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Consequences of Congruence for GARCH Modelling[1]

1. Introduction

In 1981 Granger formulated the idea of congruence, as similarity of main properties of the endogenous variable and main properties of exogenous variables. Zielinski (1984) introduced the idea of dynamic congruent modelling. Congruence of the model in the meaning of Zieliński means that harmonic structure of the endogenous process is congruent with the joint harmonic structure of explanatory processes and the error term, which is independent of explanatory processes. The idea of dynamic congruent modelling is based on capturing information about the internal structure of processes at the model specification stage and building the congruent model on a basis of white noise error terms equation. Models which were built according to the procedure of dynamic congruent modelling are often better models concerning statistical properties, if only the internal structures of analysed processes are correctly discovered and specified in the model. The use of information about the internal structure of economic processes was also a basis for the idea of congruent modelling introduced by Granger (1990).

The concept of congruence of conditional variances is not mentioned in those two ideas of modelling. In this article the idea of congruence is extended to conditional variances. In this paper congruence of the model in conditional variances means equality of conditional variance of endogenous variable with conditional variance of linear function of explanatory processes and error term. It is assumed that all analysed processes have constant and finite unconditional variances. A model is congruent in conditional variance if the unconditional

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variance of the endogenous variable is equal to the unconditional variance of linear function of explanatory processes and the error term and there is congruence of harmonic structure of the endogenous process squared with harmonic structure of the square of linear function of explanatory processes and the error term[2].

The article is laid out in four sections. Section 2 introduces the models congruent in variance for GARCH processes. Section 3 describes consequences of modelling which is not congruent in variance and introduces a new volatility measures. Section 4 concludes.

2. Models Which Are Congruent in Conditional Variance

The GARCH(p,q) process can be written as:

$$\varepsilon_t \big| \psi_{t-1} \sim D(0, h_t), \tag{1}$$

$$h_t = \alpha_o + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j h_{t-j}, \tag{2}$$

where ψ_{t-1} is the set of information available at time t–1 and $D(0,h_t)$ is probability distribution (usually the normal or Student–t distribution) with zero mean and variance h_t.

Let ε_{yt}, $\varepsilon_{x_s t}$ $(s = 1,2,...,k)$ and ε_t be white noise GARCH processes (with zero mean, constant unconditional variance and no autocorrelation). The model:

$$\varepsilon_{yt} = \sum_{s=1}^{k} \rho_s \varepsilon_{x_s t} + \varepsilon_t, \tag{3}$$

where $E(\varepsilon_t \varepsilon_{x_s t}) = 0$, is congruent, when harmonic structures of the left and right hand side terms of the equation are identical.

If ε_{yt} is an error process in the model describing internal structure of dependent variable, and $\varepsilon_{x_s t}$ are error processes in the models explaining internal structures of explanatory variables , then congruent model may be constructed in a traditional way using in equation (3) internal structure of the processes (see Talaga and Zieliński, 1986). Returns of financial processes are often stationary and usually they may be sufficiently described by autoregressive models with low order of autoregression.

[2] The idea of congruence can be extended to higher conditional moments of a distribution, however in this case interpretation is not easy.

Presented congruent models do not have to be congruent in conditional variance. First let us consider an example with one explanatory variable. Let ε_{xt} and ε_{yt} be GARCH processes of orders: (p_1, q_1) and (p_2, q_2). If ε_{yt}, ε_{xt} and ε_t are white noise processes, then model:

$$\varepsilon_{yt} = \rho\varepsilon_{xt} + \varepsilon_t \tag{4}$$

is congruent.

Model (4) is congruent in conditional variance, if unconditional variance ε_{yt} is equal to unconditional variance $\rho\varepsilon_{xt} + \varepsilon_t$, and harmonic structures of the processes ε_{yt}^2 and $(\rho\varepsilon_{xt} + \varepsilon_t)^2$ are identical, that is if the model:

$$\varepsilon_{yt}^2 = \rho^2\varepsilon_{xt}^2 + 2\rho\varepsilon_{xt}\varepsilon_t + \varepsilon_t^2 \tag{5}$$

is congruent.

The equality of unconditional variances of ε_t and $\rho\varepsilon_{xt} + \varepsilon_t$ is required in every congruent model of type (4).

Assuming $v_t = \varepsilon_t^2 - h_t$, GARCH$(p,q)$ model may be written as an ARMA (*m*,*p*) model for ε_t^2 (where *m* = *max* {*p*,*q*}). It follows that ε_{xt}^2 and ε_{yt}^2 may be presented as:

$$\varepsilon_{xt}^2 = \alpha_{01} + \sum_{i=1}^{q_1}\alpha_{i1}\varepsilon_{xt-i}^2 + \sum_{j=1}^{p_1}\beta_{j1}\varepsilon_{xt-j}^2 - \sum_{j=1}^{p_1}\beta_{j1}v_{xt-j} + v_{xt}\,, \tag{6}$$

$$\varepsilon_{yt}^2 = \alpha_{02} + \sum_{i=1}^{q_2}\alpha_{i2}\varepsilon_{yt-i}^2 + \sum_{j=1}^{p_2}\beta_{j2}\varepsilon_{yt-j}^2 - \sum_{j=1}^{p_2}\beta_{j2}v_{yt-j} + v_{yt}\,, \tag{7}$$

where v_{yt}, v_{xt} are white noise processes (if ε_{yt} and ε_{xt} have finite fourth order moments). Most financial processes have finite fourth order moments. The exceptions are mainly financial instruments from money market and less liquid instruments from other markets.

The structure of ε_t^2 will depend on internal structure of ε_{yt}^2 and ε_{xt}^2 processes. Conditional variance of the error term ε_t in equation (4) may then be described using GARCH(p_3, q_3) model. The values of p_3 and q_3 will depend on the character of the relationship between ε_{yt}^2 and ε_{xt}^2 and the properties of these processes (more about the properties of processes which are a sum of other autoregressive processes may be found in Kufel, Piłatowska and Zieliński, 1996, Stawicki and Górka, 1996). If $B_1(u)Y_{1t} = A_1(u)\varepsilon_{1t}$ and

$B_2(u)Y_{2t} = A_2(u)\varepsilon_{2t}$ are independent ARMA processes of degree (p_1, q_1) and (p_2, q_2) respectively, then

$$Y_t = Y_{2t} - Y_{1t} = B_2^{-1}(u)A_2(u)\varepsilon_{2t} - B_1^{-1}(u)A_1(u)\varepsilon_{1t} \tag{8}$$

and

$$B_1(u)B_2(u)Y_t = B_1(u)A_2(u)\varepsilon_{2t} - B_2(u)A_1(u)\varepsilon_{1t}. \tag{9}$$

Thus Y_t process is of $\mathrm{ARMA}(p,q)$ type, where $p = p_1 + p_2$ and $q = \max(p_1 q_2, p_2 q_1)$. If some parameters are close to zero, then the process may be identified as a process with fewer lags. In the case of financial time series there is usually a great number of observations, thus Y_t will almost always be an identifiable ARMA process. Similarly, when ε_{yt}^2 and ε_{xt}^2 are independent then the variance ε_t in equation (4) may be presented as a $\mathrm{ARMA}(p,q)$ process. If equation (4) is a model for error processes in the models describing internal structure of dependent and explanatory variables, then only correct specification of model (4) and GARCH model for ε_t will assure congruence in variance of model (4). In specification one should properly describe internal structure of dependent and explanatory variables (in order to receive white noise properties) and properly describe orders in GARCH model.

One reaches similar conclusions from a general model for k explanatory variables (equation (3)), assuming $E(\varepsilon_{x_s t}\varepsilon_{x_{s'} t}) = 0$. When explanatory variables are correlated then variance of ε_t will also depend on covariances of particular explanatory variables.

It is worth to mention that there is no possibility to create a classical congruent model between GARCH processes ε_{yt} and ε_{xt} on the basis of standardised processes. Let z_{yt}, z_{xt} and z_t be standardised white noise processes with constant conditional variances:

$$z_{yt} = \varepsilon_{yt} / h_{yt}, \ z_{xt} = \varepsilon_{xt} / h_{xt}, \ z_t = \varepsilon_t / h_t. \tag{10}$$

The model:

$$z_{yt} = \rho z_{xt} + z_t \tag{11}$$

is congruent. Substituting (10) in equation (11) and multiplying the equation by h_{yt} one obtains:

$$\varepsilon_{yt} = \rho \frac{h_{yt}}{h_{xt}} \varepsilon_{xt} + \varepsilon_t \frac{h_{yt}}{h_t}. \tag{12}$$

Substituting $\rho \frac{h_{yt}}{h_{xt}} = \rho_t^*$ and $\varepsilon_t \frac{h_{yt}}{h_t} = \varepsilon_t^*$ one acquires a congruent model:

$$\varepsilon_{yt} = \rho_t^* \varepsilon_{xt} + \varepsilon_t^* , \tag{13}$$

with stochastic parameter ρ_t^*.

Obviously there exists classical congruent model between processes: ε_{yt} and $\frac{\varepsilon_{xt}}{h_{xt} h_{yt}^{-1}}$ that is between ε_{yt} and properly standardized process ε_{xt}.

3. Congruent Models in Conditional Variance – Propositions of New Volatility Measures

Volatility is an important quantity in many financial analyses, eg. derivatives pricing, capital asset pricing, analysis of the flow of information between markets and financial instruments. Volatility of conditional variance of regression model error term lowers the efficiency of parameter estimators obtained with least squares method, and covariance matrix of estimators $\sigma^2 (X'X)^{-1}$ is not valid if the regression involves lagged dependent variable. Standard testing for parameters statistical significance may in this case lead to wrong results. The alternative, more suitable procedures include the use of other estimation methods which are robust to changes of variance or describing the changes of error term variance directly in the model for example with GARCH or SV specifications. If the model which serves as a tool for creation of volatility measure is not congruent in conditional variance, then the variance is usually underestimated. Thus the congruence in variance of a model is important in every analysis using volatility forecasts.

Let y_t describe $\mathrm{AR}(r)$–$\mathrm{GARCH}(p,q)$ process:

$$y_t = \phi_0 + \sum_{i=1}^{r} \phi_i y_{t-i} + \varepsilon_t , \tag{14}$$

$$\varepsilon_t \big| \psi_{t-1} \sim N(0, h_t) , \tag{15}$$

$$h_t = \alpha_o + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j h_{t-j} . \tag{16}$$

The estimates of y_t variability obtained on a basis of the conditional variance of ε_t will be underestimated, because they do not account for changes of the autoregressive process. Variability of y_t is obviously greater than variability of error term ε_t. In empirical applications it may appear that inclusion of variability resulting from autoregressive part of the model is important (for example in applications of Monte Carlo methods in pricing of derivatives or calculating the Value at Risk). One can use conditions of model congruence in conditional variance and apply a new volatility measure:

$$hs_t = \frac{h_t}{1-\sum_{i=1}^{r}\phi_i^2}\ , \tag{17}$$

Proposed volatility measure is not a „pure" conditional variance, because it is rescaled up by a autoregressive variability.

Similarly the estimates of variability ε_{yt} based on conditional variance ε_t in equation (3) will be underestimated, because changes of explanatory variables $\varepsilon_{x_s t}$ are not accounted for. Let $E(\varepsilon_{x_s t}, \varepsilon_{x_{s'} t}) = 0$. Proposed volatility measure for ε_{yt} may be written as:

$$hs_{yt} = \sum_{s=1}^{k}\rho_s^2 h_{x_s} + h_t\ , \tag{18}$$

where h_{x_s} $(s = 1,2,...,k)$ and h_t are conditional variances of explanatory variables and the error term respectively.

If model (3) is not congruent in conditional variance, that is if ε_t^2 does not account for internal structure of ε_{yt}^2 and $\varepsilon_{x_{st}}^2$ ($s = 1,2,...,k$), then the estimates of variability generated by this model will be incorrect (underestimated or overestimated).

4. Conclusions

In the paper the concept of model congruence has been extended to conditional variances. The implications of neglecting the information about the internal structure of dependent variable or explanatory variables have been indicated. New measures of variability have been proposed. Empirical application of proposed measures for financial analyses, in particular for volatility forecasting is left for future research.

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