

Duality in Fractional Programming Involving Locally Arcwise Connected and Related Functions

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Abstract

A nonlinear fractional programming problem is considered, where the functions involved are differentiable with respect to an arc. Necessary and sufficient optimality conditions are obtained in terms of the right differentials with respect to an arc of the functions. A dual is formulated and duality results are proved using concepts of locally arcwise connected, locally Q-connected and locally P-connected functions. Our results generalize the results obtained by Lyall, Suneja and Aggarwal [9], Kaul and Lyall [6] and Kaul et.al. [7]

Keywords : Generalized convexity; fractional programming; optimality conditions; duality

Mathematics Subject Classification 2000 :

Primary : 90 C 32; Secondary : 26 B 25

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1. Introduction

Recently, research efforts have been to obtain various generalizations of convex functions. By weakening certain properties of convex functions, different authors built new classes of functions. Between these we recall the class of semilocally convex, locally arcwise connected and related functions. Ewing [2] defined semilocally convex functions which he applied it to derive sufficient optimality conditions for variational and control problems. Kaul and Kaur [3] defined semilocally quasiconvex and semilocally pseudoconvex functions. Kaul and Kaur [4] derived sufficient optimality criteria for a class of nonlinear programming problems by using generalized semilocally functions. Optimality conditions and duality results were given by Kaul and Kaur [5] for a nonlinear programming problem where the functions involved are semidifferentiable and generalized semilocally. Optimality conditions and duality results were given by Lyall et al.[9] for a fractional programming problem involving semilocally convex and related functions. Kaul et al. [7] defined locally arcwise connected sets which include arcwise connected sets [1] and locally starshaped sets [2]. Also, they introduced locally connected functions and locally Q-connected functions on a locally connected set and studied some local-global minimum properties satisfied by such functions. Kaul and Lyall [6] defined locally P-connected functions and studied properties of these functions and locally connected (Q-connected) functions based on the concept of right differentiability of a function with respect to an arc. Results regarding the solution of nonlinear programming problem involving locally P-connected functions and sufficient optimality criteria for such a programming problem are derived. These results are extended to the multiple objective programming by Lyall et al. [8] which have obtained Fritz John type necessary optimality criteria for a non-linear programs and formulated a Mond-Weir type dual together with weak and strong duality results. A proper weak minimum is defined and duality results are established by using this concept.

In this paper, a nonlinear fractional programming problem is considered, where the functions involved are differentiable with respect to an arc. Necessary and sufficient optimality conditions are obtained. A dual is formulated and duality results are proved using concepts of locally arcwise connected, locally Q-connected and locally P-connected functions.

Due to the fact the class of locally arcwise connected functions is larger than the class of semilocally convex functions it results that this paper generalizes the

works of Lyall, Suneja and Aggarwal [9], Kaul and Lyall [6] and Kaul et.al. [7].

The organization of the remainder of this paper is as follows. In Section 2, we shall introduce the notation and definitions which are used throughout the paper. In Section 3, we shall give necessary optimality criteria for a nonlinear fractional programming problem. In Section 4, we shall give sufficient optimality criteria. In Section 5, a dual is formulated and duality results of weak and strong duality for the pair of primal and dual programs are proved.

2. Preliminaries

In this section, we introduce the notation and definitions which are used throughout the paper.

Let \mathbb{R}^n be the n -dimensional Euclidean space and \mathbb{R}_+^n its nonnegative orthant, i.e., $\mathbb{R}_+^n = \{x \in \mathbb{R}^n; x_j = 0; j = 1; \dots; n\}$. Throughout the paper, the following conventions for vectors in \mathbb{R}^n will be followed :

- $x > y$ if and only if $x_i > y_i$ ($i = 1; \dots; n$);
- $x = y$ if and only if $x_i = y_i$ ($i = 1; \dots; n$);
- $x \leq y$ if and only if $x_i \leq y_i$ ($i = 1; \dots; n$); but $x \notin y$.

Throughout the paper, all definitions, theorems, lemmas, corollaries, remarks are numbered consecutively in a single numeration system in each section.

Let $X^0 \subset \mathbb{R}^n$ be a nonempty and compact subset of \mathbb{R}^n .

Definition 2.1. Let $\bar{x}; x \in X^0$: A continuous mapping $H_{\bar{x};x} : [0; 1] \rightarrow \mathbb{R}^n$ with

$$H_{\bar{x};x}(0) = \bar{x}; H_{\bar{x};x}(1) = x$$

is called an arc from \bar{x} to x .

Definition 2.2. We say that $X^0 \subset \mathbb{R}^n$ is a locally arcwise connected set at \bar{x} ($\bar{x} \in X^0$) (X^0 is LAC, for short) if for any $x \in X^0$ there exist a positive number $a(x; \bar{x})$, with $0 < a(x; \bar{x}) \leq 1$, and a continuous arc such that $H_{\bar{x};x}(\lambda) \in X^0$ for any $\lambda \in (0; a(x; \bar{x}))$.

We say that X^0 is locally arcwise connected if X^0 is locally arcwise connected at any $x \in X^0$:

The concept of locally arcwise connected sets is a generalization of arcwise connected sets considered by Avriel and Zang [1] and locally starshaped sets considered by Ewing [2], since the function $H_{\bar{x};x}(\lambda) = (1 - \lambda)\bar{x} + \lambda x$ is an arc in the sense of Definition 2.1.

Definition 2.3.[7]. Let $f : X^0 \rightarrow \mathbb{R}$ be a function, where $X^0 \subset \mathbb{R}^n$ is a locally arcwise connected set at $\bar{x} \in X^0$, with the corresponding function $H_{\bar{x};x}(\delta)$ and a maximum positive number $a(x; \bar{x})$ satisfying the required conditions. We say that f is:

(i₁) locally arcwise connected (f is LAC, for short) at \bar{x} if for any $x \in X^0$, there exists a positive number $d(x; \bar{x}) \leq a(x; \bar{x})$ such that

$$f(H_{\bar{x};x}(\delta)) \leq \delta f(x) + (1 - \delta)f(\bar{x}); \quad 0 < \delta < d(x; \bar{x}); \quad (2.1)$$

(i₂) locally Q-connected (LQCN) at \bar{x} if for any $x \in X^0$, there exists a positive number $d(x; \bar{x}) \leq a(x; \bar{x})$ such that

$$f(x) \leq f(\bar{x}) + \frac{3}{4} \delta (f(x) - f(\bar{x})) \quad \text{and} \quad f(H_{\bar{x};x}(\delta)) \leq f(\bar{x});$$

$$0 < \delta < d(x; \bar{x})$$

The function f is said to be strictly locally arcwise connected (SLAC) at $\bar{x} \in X^0$ if for each $x \in X^0; x \neq \bar{x}$ the inequality (2.1) is strict.

If f is LAC (SLAC) at each $\bar{x} \in X^0$ then f is said to be LAC(SLAC) on X^0 :

If f is LQCN at each $\bar{x} \in X^0$ then f is said to be LQCN on X^0 :

It is clear from this definition that every semilocally convex (quasi-convex) function [2][3] is a locally arcwise connected (Q-connected) function but not conversely. Also, every arcwise connected (Q-connected) function [1] is a locally connected (Q-connected) function.

Thus, the class of locally arcwise connected functions includes the class of semilocally convex functions and the class of arcwise connected functions.

Definition 2.4.[6]. Let $f : X^0 \rightarrow \mathbb{R}$ be a function, where $X^0 \subset \mathbb{R}^n$ is a locally arcwise connected set at $\bar{x} \in X^0$, with the corresponding function $H_{\bar{x};x}(\delta)$ and a maximum positive number $a(x; \bar{x})$ satisfying the required conditions. The right differential of f at \bar{x} with respect to the arc $H_{\bar{x};x}(\delta)$ is given by

$$(df)^+(\bar{x}; H_{\bar{x};x}(0^+)) = \lim_{\delta \rightarrow 0^+} \frac{1}{\delta} [f(H_{\bar{x};x}(\delta)) - f(\bar{x})]$$

provided the limit exists and it is finite.

If f is differentiable at any $\bar{x} \in X^0$, then f is said to be differentiable on X^0 .

According to Avriel and Zang [1], $(df)^+(\bar{x}; H_{\bar{x};x}(0^+))$ may also be called directional derivated of f with respect to the arc $H_{\bar{x};x}(\delta)$ at $\delta = 0$.

Definition 2.5. Let $f : X^0 \rightarrow \mathbb{R}$ be a function on a locally arcwise connected set $X^0 \subset \mathbb{R}^n$ for which the right differential at \bar{x} with respect to the arc $H_{\bar{x},x}(\cdot)$ exists. We say that f is locally P-connected (LPCN) at $\bar{x} \in X^0$ if

$$(df)^+(\bar{x}; H_{\bar{x},x}(0^+)) = 0 \implies f(x) = f(\bar{x})$$

If f is LPCN at each $\bar{x} \in X^0$ then f is said to be LPCN on X^0 :

It is clear from this definition that every semilocally pseudoconvex function [3] is locally P-connected function and every differentiable P-connected function [1] is locally P-connected function.

The following theorem is an easy consequence of the above definitions.

Theorem 2.6. [6] Let $f : X^0 \rightarrow \mathbb{R}$ be a function on X^0 (a locally arcwise connected set) for which the right differential at \bar{x} with respect to the arc $H_{\bar{x},x}(\cdot)$ exists.

a) The function f is locally arcwise connected at $\bar{x} \in X^0$ if and only if

$$f(x) \leq f(\bar{x}) = (df)^+(\bar{x}; H_{\bar{x},x}(0^+)) \quad (2.2)$$

b) The function f is strictly locally arcwise connected at $\bar{x} \in X^0$, if and only if

$$f(x) \leq f(\bar{x}) > (df)^+(\bar{x}; H_{\bar{x},x}(0^+)); \forall x \in X^0$$

c) If f is locally Q-connected at \bar{x} , then

$$f(x) \leq f(\bar{x}) \implies (df)^+(\bar{x}; H_{\bar{x},x}(0^+)) \leq 0$$

In [6] Kaul and Lyall presented various properties of locally connected functions, locally Q-connected functions, locally P-connected functions in terms of the right differential of the function with respect to an arc $H_{\bar{x},x}$.

In what follows we give other new properties of the above defined functions.

Theorem 2.7. Let $f, g : X^0 \rightarrow \mathbb{R}_+$ be such that f and g are locally arcwise connected functions on X^0 (a locally arcwise connected set in \mathbb{R}^n) with respect to the same function H for each pair of points and g is strictly positive and finite function on X^0 : Then the ratio $h = \frac{f}{g}$ is a locally Q-connected function on X^0 .

Proof. Let $x^1, x^2 \in X^0$ such that

$$h(x^2) \leq h(x^1) \quad (2.3)$$

Since X^0 is a LAC set, then there exist a maximum positive number $a(x^1, x^2) \leq 1$ and an arc H_{x^1, x^2} such that $H_{x^1, x^2}(s) \in X^0$ for $0 < s < a(x^1, x^2)$: Since f and g are locally arcwise connected functions on X^0 , there exist $0 < d_1(x^1, x^2) \leq a(x^1, x^2)$ and $0 < d_2(x^1, x^2) \leq a(x^1, x^2)$ such that

$$f(H_{x^1, x^2}(s)) \leq s f(x^2) + (1-s)f(x^1); \quad 0 < s < d_1(x^1, x^2)$$

and

$$g(H_{x^1, x^2}(s)) = s g(x^2) + (1-s)g(x^1); \quad 0 < s < d_2(x^1, x^2):$$

Let $d_3(x^1, x^2) = \min\{d_1(x^1, x^2), d_2(x^1, x^2)\}$: Then

$$\begin{aligned} h(H_{x^1, x^2}(s)) - h(x^1) &= \frac{f(H_{x^1, x^2}(s)) - f(x^1)}{g(H_{x^1, x^2}(s)) - g(x^1)} \leq \\ &\leq \frac{g(x^1)[(1-s)f(x^1) + s f(x^2)] - f(x^1)[(1-s)g(x^1) + s g(x^2)]}{g(x^1)g(H_{x^1, x^2}(s))} = \\ &= \frac{s g(x^2)[h(x^2) - h(x^1)]}{g(H_{x^1, x^2}(s))} \leq 0 \text{ (using (2.3))} \end{aligned}$$

for $0 < s < d_3(x^1, x^2)$ because $H_{x^1, x^2}(s) \in X^0$ for $0 < s < d_3(x^1, x^2) \leq a(x^1, x^2)$: Thus, we have $h(H_{x^1, x^2}(s)) \leq h(x^1)$ for all $0 < s < d_3(x^1, x^2)$, i.e. h is a locally Q-connected function on X^0 .

A similar result is derived in [7] using a different approach.

Corollary 2.8. If f is locally arcwise connected and non-negative function on X^0 and g is locally arcwise connected, strictly positive and finite function on X^0 , then h is locally Q-connected function.

Corollary 2.9. If g is strictly positive and finite function on X^0 , then $\frac{1}{g}$ is locally Q-connected on X^0 if and only if g is locally Q-connected on X^0 .

Theorem 2.10. Let f be a locally arcwise connected and non-negative function on a locally arcwise connected set X^0 , g be a locally arcwise connected, strictly negative and finite function X^0 with respect to the same arc H for each pair of points. Then $h = \frac{f^2}{g}$ is locally arcwise connected on X^0 .

Proof. Let $x^1; x^2 \in X^0$. Since X^0 is a LAC set, then there exist a maximum positive number $a(x^1; x^2) \leq 1$ and an arc $H_{x^1; x^2}$ such that $H_{x^1; x^2}(\lambda) \in X^0$ for $0 < \lambda < a(x^1; x^2)$: Since f and g are locally arcwise connected functions on X^0 , there exist $0 < d_1(x^1; x^2) \leq a(x^1; x^2)$ and $0 < d_2(x^1; x^2) \leq a(x^1; x^2)$ such that

$$f(H_{x^1; x^2}(\lambda)) \leq \lambda f(x^2) + (1 - \lambda)f(x^1); \quad 0 < \lambda < d_1(x^1; x^2) \quad (2.4)$$

and

$$g(H_{x^1; x^2}(\lambda)) \geq \lambda g(x^2) + (1 - \lambda)g(x^1); \quad 0 < \lambda < d_2(x^1; x^2); \quad (2.5)$$

Let $d_3(x^1; x^2) = \min\{d_1(x^1; x^2); d_2(x^1; x^2)\}$:

Now, by (2.4) and (2.5) we have

$$\begin{aligned} h(H_{x^1; x^2}(\lambda)) &\leq [(1 - \lambda)h(x^1) + \lambda h(x^2)] = \\ &= \frac{f^2(H_{x^1; x^2})}{g(H_{x^1; x^2})} \leq (1 - \lambda) \frac{f^2(x^1)}{g(x^1)} + \lambda \frac{f^2(x^2)}{g(x^2)} \leq \\ &\leq \frac{[(1 - \lambda)f(x^1) + \lambda f(x^2)]^2}{(1 - \lambda)g(x^1) + \lambda g(x^2)} \leq (1 - \lambda) \frac{f^2(x^1)}{g(x^1)} + \lambda \frac{f^2(x^2)}{g(x^2)} = \\ &= \frac{\lambda(1 - \lambda)[f(x^1)g(x^2) - f(x^2)g(x^1)]^2}{g(x^1)g(x^2)[(1 - \lambda)g(x^1) + \lambda g(x^2)]} \leq 0 \end{aligned}$$

for $0 < \lambda < d_3(x^1; x^2)$ because $g(x^1) > 0$ and $g(x^2) > 0$. Thus h is locally arcwise connected on X^0 .

3. Necessary Optimality Criteria

Consider the nonlinear fractional programming problem

$$\text{Max } q(x) = \frac{f(x)}{g(x)}$$

(P) subject to

$$\begin{aligned} h(x) &\leq 0 \\ x &\in X^0 \end{aligned}$$

where

i) $X^0 \subset \mathbb{R}^n$ is a non-empty locally arcwise connected set,

ii) $f : X^0 \rightarrow \mathbb{R}; f(x) = 0; \forall x \in X^0;$

iii) $g : X^0 \rightarrow \mathbb{R}; g(x) > 0; \forall x \in X^0;$

iv) $h = (h_i)_{1 \leq i \leq m} : X^0 \rightarrow \mathbb{R}^m;$

v) the right differentials of $f; g$ and $h_j (j = 1; \dots; m)$ at \bar{x} with respect to the arc H exist.

Let $X = \{x \in X^0 \mid h(x) \leq 0\}$ be the set of all feasible solutions to (P).

If $h_i (i = 1; \dots; m)$ is a locally arcwise connected function on X^0 , then X is a locally arcwise connected set (Kaul and Lyall [6]).

Let $N(\bar{x})$ denote the neighbourhood of $\bar{x} \in \mathbb{R}^n$ and $S(\bar{x}; \epsilon)$ open sphere of center \bar{x} and radius ϵ ; i.e.,

$$S(\bar{x}; \epsilon) = \{x \in \mathbb{R}^n \mid \|x - \bar{x}\| < \epsilon\}$$

Definition 3.1. a) \bar{x} is said to be a local maximum solution to Problem (P) if $\bar{x} \in X$ and there exists a neighbourhood $N(\bar{x})$ such that $x \in N(\bar{x}) \setminus X \Rightarrow f(x) = f(\bar{x})$:

b) \bar{x} is said to be a maximum solution to Problem (P) if $\bar{x} \in X$ and $f(\bar{x}) = \max_{x \in X} f(x)$:

For $\bar{x} \in X$ we denote

$$I = I(\bar{x}) = \{i \mid h_i(\bar{x}) = 0\};$$

$$J = J(\bar{x}) = \{i \mid h_i(\bar{x}) < 0\};$$

and

$$h_I = (h_i)_{i \in I} :$$

Obviously $I \cap J = \emptyset$.

In order to pass from Fritz John conditions to the Kuhn-Tucker conditions we need the following generalized constraint qualification.

Definition 3.2. We say that the function h satisfies the generalized Slater's constraint qualification (GSQ) at $\bar{x} \in X$, if h_i is locally P-connected at \bar{x} and there exists an $\hat{x} \in X$ such that $h_i(\hat{x}) < 0$ for $i \in I$.

Lemma 3.3. Let $\bar{x} \in X$ be a (local) maximum solution to Problem (P). We assume that h_i is continuous at \bar{x} for any $i \in J$, and that the right differentials of $f; g$ and h_i at \bar{x} with respect to the arc $H_{\bar{x};x}$ exist. Then the system

$$(df)^+(\bar{x}; H_{\bar{x};x}(0^+)) > 0 \quad (3.1)$$

$$(dg)^+(\bar{x}; H_{\bar{x};x}(0^+)) < 0 \quad (3.2)$$

$$(dh_i)^+(\bar{x}; H_{\bar{x};x}(0^+)) < 0 \quad (3.3)$$

has no solution $x \in X^0$.

Proof. Let $\bar{x} \in X$ be a (local) maximum solution to Problem (P). We assume ad absurdum that the system (3.1)-(3.3) has a solution $x^0 \in X^0$ i.e.

$$(df)^+(\bar{x}; H_{\bar{x};x^0}(0^+)) > 0 \quad (3.4)$$

$$(dg)^+(\bar{x}; H_{\bar{x};x^0}(0^+)) < 0 \quad (3.5)$$

$$(dh_i)^+(\bar{x}; H_{\bar{x};x^0}(0^+)) < 0 \quad (3.6)$$

Consider the function

$$\varphi_1(\bar{x}; H_{\bar{x};x^0}(0^+); \delta) = f(H_{\bar{x};x^0}(\delta)) - f(\bar{x})$$

which vanishes at $\delta = 0$:

The right differential of $\varphi_1(\bar{x}; H_{\bar{x};x^0}(0^+); \delta)$ with respect to δ at $\delta = 0$ is given by

$$\begin{aligned} & \lim_{\delta \downarrow 0^+} \frac{\varphi_1(\bar{x}; H_{\bar{x};x^0}(0^+); \delta) - \varphi_1(\bar{x}; H_{\bar{x};x^0}(0^+); 0)}{\delta} = \\ & = \lim_{\delta \downarrow 0^+} \frac{f(H_{\bar{x};x^0}(\delta)) - f(\bar{x})}{\delta} = (df)^+(\bar{x}; H_{\bar{x};x^0}(0^+)) > 0 \text{ (using (3.4)).} \end{aligned}$$

Therefore there exists δ_1 such that

$$\varphi_1(\bar{x}; H_{\bar{x};x^0}(0^+); \delta) > 0; \quad \delta \in (0; \delta_1)$$

i.e.

$$f(H_{\bar{x};x^0}(\delta)) > f(\bar{x}); \quad \delta \in (0; \delta_1) \quad (3.7)$$

Similarly, if we consider

$$\mathbb{C}_2 \mathbf{x}; H_{\mathbf{x}; \mathbf{x}^0} \mathbb{I}^+ \mathbb{C}; \mathbb{C} = g(\mathbf{x}) \mathbb{I} g(H_{\mathbf{x}; \mathbf{x}^0}(\cdot)); \mathbb{C}$$

and

$$\mathbb{C}_3 \mathbf{x}; H_{\mathbf{x}; \mathbf{x}^0} \mathbb{I}^+ \mathbb{C}; \mathbb{C} = h_1(\mathbf{x}) \mathbb{I} h_1(H_{\mathbf{x}; \mathbf{x}^0}(\cdot)); \mathbb{C}$$

and using (3.5) and (3.6) we have

$$g(H_{\mathbf{x}; \mathbf{x}^0}(\cdot)) < g(\mathbf{x}) \mathbb{I} \mathbb{C} \mathbb{I} 2 (0; \pm_2) \quad (3.8)$$

and

$$h_1(H_{\mathbf{x}; \mathbf{x}^0}(\cdot)) < h_1(\mathbf{x}) \mathbb{I} \mathbb{C} \mathbb{I} 2 (0; \pm_3) \quad (3.9)$$

>From (3.9) and the definition of I , it results

$$h_1(H_{\mathbf{x}; \mathbf{x}^0}(\cdot)) < 0 \mathbb{I} \mathbb{C} \mathbb{I} 2 (0; \pm_3):$$

Also, $h_i(\mathbf{x}) < 0$ for $i \in J$ and from the continuity of h_i at \mathbf{x} , there exists $\pm_i^{\#} > 0$ such that

$$h_i(H_{\mathbf{x}; \mathbf{x}^0}(\cdot)) < 0 \mathbb{I} \mathbb{C} \mathbb{I} 2 (0; \pm_i^{\#}); i \in J: \quad (3.10)$$

Let $\pm^{\#} = \min(\pm_1; \pm_2; \pm_3; \pm_i^{\#} (i \in J))$: For $\mathbb{C} \mathbb{I} 2 (0; \pm^{\#})$ we have

$$H_{\mathbf{x}; \mathbf{x}^0}(\cdot) \mathbb{I} 2 S(\mathbf{x}; \pm^{\#}) \mu N(\mathbf{x}): \quad (3.11)$$

By the choosing of $\pm^{\#}$ from (3.7), (3.8), (3.9) and (3.10) it results

$$f(H_{\mathbf{x}; \mathbf{x}^0}(\cdot)) > f(\mathbf{x}) \mathbb{I} \mathbb{C} \mathbb{I} 2 (0; \pm^{\#}) \quad (3.12)$$

$$g(H_{\mathbf{x}; \mathbf{x}^0}(\cdot)) < g(\mathbf{x}) \mathbb{I} \mathbb{C} \mathbb{I} 2 (0; \pm^{\#}) \quad (3.13)$$

$$h_1(H_{\mathbf{x}; \mathbf{x}^0}(\cdot)) < 0 \mathbb{I} \mathbb{C} \mathbb{I} 2 (0; \pm^{\#}) \quad (3.14)$$

$$h_J(H_{\mathbf{x}; \mathbf{x}^0}(\cdot)) < 0 \mathbb{I} \mathbb{C} \mathbb{I} 2 (0; \pm^{\#}): \quad (3.15)$$

>From (3.11), (3.14) and (3.15), it follows that

$$H_{\mathbf{x}; \mathbf{x}^0}(\cdot) \mathbb{I} 2 N(\mathbf{x}) \setminus X; \mathbb{C} \mathbb{I} 2 (0; \pm^{\#})$$

and from (3.12) and (3.13), we have

$$q(H_{\mathbf{x}; \mathbf{x}^0}(\cdot)) > q(\mathbf{x})$$

which contradicts the assumption that \bar{x} is a local maximum solution for (P). Therefore, there exist no $x \in X^0$ such that

$$\begin{aligned} (df)^+(\bar{x}; H_{\bar{x}; \bar{x}}(0^+)) &> 0 \\ (dg)^+(\bar{x}; H_{\bar{x}; \bar{x}}(0^+)) &< 0 \\ (dh_1)^+(\bar{x}; H_{\bar{x}; \bar{x}}(0^+)) &< 0; \end{aligned}$$

The proof is complete.

Now we have the following Fritz John type necessary optimality criteria.

Theorem 3.4. Let us suppose that $h_i, i \in J$, is continuous at \bar{x} . Assume also that $(df)^+(\bar{x}; H_{\bar{x}; \bar{x}}(0^+)); (dg)^+(\bar{x}; H_{\bar{x}; \bar{x}}(0^+)); (dh_1)^+(\bar{x}; H_{\bar{x}; \bar{x}}(0^+))$ are convex functions of x and X^0 is convex. If \bar{x} is a (local) maximum solution to Problem (P), then there exist $\bar{u}_0 \in \mathbb{R}; \bar{u} \in \mathbb{R}^m; \bar{\lambda}_i \in \mathbb{R}$ such that $(\bar{x}; \bar{u}_0; \bar{\lambda}; \bar{u})$ satisfies the following conditions:

$$\begin{aligned} \bar{u}_0 (df)^+(\bar{x}; H_{\bar{x}; \bar{x}}(0^+)) + \sum_{i \in I} \bar{\lambda}_i (dg)^+(\bar{x}; H_{\bar{x}; \bar{x}}(0^+)) + \\ + \sum_{i \in J} \bar{u}_i (dh_i)^+(\bar{x}; H_{\bar{x}; \bar{x}}(0^+)) = 0; \quad \forall x \in X^0 \end{aligned} \quad (3.16)$$

$$\bar{u} \nabla h(\bar{x}) = 0 \quad (3.17)$$

$$h(\bar{x}) \leq 0 \quad (3.18)$$

$$\bar{u}_0; \bar{\lambda}_i; \bar{u}_i = 0; \bar{u}_0; \bar{\lambda}_i; \bar{u}_i \in \mathbb{R} \quad (3.19)$$

Proof. Let $\bar{x} \in X$ be a (local) maximum solution to Problem (P). Since the conditions of Lemma 3.3 are satisfied, we get that the system (3.1)-(3.3) has no solution $x \in X^0$: Since $(df)^+(\bar{x}; H_{\bar{x}; \bar{x}}(0^+)); (dg)^+(\bar{x}; H_{\bar{x}; \bar{x}}(0^+)); (dh_1)^+(\bar{x}; H_{\bar{x}; \bar{x}}(0^+))$ are convex functions of x , therefore by Theorem of alternative for the convex functions ([10]), there exist $\bar{u}_0 \in \mathbb{R}; \bar{u}_i \in \mathbb{R} (i \in I); \bar{\lambda}_i \in \mathbb{R}$, such that

$$\begin{aligned} \bar{u}_0 (df)^+(\bar{x}; H_{\bar{x}; \bar{x}}(0^+)) + \sum_{i \in I} \bar{\lambda}_i (dg)^+(\bar{x}; H_{\bar{x}; \bar{x}}(0^+)) + \\ + \sum_{i \in J} \bar{u}_i (dh_i)^+(\bar{x}; H_{\bar{x}; \bar{x}}(0^+)) = 0; \quad \forall x \in X^0 \end{aligned} \quad (3.20)$$

$$\bar{u}_0; \bar{\lambda}_i; \bar{u}_i = 0; \bar{u}_0; \bar{\lambda}_i; \bar{u}_i \in \mathbb{R} \quad (3.21)$$

If we define $\hat{u}_j = 0$, by (3.20), we get (3.16). Since $h_i(\bar{x}) = 0$ then for $\hat{u} = (\hat{u}_1; \hat{u}_j)$ we have

$$\hat{u}^T h(\bar{x}) = 0 \quad (3.22)$$

i.e. the relation (3.17).

The relation (3.18) results from $\bar{x} \in X$. The proof is complete.

Now, we consider the parametric problem

$$(P_s) \quad \begin{aligned} & \text{Max } f(\bar{x}) - \lambda_s g(\bar{x}); \quad \lambda_s \in \mathbb{R} \quad (\lambda_s \text{ a parameter}) \\ & \text{subject to} \\ & \quad h(\bar{x}) \leq 0 \\ & \quad \bar{x} \in X^0 \end{aligned}$$

It is well known that (P_s) is closely related to Problem (P).

The following lemma is well known in fractional programming [13] and establishes a connection between the fractional programming problem (P) and a certain parametric programming problem (P_s) :

Lemma 3.5. \bar{x} is an optimal solution to Problem (P) if and only if it is optimal solution to Problem (P_{λ_s}) with $\lambda_s = f(\bar{x})/g(\bar{x})$:

The next Theorem is a Kuhn-Tucker type necessary optimality criteria and results from Lemma 3.5 and Theorem 3.4.

Theorem 3.6. Let us suppose that h_i is continuous at \bar{x} for $i \in J$. Assume also that $\lambda_s (df)^+(\bar{x}; H_{\bar{x}, \bar{x}}(0^+))$; $(dg)^+(\bar{x}; H_{\bar{x}, \bar{x}}(0^+))$; $(dh_i)^+(\bar{x}; H_{\bar{x}, \bar{x}}(0^+))$ are convex functions of x and X^0 is convex and h satisfies GSQ at \bar{x} . If \bar{x} is a (local) maximum solution to Problem (P), then there exist $\lambda_s \in \mathbb{R}$; $\hat{u} \in \mathbb{R}^m$ such that

$$\begin{aligned} & \lambda_s (df)^+(\bar{x}; H_{\bar{x}, \bar{x}}(0^+)) + \sum_{i \in J} \hat{u}_i (dh_i)^+(\bar{x}; H_{\bar{x}, \bar{x}}(0^+)) + \\ & + \alpha (dh)^+(\bar{x}; H_{\bar{x}, \bar{x}}(0^+)) = 0; \quad \forall \bar{x} \in X^0 \end{aligned} \quad (3.23)$$

$$f(\bar{x}) - \lambda_s g(\bar{x}) = 0 \quad (3.24)$$

$$\hat{u}^T h(\bar{x}) = 0 \quad (3.25)$$

$$h(\bar{x}) \leq 0 \quad (3.26)$$

$$\hat{u}_i \geq 0; \quad \hat{u}_i = 0; \quad \hat{u}_i \leq 0 \quad (3.27)$$

4. Sufficient optimality criteria

Theorem 4.1. Let $\bar{x} \in X \subset \mathbb{R}^n; \bar{u} \in \mathbb{R}^m$. Let f, g and h be locally arcwise connected at \bar{x} , with respect to a same arc $H_{\bar{x},x}$. We assume that at \bar{x} ; there exist the right differentials with respect to the arc $H_{\bar{x},x}$ of f, g and h and $(\bar{x}; \bar{u})$ satisfies the following conditions:

$$\bar{u}^T (df)^+(\bar{x}; H_{\bar{x},x}(0^+)) + \bar{u}^T (dh)^+(\bar{x}; H_{\bar{x},x}(0^+)) = 0; \quad \forall x \in X \quad (4.1)$$

$$(dg)^+(\bar{x}; H_{\bar{x},x}(0^+)) = 0; \quad \forall x \in X \quad (4.2)$$

$$\bar{u}^T h(\bar{x}) = 0 \quad (4.3)$$

$$h(\bar{x}) \leq 0 \quad (4.4)$$

$$\bar{u} = 0; \bar{u} \notin 0 \quad (4.5)$$

Then \bar{x} is a maximum solution to Problem (P).

Proof. Let $(\bar{x}; \bar{u})$ satisfy conditions (4.1)-(4.5). Relation (4.4) yields that $\bar{x} \in X$, hence \bar{x} is a feasible solution to Problem (P). The function f is locally arcwise connected. Therefore, for any $x \in X$, Theorem 2.6, yields.

$$f(x) - f(\bar{x}) \leq (df)^+(\bar{x}; H_{\bar{x},x}(0^+))$$

or

$$\begin{aligned} f(\bar{x}) - f(x) &= \bar{u}^T (df)^+(\bar{x}; H_{\bar{x},x}(0^+)) = \bar{u}^T (dh)^+(\bar{x}; H_{\bar{x},x}(0^+)) \quad (\text{by (4.1)}) \\ &= \bar{u}^T [h(x) - h(\bar{x})] \quad (\text{since } h \text{ is locally arcwise connected at } \bar{x}) \\ &= \bar{u}^T h(x) \quad (\text{by (4.3)}) \\ &= 0 \quad (\text{by (4.5)}). \end{aligned}$$

Thus

$$f(\bar{x}) = f(x) \quad \forall x \in X: \quad (4.6)$$

Since g is locally arcwise connected, by Theorem 2.6, it results that

$$g(x) - g(\bar{x}) = (dg)^+(\bar{x}; H_{\bar{x},x}(0^+)) = 0 \quad (\text{by (4.2)})$$

Therefore

$$g(x) = g(\bar{x}) \quad \forall x \in X: \quad (4.7)$$

Thus, from (4.6) and (4.7), it follows that

$$q(x) = g(x); \quad 8 \times 2 \times X:$$

Hence, \bar{x} is an optimal solution to Problem (P).

Corollary 4.2. Let $\bar{x} \in X^0 \subset \mathbb{R}^n; \bar{u}_0 \in \mathbb{R}; \bar{u} \in \mathbb{R}^m$. Let $f; g$ and h be locally arcwise connected at \bar{x} ; with respect to the same arc $H_{\bar{x};\bar{x}}$. We assume that at \bar{x} ; there exist the right differentials of $f; g$ and h with respect to the arc $H_{\bar{x};\bar{x}}$ and $(\bar{x}; \bar{u}_0; \bar{u})$ satisfies the following conditions:

$$\sum_i \bar{u}_0 (df)^+(\bar{x}; H_{\bar{x};\bar{x}}) \bar{i}_0^{++} + \bar{u} (dh)^+(\bar{x}; H_{\bar{x};\bar{x}}) \bar{i}_0^{++} = 0; \quad 8 \times 2 \times X \quad (4.8)$$

$$(dg)^+(\bar{x}; H_{\bar{x};\bar{x}}) \bar{i}_0^{++} = 0; \quad 8 \times 2 \times X \quad (4.9)$$

$$\bar{u} \nabla h(\bar{x}) = 0 \quad (4.10)$$

$$h(\bar{x}) \leq 0 \quad (4.11)$$

$$\bar{u} = 0; \bar{u} \leq 0 \quad (4.12)$$

$$\bar{u}_0 > 0; \quad (4.13)$$

Then \bar{x} is a maximum solution to Problem (P).

Proof. Since $\bar{u}_0 > 0$ (by (4.13)), therefore $(\bar{x}; \bar{u} = \bar{u}_0)$ satisfies conditions (4.1)-(4.5) of Theorem 4.1 and hence \bar{x} is an optimal solution to Problem (P).

Remark 4.3. In the statement of Theorem 4.1 and Corollary 4.2 it suffices to assume only the local arcwise connectivity of h_i instead of h at \bar{x} .

Theorem 4.4. Let $\bar{x} \in X^0; \bar{u}_0 \in \mathbb{R}; \bar{u} \in \mathbb{R}^m$: Let f and g locally arcwise connected and h strictly locally arcwise connected at \bar{x} ; with respect to the same arc $H_{\bar{x};\bar{x}}$. We assume that at \bar{x} ; there exist the right differential with respect to the arc $H_{\bar{x};\bar{x}}$ of $f; g$ and h and $(\bar{x}; \bar{u}_0; \bar{u})$ satisfies conditions (4.8)-(4.13).

Then \bar{x} is a maximum solution to Problem (P).

Proof. From the relations (4.10) and (4.11) we obtain $\bar{u}_i = 0$ for $i \in J$ and thus (4.8) may be written as

$$\sum_i \bar{u}_0 (df)^+(\bar{x}; H_{\bar{x};\bar{x}}) \bar{i}_0^{++} + \bar{u}_1 (dh_1)^+(\bar{x}; H_{\bar{x};\bar{x}}) \bar{i}_0^{++} = 0; \quad 8 \times 2 \times X \quad (4.14)$$

From (4.12) and (4.13), we obtain

$$(\dot{u}_0; \dot{u}_1) = 0; (\dot{u}_0; \dot{u}_1) \notin 0 \quad (4.15)$$

and from (4.14) and (4.15), we obtain that the system

$$\begin{aligned} (df)^+(\bar{x}; H_{\bar{x};\bar{x}}(0^+)) &> 0 \\ (dh_1)^+(\bar{x}; H_{\bar{x};\bar{x}}(0^+)) &< 0 \end{aligned} \quad (4.16)$$

has no solution $x \in X$: We can infer that $f(x) \not\leq f(\bar{x}) \quad \forall x \in X$. Indeed, if there exists $x^0 \in X$ such that $f(x^0) > f(\bar{x})$, then

$$h_1(x^0) \leq h_1(\bar{x}) :$$

>From the locally arcwise connectivity of f and strict locally arcwise connectivity of h_1 at \bar{x} we have

$$\begin{aligned} 0 < f(x^0) - f(\bar{x}) &\leq (df)^+(\bar{x}; H_{\bar{x};\bar{x}}(0^+)) \\ 0 = h_1(x^0) - h_1(\bar{x}) &> (dh_1)^+(\bar{x}; H_{\bar{x};\bar{x}}(0^+)) \end{aligned}$$

i.e. the system

$$\begin{aligned} (df)^+(\bar{x}; H_{\bar{x};\bar{x}}(0^+)) &> 0 \\ (dh_1)^+(\bar{x}; H_{\bar{x};\bar{x}}(0^+)) &< 0 \end{aligned}$$

has a solution x^0 , which is a contradiction to (4.16). Therefore,

$$f(x) \leq f(\bar{x}) \quad \forall x \in X: \quad (4.17)$$

Similar as in the proof of Theorem 4.1 from the locally arcwise connectivity of g , it results

$$g(x) = g(\bar{x}) \quad \forall x \in X: \quad (4.18)$$

Combining (4.17) and (4.18), we conclude that

$$q(x) \leq q(\bar{x}) \quad \forall x \in X:$$

Hence, \bar{x} is an optimal solution to Problem (P). This completes the proof of the theorem.

Theorem 4.5. Let $\bar{x} \in X^0; \bar{u} \in \mathbb{R}^m$. Let f, g be locally arcwise connected and h_i be locally Q-connected at \bar{x} with respect to the same arc $H_{\bar{x}, \bar{x}}$. We assume that at \bar{x} ; there exist the right differential with respect to the arc $H_{\bar{x}, \bar{x}}$ of $f; g$ and h and $(\bar{x}; \bar{u}_0; \bar{u})$ satisfies conditions (4.1)-(4.5). Then \bar{x} is a maximum solution to Problem (P).

Proof. From the relations (4.3) and (4.4) we obtain $\bar{u}_i = 0$ for $i \in J$. Also, $h_i(\bar{x}) \leq 0 = h_i(\bar{x}); \bar{x} \in X$. Since h_i is locally Q-connected at \bar{x} , therefore by Theorem 2.6 it follows $(dh_i)^+(\bar{x}; H_{\bar{x}, \bar{x}}(0^+)) \leq 0; \bar{x} \in X$: From this inequality and $\bar{u}_i = 0$ we get $\bar{u}_i (dh_i)^+(\bar{x}; H_{\bar{x}, \bar{x}}(0^+)) \leq 0; \bar{x} \in X$. Also, $\bar{u}_j (dh_j)^+(\bar{x}; H_{\bar{x}, \bar{x}}(0^+)) = 0; \bar{x} \in X$ and because $\bar{u}_i = 0$ for $i \in J$ we have $\bar{u} (dh)^+(\bar{x}; H_{\bar{x}, \bar{x}}(0^+)) \leq 0; \bar{x} \in X$:

By (4.1) we have $\bar{u}_j (df)^+(\bar{x}; H_{\bar{x}, \bar{x}}(0^+)) = 0; \bar{x} \in X$ and from f locally arcwise connected and Theorem 2.6 it results

$$f(\bar{x}) \leq f(\bar{x}); \bar{x} \in X:$$

Also, by (4.2)

$$g(\bar{x}) = g(\bar{x}); \bar{x} \in X:$$

Hence $q(\bar{x}) \leq q(\bar{x}); \bar{x} \in X$, i.e. \bar{x} is an optimal solution to Problem (P).

5. Duality

For Problem (P) we consider the dual problem

$$\min v(\bar{u}) = \bar{u}_0$$

(D) subject to

$$\bar{u}_j (df)^+(\bar{y}; H_{\bar{y}, \bar{x}}(0^+)) + \bar{u}_j (dg)^+(\bar{y}; H_{\bar{y}, \bar{x}}(0^+)) + \bar{u} (dh)^+(\bar{y}; H_{\bar{y}, \bar{x}}(0^+)) = 0; \bar{x} \in X \quad (5.1)$$

$$f(\bar{y}) \leq g(\bar{y}) \leq 0 \quad (5.2)$$

$$\bar{u} \in h(\bar{y}) = 0 \quad (5.3)$$

$$\bar{u} = 0; \bar{y} \in X^0; \bar{u} \in \mathbb{R}^m; \bar{u}_0 \in \mathbb{R}; \bar{u}_0 = 0: \quad (5.4)$$

Let T denote the set of all feasible solutions to Problem (D).

Theorem 5.1. (Weak Duality). Let $x \in X$ and $(y; \gamma; u) \in T$: If $j; f; g$ and h are locally arcwise connected, with respect to a same arc, then

$$q(x) \leq v(\gamma):$$

Proof. Locally arcwise connectivity of f and Theorem 2.6, yield

$$f(x) - f(y) \in (df)^+_{y; H_{y;x}} \mathbf{i} 0^+$$

$$\in \gamma (dg)^+_{y; H_{y;x}} \mathbf{i} 0^+ + u (dh)^+_{y; H_{y;x}} \mathbf{i} 0^+ \quad (\text{using (5.1)})$$

$\in \gamma fg(x) - g(y)g + u fh(x) - h(y)g$ (by locally arcwise connectivity of g and h):

Or

$$f(x) - \gamma g(x) \in f(y) - \gamma g(y)g + u fh(x) - h(y)g \in 0$$

using (5.2), (5.3) and (5.4) and $x \in X$. Thus

$$\frac{f(x)}{g(x)} \leq \gamma$$

i.e.

$$q(x) \leq v(\gamma):$$

The weak duality theorem take place in weaker conditions on $f; g$ and h .

Theorem 5.2. If $x \in X$ and $(y; \gamma; u) \in T$ and $j; f + \gamma g + uh$ is locally P-connected, then $q(x) \leq v(\gamma)$:

Proof. Let $x \in X$ and $(y; \gamma; u) \in T$. The relation (5.1) can be written under the form

$$(d(j; f + \gamma g + uh))^+_{y; H_{y;x}} \mathbf{i} 0^+ = 0; \forall x \in X:$$

Since $j; f + \gamma g + uh$ is locally P-connected we have

$$(j; f + \gamma g + uh)(x) = (j; f + \gamma g + uh)(y)$$

i.e.

$$f(x) - \gamma g(x) \in f(y) - \gamma g(y)g + u fh(x) - h(y)g:$$

The proof follows as in Theorem 5.1.

Corollary 5.3. Let $\bar{x} \in X$ and $(\bar{x}; \bar{u}) \in T$ such that $q(\bar{x}) = v(\bar{u})$. If the hypotheses of either Theorem 5.1 or Theorem 5.2 are satisfied, then \bar{x} is an optimal solution to Problem (P) and $(\bar{x}; \bar{u})$ is an optimal solution to Problem (D).

Proof. According to Theorems 5.1 and 5.2, for each $x \in X$ we have

$$q(x) \leq v(\bar{u}) = q(\bar{x})$$

and hence \bar{x} is an optimal solution to Problem (P). Also if $(\bar{x}; \bar{u}) \in T$, then according to Theorems 5.1 and 5.2, we have

$$v(\bar{u}) = q(\bar{x}) = v(\bar{u})$$

and hence $(\bar{x}; \bar{u})$ is an optimal solution to Problem (D).

Theorem 5.4. Let \bar{x} be a (local) optimal solution to (P). Let $h_i, i \in J$ be continuous at \bar{x} and let $(df)^+(\bar{x}; H_{\bar{x}; \bar{x}}(0^+)); (dg)^+(\bar{x}; H_{\bar{x}; \bar{x}}(0^+)); (dh_i)^+(\bar{x}; H_{\bar{x}; \bar{x}}(0^+))$ be convex functions of x . If h satisfies GSQ at \bar{x} , then there exists $(\bar{x}; \bar{u}) \in T$ such that $q(\bar{x}) = v(\bar{u})$. Moreover, if either $(i) f; g; h$ are locally arcwise connected or $(ii) f + \sum g + \sum u h$ is locally P-connected for any $(y; u) \in T$, then $(\bar{x}; \bar{u})$ is an optimal solution to (D).

Proof. Since \bar{x} satisfies the conditions of Theorem 3.6, there exist $\bar{u} \in \mathbb{R}^1; \bar{u} \in \mathbb{R}^m$ such that $(\bar{x}; \bar{u})$ is feasible to (D) and $q(\bar{x}) = v(\bar{u})$. Hence, by Corollary 5.3, it follows that $(\bar{x}; \bar{u})$ is optimal to (D).

Remark 5.5. Since the class of locally arcwise connected functions includes the class of semilocally convex functions and the class of arcwise connected functions, our results generalize those of Lyall, Suneja and Aggarwal [9], Kaur and Lyall [6] and Kaul et al.[7].

References

- [1] M.Avriel and I.Zang : Generalized arcwise-connected functions and characterizations of local-global minimum properties. J.Optim.Theory Appl. 32(1980)4, 407-425.
- [2] G.M.Ewing: Sufficient conditions for global minima of suitably convex functions from variational and control theory. SIAM Rev. 19 (1977) 2, 202-220.

- [3] R.N.Kaul and Surjeet Kaur: Generalizations of convex and related functions. *European J. Oper. Res.* 9 (1982), 369-377.
- [4] R.N.Kaul and Surjeet Kaur: Sufficient optimality conditions using generalized convex functions. *Opsearch* 19 (1982) 4, 212-224.
- [5] R.N.Kaul and Surjeet Kaur: Generalized convex functions - properties, optimality and duality. Technical report SOL, 84-4, Stanford University, California, 1984.
- [6] R.N.Kaul and Vinod Lyall : Locally connected functions and optimality. *Indian J. Pure Appl. Math.* 22(1991)2, 99-108.
- [7] R.N.Kaul, Vinod Lyall and Surjeet Kaur : Locally connected set and functions. *J.Math.Anal.Appl.* 134(1988)1, 30-45.
- [8] V.Lyall, S.K.Suneja and Sunila Aggarwal : Fritz John optimality and duality for non-convex programs. *J.Math.Anal.Appl.* 212(1997)1, 38-50.
- [9] V.Lyall, S.Suneja and S.Aggarwal : Optimality and duality in fractional programming involving semilocally convex and related functions. *Optimization* 41(1997) 3, 237-255.
- [10] O.L.Mangasarian : *Nonlinear Programming*, McGraw-Hill Book Company, New York, 1969.
- [11] V. Preda, I.M. Stancu-Minasian and A. Băţătorescu: Optimality and duality in nonlinear programming involving semilocally preinvex and related functions. *J. Inform. Optim. Sci.* 17(1996) 3, 585-596.
- [12] V. Preda and I.M. Stancu-Minasian: Duality in multiple objective programming involving semilocally preinvex and related functions. *Glasnik Matematicki.* 32 (52) (1997) 1, 153-165.
- [13] I.M.Stancu-Minasian : *Fractional Programming. Theory, Methods and Applications.* Kluwer Academic Publishers, Dordrecht, 1997.
- [14] I.M.Stancu-Minasian : Optimality conditions and duality in fractional programming involving semilocally preinvex and related functions. (submitted).

- [15] S.K.Suneja and Sudha Gupta: Duality in nonlinear programming involving semilocally convex and related functions. Optimization 28 (1993) 1, 17-29.