a(.)$ because it holds under $q(.)$.

Next suppose that $\theta \in [\theta_1, \hat{\theta}]$. Then $q_a(\theta) = \min\{q^*(\theta), q^*(\hat{\theta})\}$. Then, since $g(\theta') \leq q^*(\theta')$ and $q_a(\theta) \geq q^*(\theta_1)$, the inequality $U(\theta_1) \geq u(g(\theta'), \theta_1) - u(g(\theta'), \theta')$ $\forall\theta' \in [0, \hat{\theta}]$ implies that $U(\theta) \geq u(g(\theta'), \theta) - u(g(\theta'), \theta')$ $\forall\theta \in [\theta_1, \hat{\theta}]$ and $\theta' \in [0, \theta_1]$.

Similarly, $\forall\theta' \in [\theta_1, \theta_3]$, we have $U(\theta) \geq \int_{\theta'}^\theta u_\theta(q^*(s), s)ds \geq u(g(\theta'), \theta) - u(g(\theta'), \theta')$. It also follows immediately that $U(\theta) \geq u(g(\theta'), \theta) - u(g(\theta'), \theta')$ $\forall\theta' \in (\hat{\theta}, \theta_3)$ if $\theta' \in (\hat{\theta}, \theta_3)$ and $g(\theta') \leq q^*(\hat{\theta})$.

Finally, if $\theta \in (\hat{\theta}, \theta_3)$ and $\theta' \in (\hat{\theta}, \theta_3)$ and $g(\theta') > q^*(\hat{\theta})$, we can use $U(\theta_3) \geq u(g(\theta'), \theta_3) - u(g(\theta'), \theta')$ and $q_a(\theta) = q^*(\hat{\theta})\ \forall \theta \in [\hat{\theta}, \theta_3]$ to show that $U(\theta) \geq u(g(\theta'), \theta) - u(g(\theta'), \theta')$. So, all the incentive constraints (7) hold when we replace $q(.)$ with $q_a(.)$.

To prove **Claim 2**, focus on the first integral in (6). Note that $\int_0^1 (u(q(\theta), \theta) - c(q(\theta)) f(\theta) d\theta < \int_0^1 (u(q_a(\theta), \theta) - c(q_a(\theta)) f(\theta) d\theta$, because $u(q(\theta), \theta) - c(q(\theta)) \leq u(q_a(\theta), \theta) - c(q_a(\theta))$, $\forall \theta \in [0, 1]$ and the inequality is strict $\forall \theta \in (\theta_1, \theta_2)$. If there exists $\hat{\theta}$ satisfying (45), then we have:

$$\int_0^1 (u_\theta(q_a(\theta), \theta) - u_\theta(q(\theta), \theta))(1 - F(\theta)) d\theta = \int_{\theta_1}^{\theta_3} (u_\theta(\min\{q^*(\theta), q^*(\hat{\theta})\}, \theta) - u_\theta(q(\theta), \theta))(1 - F(\theta)) d\theta$$

$$\leq (1 - F(\theta_2)) \int_{\theta_1}^{\theta_3} (u_\theta(\min\{q^*(\theta), q^*(\hat{\theta})\}, \theta) - u_\theta(q(\theta), \theta)) d\theta = 0$$

where the first equality holds by definition of $q_a(.)$, the inequality holds because $\min\{q^*(\theta), q^*(\hat{\theta})\} = q^*(\hat{\theta}) > q(\theta)\ \forall \theta \in (\theta_1, \theta_2)$ and $\min\{q^*(\theta), q^*(\hat{\theta})\} = q^*(\hat{\theta}) < q(\theta)\ \forall \theta \in (\theta_2, \theta_3)$, and the last equality holds by (45). The same result obtains if there is no $\hat{\theta}$ satisfying (45). To see this simply replace $\hat{\theta}$ by 1. After this modification the value of the first integral in (6) increases, i.e. the firm's expected profits go up. Q.E.D.

Proof of Lemma 9. Suppose that $q(\theta)$ is a solution to Problem (17) on the domain $C_p^1([0, 1])$, but there exists an admissible schedule $\hat{q}(\theta) \in C([0, 1]) \setminus C_p^1([0, 1])$ s.t. the objective function in (17) takes a strictly higher value under $\hat{q}(\theta)$ than under $q(\theta)$.

By the Stone-Weierstrass theorem, the space of continuously differentiable functions $C^1([0, 1])$, which is a subspace of $C_p^1([0, 1])$ is dense in $C([0, 1])$. Therefore, $C_p^1([0, 1])$ is dense in $C([0, 1])$. So, there exists a sequence $\tilde{q}_n(\theta) \in C_p^1([0, 1])$ converging to $\hat{q}(\theta)$ in the *sup*−norm. The objective function (17) is continuous in the *sup*-norm. Therefore, $\exists N > 0$ s.t. $\forall n \geq N$ (17) takes a strictly higher value under $\tilde{q}_n(\theta)$ than under $q(\theta)$. This contradicts the hypothesis that $q(\theta)$ is a solution on $C_p^1([0, 1])$. *Q.E.D.*

Proof of Lemma 10. The Lemma will be established in a sequence of steps.
Step 1. $\exists \theta_1, \theta_2 \in [0, 1]$, $\theta_1 < \theta_2$, s.t. Case 1 applies i.e. $q(\theta) < q^*(r(\theta))$ on (θ_1, θ_2).

For suppose not. Then by continuity, $q(\theta) \geq q^*(r(\theta))\ \forall \theta \in [0, 1]$. This implies that for any θ s.t. $q'(\theta) > 0$, $q(\theta)$ is given by the solution to (36) with some constant of integration k_2.

Since $q(1) = q^*(1)$, (36) implies that $k_2 = 0$. Then, however, (36) cannot hold as an equality near $\theta = 0$ if $q(\theta) > 0\ \forall \theta > 0$. To see this, recall that $u_q(q, 0) = 0\ \forall q \geq 0$, and so $u_q(q, \theta) = \int_0^\theta u_{q\theta}(q, s) ds$. Therefore, $u_q(0, \theta) < u_{\theta q}(0, \theta) \frac{1 - F(\theta)}{f(\theta)}$ when θ is sufficiently small. Consequently, θ_0 solving $u_q(0, \theta_0) = u_{\theta q}(0, \theta_0) \frac{1 - F(\theta_0)}{f(\theta_0)}$ is strictly positive, and we must have $q(\theta) = 0\ \forall \theta \leq \theta_0$.
Step 2. $\exists \theta_l \in (0, 1)$ s.t. $q(\theta) < q^*(r(\theta))$, i.e. Case 1 applies, on $(0, \theta_l)$.

Suppose not. Then $\exists \theta_a, \theta_b \in (0, 1]$, s.t. Case 2 applies on $(0, \theta_a)$ and Case 1 applies on (θ_a, θ_b). So, $q(\theta_a) = q^*(r(\theta_a))$. Since $r(\theta_a) < \theta_a$, we have $u_q(q(\theta_a), \theta_a) > u_q(q(\theta_a), r(\theta_a)) = c'(q(\theta_a))$. Hence, since (36) holds on $(0, \theta_a)$, the constant of integration k_2 on this interval needs to satisfy $k_2 > -1$. Therefore, by the same argument as in Step 2, $\tilde{\theta}$ solving $u_q(0, \tilde{\theta}) = u_{\theta q}(0, \tilde{\theta}) \frac{1 + k_2 - F(\tilde{\theta})}{f(\tilde{\theta})}$ is strictly positive, and we must have $q(\theta) = 0\ \forall \theta \leq \tilde{\theta}$. Contradiction.
Step 3. $\exists \theta_n \in (\theta_l, 1)$ s.t. $q(\theta) > q^*(r(\theta))$, i.e. Case 2 applies, on $(\theta_n, 1)$.

Since $r(\theta) < \theta\ \forall \theta \in (0, 1]$, we have $q^*(r(\theta)) \leq q^*(r(1)) < q^*(1) = q(1)$. Then by continuity of $q(.)$ it follows that $q^*(r(\theta)) < q(\theta)$ for all θ sufficiently close to 1. This establishes the result.

Step 4. $\exists \theta_h \in [\theta_n, 1)$ s.t. $q(.)$ is strictly increasing on $[\theta_h, 1)$.

Suppose otherwise. Since by Lemma (1) $q(1) = q^*(1)$, $\exists \theta^f \in (0,1]$ s.t. $q(\theta) = q^*(1)$ $\forall \theta \in [\theta^f, 1]$. Also, since $q(.)$ is continuous and $q(0) = 0$, $\exists \theta^l$ s.t. $q(\theta^l) = q^*(\theta^l)$ and $q(\theta) > q^*(\theta)$ $\forall \theta \in (\theta^l, 1)$.

Let us show that the value of the objective can be strictly increased by replacing the quantity schedule $q(.)$ with modified quantity schedule $q_m(.)$ s.t. $q_m(\theta) = q(\theta)$ $\forall \theta \in [0, \theta^l]$, and $q^m(\theta) = q^*(\theta)$ $\forall \theta \in [\theta^l, 1]$.

First, from the definition of $r(\theta)$ it follows that $r(\theta) \leq \theta$ $\forall \theta \in [0,1]$.[35] Therefore, $\forall \theta \in (\theta^l, 1)$, $q(\theta) > q^m(\theta) \geq q^*(r(\theta))$, i.e. the solution is in Case 2 and $g(\theta) = q^*(\theta)$ $\forall \theta \in [r(\theta), 1]$. Thus, under both the original and the modified quantity schedule $g(\theta)$ is the same $\forall \theta \in [0,1]$.

Inspecting the objective in (6) it is easy to see that its value increases as a result of this modification, because the value of the first integral goes up, while the second integral remains unchanged.

Step 5 $q(\theta) = q^{sb}(\theta)$ on $[\theta_h, 1]$.

Since $q(.)$ is strictly increasing and is in Case 2 on $[\theta_h, 1)$, it is given by the solution to (36). From $q(1) = q^*(1)$, it follows that $k_2 = 0$ on this interval, and hence $q(\theta) = q^{sb}(\theta)$. Q.E.D.

Proof of Lemma 11. Define $z(\theta) = q(\theta) - q^*(r(\theta))$ and let $\theta_1 = \min\{\theta > 0 : z(\theta) \geq 0\}$. It follows from Lemma 10 that θ_1 is well defined. Let $\theta_2 \equiv \max\{\theta > \theta_1 : z(\hat{\theta}) \geq 0,\ \forall \hat{\theta} \in [\theta_1, \theta]\}$. Suppose that, contrary to the statement of Lemma 11, $\theta_2 < 1$. By definition of θ_1 and θ_2, $z'(\theta_1) \geq 0$ and $z'(\theta_2) \leq 0$. At the same time, the definition of $q^*(\cdot)$, equation (12), and the fact that (36) holds for all $\theta \in [\theta_1, \theta_2]$ yields $z'(\theta) = q'(\theta)(1 - \mu(\theta))$, where

$$\mu(\theta) = \frac{1}{c''(q(\theta)) - u_{qq}(q(\theta), r(\theta))} \frac{u_{q\theta}(q(\theta), r(\theta))}{u_\theta(q(\theta), r(\theta))} \frac{[1 + k_2 - F(\theta)] u_{q\theta}(q(\theta), \theta)}{f(\theta)}$$

Let $\omega(\theta)$, $\nu(\theta)$, and $\eta(\theta)$ respectively denote the first, second and third term in the above expression for $\mu(\theta)$. We may then compute:

$$\begin{aligned}
\frac{\nu'}{\nu} &= \frac{u_{qq\theta} u_\theta - u_{q\theta}^2}{u_{q\theta} u_\theta} q' + \frac{u_{q\theta\theta} u_\theta - u_{q\theta} u_{\theta\theta}}{u_{q\theta} u_\theta} r' \\
\frac{\omega'}{\omega} &= \frac{c''' - u_{qqq}}{c'' - u_{qq}} q' - \frac{u_{qq\theta}}{c'' - u_{qq}} r' \\
\frac{\eta'}{\eta} &= \frac{u_{qq\theta}}{u_{q\theta}} q' + \frac{\left(\frac{1+k_2-F(\theta)}{f(\theta)}\right)'}{\left(\frac{1+k_2-F(\theta)}{f(\theta)}\right)} + \frac{u_{q\theta\theta}}{u_{q\theta}}
\end{aligned}$$

Thus

$$\frac{\mu'}{\mu} = \left(\frac{u_{qq\theta}}{u_{q\theta}}(q,r) + \frac{u_{qq\theta}}{u_{q\theta}}(q,\theta) - \frac{u_{q\theta}}{u_\theta}(q,r) - \frac{c''' - u_{qqq}}{c'' - u_{qq}}(q,r) \right) q' +$$
$$\left(\frac{u_{q\theta\theta}}{u_{q\theta}}(q,r) - \frac{u_{\theta\theta}}{u_\theta}(q,r) + \frac{u_{qq\theta}}{c'' - u_{qq}}(q,r) \right) r' + \frac{\left(\frac{1+k_2-F(\theta)}{f(\theta)}\right)'}{\left(\frac{1+k_2-F(\theta)}{f(\theta)}\right)} + \frac{u_{q\theta\theta}}{u_{q\theta}}(q,\theta)$$

[35] This also follows from the differential equation (18) and the initial condition $r(0) = 0$, because by (18) $r'(\theta) < 0$ if $r(\theta) > \theta$.

The hypothesis that $\max_{\theta\in[0,1],q\in[0,q^*(1)]} \frac{u_{qq\theta}(q,\theta)}{u_{q\theta}(q,\theta)} \leq \min\{M, N\}$ implies that the terms multiplying q' and r' in the above expression are nonpositive. Furthermore, the hypothesis that $f(\theta)\frac{u_q(q,\theta)-c'(q)}{u_{\theta q}(\theta,q)}$ is strictly increasing, and the fact that (36) holds for all $\theta \in [\theta_1, \theta_2]$, imply that the sum of the last two terms in the above expression is strictly negative. We therefore have $\mu'(\theta) < 0\ \forall\theta \in [\theta_1, \theta_2]$, and hence $\mu(\theta_2) < \mu(\theta_1)$. The fact that $z'(\theta_1) = q'(\theta_1)(1 - \mu(\theta_1)) \geq 0$ then produces the contradiction that $z'(\theta_2) = q'(\theta_2)(1 - \mu(\theta_2)) > 0$. Q.E.D.

Proof of Lemma 12. The existence of at least one solution to (11) and (12) with the boundary conditions $q(0) = r(0) = 0$ and $q(1) = 1$ follows because the optimal solution, which does exists, must possess these properties. Now, suppose that there are two pairs of functions $(q_1(\theta), r_1(\theta))$ and $(q_2(\theta), r_2(\theta))$ with these properties.

Lemma 10 implies that for $i = 1, 2\ \exists\theta_i^f > 0$ s.t. $q_i(\theta_i^f) = q^*(r_i(\theta_i^f))$, and $\forall\theta \in (\theta_i^f, 1]\ q_i(\theta) \geq q^*(r_i(\theta))$, so $q_i(\theta)$ satisfies (36) with $k_2 = 0$. Suppose without loss of generality that $\theta_1^f > \theta_2^f$. (We can rule out $\theta_1^f = \theta_2^f$, because in this case $(q_1(\theta), r_1(\theta))$ would be identical to $(q_2(\theta), r_2(\theta))$.) It follows that, $q_1(\theta_1^f) = q_2(\theta_1^f)$ and $U_1(\theta_1^f) \equiv \int_0^{\theta_1^f} u_\theta(q_1(s), s)ds < U_2(\theta_1^f) \equiv \int_0^{\theta_1^f} u_\theta(q_2(s), s)ds$. Then there must exist $\tilde{\theta} \in (0, \theta_1^f]$ and $\epsilon_1 > 0$ s.t. $q_1(\theta) \geq q_2(\theta)\ \forall\theta \in [\tilde{\theta}, \theta_1^f]$ and $q_1(\theta) < q_2(\theta)$ $\forall\theta \in (\tilde{\theta} - \epsilon_1, \tilde{\theta}]$.

Therefore, $q_1'(\tilde{\theta}) \geq q_2'(\tilde{\theta})$ and there exists $\epsilon_2 > 0$ s.t. $q_1'(\theta) > q_2'(\theta)\ \forall\theta \in (\tilde{\theta} - \epsilon_2, \tilde{\theta})$. By inspection of (11), this implies that $\forall\theta \in (\tilde{\theta} - \epsilon_2, \tilde{\theta})\ q^*(r_2(\theta)) > q_1(\theta)$, and so by continuity $q^*(r_2(\tilde{\theta})) \geq q_1(\tilde{\theta})$.

Since $q_1(\theta) \geq q_2(\theta)\ \forall\theta \in [\tilde{\theta}, \theta_1^f]$ and $U_1(\theta_1^f) < U_2(\theta_1^f)$, it follows that $U_1(\tilde{\theta}) < U_2(\tilde{\theta})$, and so $r_1(\tilde{\theta}) > r_2(\tilde{\theta})$. But since $q^*(r_2(\theta)) \geq q_2(\tilde{\theta})$, $q_1(\tilde{\theta}) = q_2(\tilde{\theta})$ and $\frac{f(r)(u_q(q,r)-c'(q))}{u_\theta(q,r)}$ is increasing in r, it follows from (11) that $q_1'(\tilde{\theta}) < q_2'(\tilde{\theta})$. A contradiction. Q.E.D.

Proof of Corollary 1. Given that $u(q, \theta) = \theta q - \frac{q^2}{2}$, $c(q) = 0$, and $F(\theta) = \theta$, the first-best and second-best allocation are given by $q^*(\theta) = \theta$ and $q^{sb}(\theta) = \max\{2\theta - 1, 0\}$, respectively. Also, we can explicitly solve the defining equation for r which yields $r(\theta) = \theta - \frac{U(\theta)}{q(\theta)}$.

This example satisfies the conditions of Theorem 4 and Lemmas 11 and 12. Hence, there exists a unique switchpoint $\overline{\theta}$ such that on $[\overline{\theta}, 1]$ the solution is in Case 2 and satisfies $q(\theta) = q^{sb}(\theta) = 2\theta - 1$. On the interval $[0, \overline{\theta})$ the solution is in Case 1, and is characterized by a pair of differential equations (11) and (12) which in this case simplify to:

$$r' = \frac{q'(\theta - r)}{q} \tag{46}$$

$$q'(\alpha r + (1 - \alpha)q) = 2q \tag{47}$$

By Lemma 12 there is a unique solution to the system (46) and (47)[36] satisfying the correct boundary conditions $q(\overline{\theta}) = q^{sb}(\overline{\theta}) = q^*(r(\overline{\theta}))$. So, our goal is identify this solution and determine the boundary point $\overline{\theta}$.[37]

[36](47) provides another way to ascertain the no-exclusion result in the linear-quadratic case. If $\underline{\theta} \equiv \inf\{\theta|q(\theta > 0\} > 0$, then by definition $r(\underline{\theta}) = \underline{\theta}$. So, $\exists\theta_k \in (\underline{\theta}, 1)$ s.t. the solution is in Case 1 on $(\underline{\theta}, \theta_k)$ and has to satisfy (47). Since $q(.)$ and $r(.)$ are nonnegative and nondecreasing in θ, on this interval $q'(\theta) = \frac{2q(\theta)}{\alpha r(\theta)+(1-\alpha)q(\theta)} \leq \frac{2q(\theta)}{\alpha\underline{\theta}}$.

Pick $\theta \in (\underline{\theta}, \min\{\theta_k, \underline{\theta}+\frac{\alpha\underline{\theta}}{2}\})$. Integrating, we get $q(\theta) \leq \frac{2}{\alpha\underline{\theta}}\int_{\underline{\theta}}^{\theta} q(s)ds$. Since $\frac{2(\theta-\underline{\theta})}{\alpha\underline{\theta}} < 1$ and $q(.)$ is non-decreasing, this inequality can only hold if $q(\theta) = 0$, which contradicts the definition of $\underline{\theta} = \inf\{\theta|q(\theta) > 0\}$.

[37]It is possible to show directly that the solution switches between Cases 1 and 2 only once. Let $\overline{\theta}$ be the smallest switching point i.e. $q(\theta) < r(\theta)\ \forall\theta \in (0, \overline{\theta})$ and $q(\theta) \geq r(\theta)\ \forall\theta \in (\overline{\theta}, \tilde{\theta})$ for some $\tilde{\theta} \in (\overline{\theta}, 1)$. By continuity of the

Our strategy is to guess the structure of the solution. Inspection of the system (46) and (47) leads to the conjecture that $r(\theta) = a\theta + bq(\theta)$ on the interval $[0, \overline{\theta}]$, for some constants a and b. Applying this conjecture to (46) and (47) and rearranging we obtain:

$$\theta(\alpha a^2 + 2a - 2) = -q(\theta)(4b + a(1 - \alpha + \alpha b)) \tag{48}$$

Suppose that (48) holds as an identity,[38] i.e. $\alpha a^2 + 2a - 2 = 0$ and $(1-\alpha)a + 4b + ab\alpha = 0$. Solving for the coefficients a and b yields: $a = \frac{-1\pm\sqrt{1+2\alpha}}{\alpha}$, $\quad b = -\frac{1-\alpha}{4+\alpha a}a$.

Choose the positive root for a, so that $b = -\frac{1-\alpha}{\alpha}\frac{\sqrt{1+2\alpha}-1}{3+\sqrt{1+2\alpha}}$. By computation we can show that $a < 1$ and $a + b < 1$ $\forall \alpha > 0$, so $r(\theta) < \theta$. Let $y(\theta) = \ln \frac{q(\theta)}{\theta}$, so that $dy = \frac{dq}{q} - \frac{d\theta}{\theta}$. It follows from (47) that $\frac{dq}{q} = \frac{2}{\alpha a + (1-\alpha+\alpha b)e^y}\frac{d\theta}{\theta}$. Hence we obtain:

$$\frac{d\theta}{\theta} = dy\frac{c_0 + c_1 e^y}{c_2 - c_1 e^y}. \tag{49}$$

where $c_0 = \alpha a = \sqrt{1+2\alpha} - 1$, $c_1 = 1 - \alpha + \alpha b = \frac{4(1-\alpha)}{3+\sqrt{1+2\alpha}}$, and $c_2 = 2 - \alpha a = 3 - \sqrt{1+2\alpha}$. When $\alpha \neq 4$ so that $c_2 \neq 0$, we can integrate both sides of this equation to obtain:

$$\ln\theta = k + \frac{c_0}{c_2}y - \frac{c_0 + c_2}{c_2}\ln(|c_2 - c_1 e^y|),$$

where k is a constant of integration. Exponentiating both sides, substituting $y(\theta) = \ln\frac{q(\theta)}{\theta}$, and simplifying finally produces an implicit equation for $q(\theta)$:

$$[(2 - \alpha/2)\theta - (1-\alpha)q]^2 = h(\alpha)q^{\sqrt{1+2\alpha}-1} \tag{50}$$

where $h(\alpha) = \frac{(3+\sqrt{1+2\alpha})^2}{16}e^{k(3-\sqrt{1+2\alpha})}$. When $\alpha = 4$, (49) can be rewritten as $\frac{d\theta}{\theta} = \frac{dy}{y}\frac{1-e^y}{e^y}$, which can be solved directly to yield:

$$\theta = h(4)q - q\ln(q) \tag{51}$$

Note that $h(\alpha)$ defines a family of solutions to the system (47)-(46) reflecting the singularity of this system at the origin. To determine $h(\alpha)$, we will exploit the fact that only one member of this family satisfies the boundary condition $q(\overline{\theta}) = q^*(r(\overline{\theta})) = q^{sb}(\overline{\theta})$. Since $r(\theta) = a\theta + bq(\theta)$, the condition $q(\overline{\theta}) = q^*(r(\overline{\theta}))$ implies $q(\overline{\theta}) = \frac{a}{1-b}\overline{\theta}$. Combining this with the condition $q(\overline{\theta}) = q^{sb}(\overline{\theta}) = 2\overline{\theta} - 1$ yields $\overline{\theta} = \frac{1-b}{2-2b-a} = \frac{2}{3} + \frac{1}{3(\sqrt{1+2\alpha}+1)}$, so that $q(\overline{\theta}) = r(\overline{\theta}) = \frac{1}{3} + \frac{2}{3(\sqrt{1+2\alpha}+1)}$.[39]

For $\alpha = 4$, we substitute $\overline{\theta}(4) = 3/4$ and $q(\overline{\theta}(4)) = 2\overline{\theta}(4) - 1 = 1/2$ into $\theta = h(4)q - q\ln(q)$ to obtain $h(4) = 3/2 + \ln(1/2)$. For $\alpha \neq 4$, substituting $\overline{\theta}$ and $q(\overline{\theta})$ into (50) yields:

$$h(\alpha) = \frac{\left(1 + \frac{\alpha}{2(\sqrt{1+2\alpha}+1)}\right)^2}{\left(\frac{1}{3} + \frac{2}{3(\sqrt{1+2\alpha}+1)}\right)^{\sqrt{1+2\alpha}-1}}$$

optimal quantity schedule $q(.)$, $q(\overline{\theta}) = r(\overline{\theta})$, and $q'(\overline{\theta}) \geq r'(\overline{\theta}) = q'(\overline{\theta})\frac{\overline{\theta}-r(\overline{\theta})}{q(\overline{\theta})}$. Thus, $\overline{\theta} \leq r(\overline{\theta}) + q(\overline{\theta})$. But $q'(\theta) = 2$ $\forall\theta \in (\overline{\theta}, \tilde{\theta})$. Hence, $\theta < r(\theta) + q(\theta)$ and $q'(\theta) > r'(\theta)$ $\forall\theta \in (\overline{\theta}, \tilde{\theta})$. So, $q(\tilde{\theta}) > r(\tilde{\theta})$, i.e. the solution cannot switch to Case 2 at $\tilde{\theta}$. This implies that $\tilde{\theta} = 1$.

[38] Otherwise, $q(\theta)$ must be a linear function of θ, in which case (46) and (47) can be solved to yield $q(\theta) = \theta\frac{3+\alpha}{2\alpha}$ and $r(\theta) = \theta/2$. But $q(\theta) > r(\theta)$, and so we can rule out this possibility.

[39] Note that $\overline{\theta}$ is decreasing in α. It converges to 2/3 as α increases to infinity (almost all consumers are 'honest'), and converges to 5/6 as α decreases to 0 (almost all consumers are strategic).

Note that only the positive root of equation (50) holds as an equality at $\overline{\theta}$. So, if $\exists\theta_1 \in (0, \overline{\theta})$ s.t. $(2-\alpha/2)\theta_1 - (1-\alpha)q(\theta_1) < 0$, then by continuity $\exists\theta_2 \in (\theta_1, \overline{\theta})$ satisfying $(2-\alpha/2)\theta_2 - (1-\alpha)q(\theta_2) = 0$. But since $q(\theta_2) > 0$, (50) cannot hold at θ_2. Thus, $q(\theta)$ is a solution to:

$$\theta = \frac{(1-\alpha)}{2-\alpha/2} q + \frac{h(\alpha)^{1/2}}{(2-\alpha/2)} q^{\frac{\sqrt{1+2\alpha}-1}{2}} \qquad (52)$$

Both (51) and (52) characterize $q(.)$ as an implicit function of θ for given α. Since these equations may have multiple roots, we need to establish that $q(\theta)$ is well-defined. Start with $\alpha \neq 4$. Consider $\theta(q)$ as a function of q defined by (52) on $[0, q(\overline{\theta})]$. Substitution yields

$$r(\theta(q)) - q = a\theta(q) + bq - q = \frac{\sqrt{1+2\alpha}-1}{\alpha} \left(\frac{(1-\alpha)}{2-\alpha/2} q + \frac{h(\alpha)^{1/2}}{2-\alpha/2} q^{\sqrt{1+2\alpha}-1} \right) - \left(1 + \frac{1-\alpha}{\alpha} \frac{\sqrt{1+2\alpha}-1}{3+\sqrt{1+2\alpha}} \right) q$$

Observe that $r(\theta(q)) - q$ is strictly concave in q, and $r(\theta(0)) = 0$, while our choice of $h(\alpha)$ guarantees that $r(\theta(q(\overline{\theta}))) - q(\overline{\theta}) = 0$. Then, by strict concavity, $r(\theta(q)) - q > 0$ $\forall q \in (0, q(\overline{\theta}))$. Since $\theta(q)$ must also satisfy (47), $r(\theta(q)) - q > 0$ implies that θ is strictly increasing in q on $[0, q(\overline{\theta})]$. Therefore, on $[0, \overline{\theta}]$ (52) admits a unique increasing continuous solution $q(\theta)$ s.t. $q(\overline{\theta}) = 2\overline{\theta} - 1$ and $r(\theta) > q(\theta,)$ $\forall\theta \in (0, \overline{\theta})$. Since $\theta(q)$ is strictly concave in q, $q(\theta)$ is convex. The case $\alpha = 4$ can be handled in a similar way.

We have thus found the unique solution to (47) and (46) that satisfies the condition $q(\overline{\theta}) = q^{sb}(\overline{\theta}) = q^*(r(\overline{\theta}))$ for some $\overline{\theta}$, and hence the solution to Problem (19).

It remains to determine the optimal schedule for the 'honest' types. On the interval $[\overline{\theta}, 1]$ the solution is in Case 2, so $g(\theta) = q^*(\theta) = \theta$ for $\theta \in [r(\overline{\theta}), 1]$, where $r(\overline{\theta}) = \frac{1}{3} + \frac{2}{3(\sqrt{1+2\alpha}+1)}$. Meanwhile, on the interval $[0, \overline{\theta}]$ the solution is in Case 1, so $g(\theta) = q(r^{-1}(\theta))$ on $[0, r(\overline{\theta})]$. To determine $g(.)$ on this interval, note that $r(.)$ is strictly increasing on $[0, \overline{\theta}]$, so its inverse $r^{-1}(\theta)$ is well-defined $\forall\theta \in [0, r(\overline{\theta})]$. For $\alpha \neq 4$, we can use (52) to obtain:

$$r^{-1}(\theta) = \frac{(1-\alpha)}{2-\alpha/2} g(\theta) + \frac{h(\alpha)^{1/2}}{(2-\alpha/2)} g(\theta)^{\frac{\sqrt{1+2\alpha}-1}{2}} \qquad (53)$$

From $r(\theta) = a\theta + bq(\theta)$ it follows that $\theta = ar^{-1}(\theta) + bg(\theta)$. Substituting this into (53) and solving for θ produces (14). To show that $g(\alpha)$ is well-defined by (14) and is continuous and increasing on $[0, \frac{1}{3} + \frac{2}{3(\sqrt{1+2\alpha_1}+1)}]$, use an argument similar to the one establishing these properties for $q(\theta)$. In a similar way, we can compute $g(\theta)$ for $\alpha = 4$. *Q.E.D.*

Proof of Corollary 2: Fix some α_1 and α_2 s.t. $\alpha_1 > \alpha_2 > 0$.
Part (i). Instead of $q(\theta, \alpha)$, it is more convenient to operate with its inverse - the function $\theta(q, \alpha)$ given by (52). Since $\theta(\frac{1}{3} + \frac{2}{3(\sqrt{1+2\alpha_1}+1)}, \alpha_1) = \frac{2}{3} + \frac{1}{3(\sqrt{1+2\alpha_1}+1)} > \theta(\frac{1}{3} + \frac{2}{3(\sqrt{1+2\alpha_1}+1)}, \alpha_2)$, we need to show that $\theta(q, \alpha_1) < \theta(q, \alpha_2)$ when θ is sufficiently small, and that there exists a unique point of intersection $q_c(\alpha_1, \alpha_2) \in (0, \frac{1}{3} + \frac{2}{3(\sqrt{1+2\alpha_1}+1)})$ s.t. $\theta(q_c(\alpha_1, \alpha_2), \alpha_1) = \theta(q_c(\alpha_1, \alpha_2), \alpha_2)$.

Step 1. $\exists q_l \in (0, \frac{1}{3} + \frac{2}{3(\sqrt{1+2\alpha}+1)})$ s.t. $\theta(q, \alpha_1) < \theta(q, \alpha_2)$ $\forall q \in (0, q_l)$.

When both $\alpha_1 \neq 4$ and $\alpha_2 \neq 4$, then by (52) $\theta(q, \alpha_1) < \theta(q, \alpha_2)$ iff

$$\frac{(1-\alpha_1)}{2-\alpha_1/2} q + \frac{k(\alpha_1)^{1/2}}{2-\alpha_1/2} q^{\frac{\sqrt{1+2\alpha_1}-1}{2}} < \frac{(1-\alpha_2)}{2-\alpha_2/2} q + \frac{k(\alpha_2)^{1/2}}{2-\alpha_2/2} q^{\frac{\sqrt{1+2\alpha_2}-1}{2}}$$

Dividing both sides of this inequality by $q^{\frac{\sqrt{1+2\alpha_2}-1}{2}}$ and rearranging we obtain an equivalent inequality:

$$\frac{k(\alpha_1)^{1/2}}{2-\alpha_1/2}q^{\frac{\sqrt{1+2\alpha_1}-\sqrt{1+2\alpha_2}}{2}} < \left(\frac{(1-\alpha_2)}{2-\alpha_2/2} - \frac{(1-\alpha_1)}{2-\alpha_1/2}\right) q^{\frac{3-\sqrt{1+2\alpha_2}}{2}} + \frac{k(\alpha_2)^{1/2}}{2-\alpha_2/2}$$

Let q go to zero. Then the left-hand of the above inequality converges to zero. If $\alpha_2 > 2$, then the first term on the right-hand side converges to plus infinity because $\frac{(1-\alpha_2)}{2-\alpha_2/2} > \frac{(1-\alpha_1)}{2-\alpha_1/2}$, while the second term is bounded. If $\alpha_2 < 2$, then the first term on the right-hand side converges to zero, while the second remains is a positive and constant. So, the inequality holds when q is sufficiently small.

If $\alpha_2 = 4$, then we need to show that

$$\frac{(1-\alpha_1)}{2-\alpha_1/2}q + \frac{k(\alpha_1)^{1/2}}{2-\alpha_1/2}q^{\frac{\sqrt{1+2\alpha_1}-1}{2}} \leq q\left(3/2 + \log(1/2) - \log(q)\right)$$

It is easy to see that this inequality holds for small q after we divide both sides of it by q, and then let q go to zero.

If $\alpha_1 = 4$, then we need to show that

$$q\left(3/2 + \log(1/2) - \log(q)\right) < \frac{(1-\alpha_2)}{2-\alpha_2/2}q + \frac{k(\alpha_2)^{1/2}}{2-\alpha_2/2}q^{\frac{\sqrt{1+2\alpha_2}-1}{2}}$$

To see that this inequality holds for small q, we divide both sides of it by $q^{\frac{\sqrt{1+2\alpha_2}-1}{2}}$, and let q go to zero.

Step 2. Existence of an intersection point. $\exists q_i \in [0, \frac{1}{3} + \frac{2}{3(\sqrt{1+2\alpha_1}+1)}]$, $\epsilon_1 > 0$ and $\epsilon_2 > 0$ s.t. $\theta(q_i, \alpha_1) = \theta(q_i, \alpha_2)$, $\theta(q, \alpha_1) > \theta(q, \alpha_2)$ $\forall q \in (q_i, q_i + \epsilon_1)$, and $\theta(q, \alpha_1) < \theta(q, \alpha_2)$ $\forall q \in (q_i, q_i - \epsilon_2)$.

The existence of an intersection point q_i follows from Step 1 and from the fact that $\theta(\frac{1}{3} + \frac{2}{3(\sqrt{1+2\alpha_1}+1)}, \alpha_1) = \frac{2}{3} + \frac{1}{3(\sqrt{1+2\alpha_1}+1)} > \theta(\frac{1}{3} + \frac{2}{3(\sqrt{1+2\alpha_1}+1)}, \alpha_2)$.

Step 3. If $\theta_{qq}(q_1, \alpha_1) \leq \theta_{qq}(q_1, \alpha_2)$, then $\theta_{qq}(q_2, \alpha_1) < \theta_{qq}(q_2, \alpha_2)$ $\forall q_2 > q_1$, where $\theta_{qq}(q, \alpha)$ is the second derivative of $\theta(q, \alpha)$ with respect to q. This step follows by simple computation.

Step 4. Uniqueness of an intersection point on $[0, \frac{1}{3} + \frac{2}{3(\sqrt{1+2\alpha_1}+1)}]$.

Let q_i be the smallest q s.t. $\theta(q_i, \alpha_1) = \theta(q_i, \alpha_2)$. Then by Step 1 $\theta(q, \alpha_1) < \theta(q, \alpha_2)$ $\forall q \in (0, q_i)$, and so $\theta_q(q_i, \alpha_1) \geq \theta_q(q_i, \alpha_2)$. To finalize the proof, consider two cases.

Case 2. $\exists \epsilon_1 > 0$ s.t. $\theta(q, \alpha_1) > \theta(q, \alpha_2)$ $\forall q \in (q_i, q_i + \epsilon_1)$. Then $\theta_q(q_i, \alpha_1) > \theta_q(q_i, \alpha_2)$. If $\exists q_r \in (q_i, \frac{1}{3} + \frac{2}{3(\sqrt{1+2\alpha_1}+1)})$ s.t. $\theta(q_r, \alpha_1) = \theta(q_r, \alpha_2)$, let us choose the smallest such q_r. So, $\theta(q, \alpha_1) > \theta(q, \alpha_2)$ for $q \in (q_i, q_r)$. Therefore, $\theta_q(q_r, \alpha_1) \leq \theta_q(q_r, \alpha_2)$. But since $\theta_q(q_i, \alpha_1) > \theta_q(q_i, \alpha_2)$, we conclude that $\exists q_m \in [q_i, q_r]$ s.t. $\theta_{qq}(q_m, \alpha_1) \leq \theta_{qq}(q_m, \alpha_2)$. But then by Step 3 $\theta_{qq}(q, \alpha_1) < \theta_{qq}(q, \alpha_2)$ $\forall q > q_m$.

Consequently, $\theta_q(q, \alpha_1) < \theta_q(q, \alpha_2)$ $\forall q > q_r$, and so $\theta(q, \alpha_1) < \theta(q, \alpha_2)$ $\forall q > q_r$. But this contradicts the fact that $\theta(\frac{1}{3} + \frac{2}{3(\sqrt{1+2\alpha_1}+1)}, \alpha_1) > \theta(\frac{1}{3} + \frac{2}{3(\sqrt{1+2\alpha_1}+1)}, \alpha_2)$.

Case 1. $\exists \epsilon_1 > 0$ s.t. $\theta(q, \alpha_1) \leq \theta(q, \alpha_2)$ $\forall q \in (q_i, q_i + \epsilon_1)$.

Then $\theta_q(q_i, \alpha_1) = \theta_q(q_i, \alpha_2)$ and $\theta_{qq}(q_i, \alpha_1) \leq \theta_{qq}(q_i, \alpha_2)$, and so by Step 3, $\theta_{qq}(q, \alpha_1) < \theta_{qq}(q, \alpha_2)$ $\forall q > q_i$. Hence, $\theta_q(q, \alpha_1) < \theta_q(q, \alpha_2)$ and, consequently, $\theta(q, \alpha_1) < \theta(q, \alpha_2)$ $\forall q > q_i$. This also contradicts the fact that $\theta(\frac{1}{3} + \frac{2}{3(\sqrt{1+2\alpha_1}+1)}, \alpha_1) > \theta(\frac{1}{3} + \frac{2}{3(\sqrt{1+2\alpha_1}+1)}, \alpha_2)$.

Part (ii). Recall that $U(\theta, \alpha) \equiv \theta q(\theta, \alpha) - t^s(\theta, \alpha)$ is the total surplus of a 'strategic' consumer with valuation θ. We have established that $U(\theta, \alpha) = \int_0^\theta q(s, \alpha)ds$.

Since $q(\theta, \alpha_1) > q(\theta, \alpha_2)\ \forall \theta \in (0, \theta_c(\alpha_1, \alpha_2)$, we also have $U(\theta, \alpha_1) > U(\theta, \alpha_2)\ \forall \theta \in (0, \theta_c(\alpha_1, \alpha_2)$.

Let us show that $U(\overline{\theta}(\alpha_2), \alpha_1) > U(\overline{\theta}(\alpha_2), \alpha_2)$. Note that $\overline{\theta}(\alpha_2) > \overline{\theta}(\alpha_1)$, and $r(\overline{\theta}(\alpha_2), \alpha_2) > r(\overline{\theta}(\alpha_1), \alpha_1)$. Combining these inequalities and invoking Lemma 8, we conclude that $g(\overline{\theta}(\alpha_2), \alpha_1) = g(\overline{\theta}(\alpha_2), \alpha_2) = \overline{\theta}(\alpha_2)$.

By definition, $U(\overline{\theta}(\alpha_2), \alpha_2) = \left(\overline{\theta}(\alpha_2) - r(\overline{\theta}(\alpha_2))\right) r(\overline{\theta}(\alpha_2))$. At the same time, by Lemma 7, $U(\overline{\theta}(\alpha_2), \alpha_1) > \left(\overline{\theta}(\alpha_2) - r(\overline{\theta}(\alpha_2))\right) g(r(\overline{\theta}(\alpha_2)), \alpha_1) = \left(\overline{\theta}(\alpha_2) - r(\overline{\theta}(\alpha_2))\right) r(\overline{\theta}(\alpha_2))$. So, $U(\overline{\theta}(\alpha_2), \alpha_1) > U(\overline{\theta}(\alpha_2), \alpha_2)$.

Further, $U(\theta, \alpha_1) > U(\theta, \alpha_2)\ \forall \theta \in [\theta_c(\alpha_1, \alpha_2), \overline{\theta}(\alpha_2)]$, because $U_\theta(\theta, \alpha_1) = q(\theta, \alpha_1) < q(\theta, \alpha_2) = U_\theta(\theta, \alpha_2)$ on this interval. Finally, $U(\theta, \alpha_1) > U(\theta, \alpha_2)\ \forall \theta \in [\overline{\theta}(\alpha_2), 1]$ because $U_\theta(\theta, \alpha_1) = q(\theta, \alpha_1) = 2\theta - 1 = q(\theta, \alpha_2) = U_\theta(\theta, \alpha_2)$ on this interval.

Part (iii).

Step 1. $\exists \theta_m > 0$ s.t. $\forall \theta \in (0, \theta_m)\ g(\theta, \alpha_1) > g(\theta, \alpha_2)$.

Using an argument identical to the one in **Step 1** of Part (i) in this proof, we can prove an equivalent result - if g is small enough, then $\theta(g, \alpha_1) < \theta(g, \alpha_2)$.

Step 2. $g(\theta, \alpha_1) > g(\theta, \alpha_2)\ \forall \theta \in [r(\overline{\theta}(\alpha_1), \alpha_1), r(\overline{\theta}(\alpha_2), \alpha_2))$.

Note that $r(\overline{\theta}(\alpha_1), \alpha_1) = \frac{1}{3} + \frac{2}{3(\sqrt{1+2\alpha_1}+1)} < r(\overline{\theta}(\alpha_2), \alpha_2)) = \frac{1}{3} + \frac{2}{3(\sqrt{1+2\alpha_2}+1)}$, $g(\theta, \alpha_1) = \theta$ $\forall \theta \geq r(\overline{\theta}(\alpha_1), \alpha_1)$ and $g(\theta, \alpha_2) < \theta\ \forall \theta \in (0, r(\overline{\theta}(\alpha_2), \alpha_2))$.

Step 3. If $\exists \theta \in [\theta_m, r(\overline{\theta}(\alpha_1), \alpha_1)]$ s.t. $g(\theta, \alpha_1) \leq g(\theta, \alpha_2)$, then $\exists \theta_1, \theta_2 \in [\theta_m, r(\overline{\theta}(\alpha_1), \alpha_1)]$, $\theta_1 \leq \theta_2$ and $\delta > 0$, s.t. (i) $g(\theta_1, \alpha_1) = g(\theta_1, \alpha_2)$ and $g(\theta, \alpha_1) > g(\theta, \alpha_2)\ \forall \theta \in (\theta_1 - \delta, \theta_1)$; (ii) $g(\theta_2, \alpha_1) = g(\theta_2, \alpha_2)$ and $g(\theta, \alpha_1) > g(\theta, \alpha_2)\ \forall \theta \in (\theta_2, \theta_2+\delta)$. Consequently, $g_\theta(\theta_1, \alpha_1) \leq g_\theta(\theta_1, \alpha_2)$ and $g_\theta(\theta_2, \alpha_1) \geq g_\theta(\theta_2, \alpha_2)$. The proof of this step is obvious.

Step 4. $g_\theta(\theta, \alpha) = \frac{q(r^{-1}(\theta,\alpha),\alpha)}{r^{-1}(\theta,\alpha)-\theta} = \frac{g(\theta,\alpha)}{r^{-1}(\theta,\alpha)-\theta}\ \forall \theta \in (0, r(\overline{\theta}(\alpha_1), \alpha_1))$.

To see this, differentiate $g(\theta, \alpha) \equiv q(r^{-1}(\theta, \alpha), \alpha)$ and use (46) to make a substitution.

Step 5. $r^{-1}(\theta_1, \alpha_1) > r^{-1}(\theta_1, \alpha_2)$.

Since $g(\theta_1, \alpha_1) = g(\theta_1, \alpha_2)$ and $g_\theta(\theta_1, \alpha_1) \leq g_\theta(\theta_1, \alpha_2)$, Step 4 implies that $r^{-1}(\theta_1, \alpha_1) \geq r^{-1}(\theta_1, \alpha_2)$.

This inequality must be strict, i.e. $r^{-1}(\theta_1, \alpha_1) \neq r^{-1}(\theta_1, \alpha_2)$. To see this, rewrite $r(\theta, \alpha) = \theta - \frac{U(\theta,\alpha)}{q(\theta,\alpha)}$ as $\theta = r^{-1}(\theta, \alpha) - \frac{U(r^{-1}(\theta,\alpha),\alpha)}{q(r^{-1}(\theta,\alpha),\alpha)}$. Therefore, if $r^{-1}(\theta_1, \alpha_1) = r^{-1}(\theta_1, \alpha_2)$, then we have: $\frac{U(r^{-1}(\theta_1,\alpha_1),\alpha_1)}{q(r^{-1}(\theta_1,\alpha_1),\alpha_1)} = \frac{U(r^{-1}(\theta_1,\alpha_2),\alpha_2)}{q(r^{-1}(\theta_1,\alpha_2),\alpha_2)}$. But $q(r^{-1}(\theta_1, \alpha_1), \alpha_1) = g(\theta_1, \alpha_1) = g(\theta_1, \alpha_2) = q(r^{-1}(\theta_1, \alpha_2), \alpha_2)$, yet $U(r^{-1}(\theta_1, \alpha_1), \alpha_1) > U(r^{-1}(\theta_1, \alpha_2), \alpha_2)$ as established above. Contradiction.

Step 6. $r^{-1}(\theta_1, \alpha_2) > \theta_c(\alpha_1, \alpha_2)$.

Note that $q(r^{-1}(\theta_1, \alpha_1), \alpha_1) = g(\theta_1, \alpha_1) = g(\theta_1, \alpha_2) = q(r^{-1}(\theta_1, \alpha_2), \alpha_2)$. But by Step 5, $r^{-1}(\theta_1, \alpha_1) > r^{-1}(\theta_1, \alpha_2)$. So, since $q(\theta, \alpha)$ is strictly increasing in θ, $q(r^{-1}(\theta_1, \alpha_2), \alpha_1) < q(r^{-1}(\theta_1, \alpha_2), \alpha_2)$, and hence $r^{-1}(\theta_1, \alpha_2) > \theta_c(\alpha_1, \alpha_2)$.

Step 7. $r^{-1}(\theta_2, \alpha_2) \geq r^{-1}(\theta_2, \alpha_1)$.

To see this, combine $g_\theta(\theta_2, \alpha_1) \equiv \frac{q(r^{-1}(\theta_2,\alpha_1),\alpha_1)}{r^{-1}(\theta_2,\alpha_1)-\theta_2} \geq g_\theta(\theta_1, \alpha_2) \equiv \frac{q(r^{-1}(\theta_1,\alpha_2),\alpha_2)}{r^{-1}(\theta_2,\alpha_2)-\theta_2}$ with the fact that $q(r^{-1}(\theta_2, \alpha_1), \alpha_1) \equiv g(\theta_2, \alpha_1) = g(\theta_2, \alpha_2) \equiv q(r^{-1}(\theta_2, \alpha_2), \alpha_2)$.

Step 8. Since $r^{-1}(\theta_2, \alpha_2) \geq r^{-1}(\theta_2, \alpha_1) > \theta_c(\theta_1, \theta_2)$, it follows that $g(\theta_2, \alpha_2) \equiv q(r^{-1}(\theta_2, \alpha_2), \alpha_2) > g(\theta_2, \alpha_1) \equiv q(r^{-1}(\theta_2, \alpha_1), \alpha_1)$. However, by assumption $g(\theta_2, \alpha_2) = g(\theta_2, \alpha_1)$. This contradiction implies that $g(\theta, \alpha_1) > g(\theta, \alpha_2)\ \forall \theta \in (0, \frac{1}{3} + \frac{2}{3(\sqrt{1+2\alpha_2}+1)})$. *Q.E.D.*