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Asset Allocation with Aversion to Parameter Uncertainty: A Minimax Regression Approach

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Asset Allocation with Aversion to Parameter Uncertainty: A Minimax Regression Approach

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Abstract

This paper takes a minimax regression approach to incorporate aversion to parameter uncertainty into the mean-variance model. The uncertainty-averse minimax mean-variance portfolio is obtained by minimizing with respect to the unknown weights the upper bound of the usual quadratic risk function over a fuzzy ellipsoidal set. Beyond the existing approaches, our methodology offers three main advantages: first, the resulting optimal portfolio can be interpreted as a Bayesian mean-variance portfolio with the least favorable prior density, and this result allows for a comprehensive comparison with traditional uncertainty-neutral Bayesian mean-variance portfolios. Second, the minimax mean-variance portfolio has a shrinkage expression, but its performance does not necessarily lie within those of the two reference portfolios. Third, we provide closed form expressions for the standard errors of the minimax mean-variance portfolio weights and statistical significance of the optimal portfolio weights can be easily conducted. Empirical applications show that incorporating aversion to parameter uncertainty leads to more stable optimal portfolios that outperform traditional uncertainty-neutral Bayesian mean-variance portfolios.

Key words : Asset allocation, estimation error, aversion to uncertainty, minimax regression, Bayesian mean-variance portfolios, least favorable prior.

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1 Introduction

The mean-variance model of Markowitz (1952) is one of the most prominent advance in guiding the practice of portfolio selection. This model provides a rigorous framework within which the investor maximizes his expected utility. Yet, the model is unfeasible in practice because the true parameters characterizing the expected utility are not known. A common remedy to this problem is to adopt a plug-in approach which consists in replacing the true unkown parameters by their sample analogues. However, sample estimates are different from the true parameters and the resulting estimation error in most cases leads to optimized mean-variance portfolios that perform poorly out-of-sample (Michaud (1989), Best and Grauer (1991), Black and Litterman (1992), DeMiguel et al. (2009)). To take care of the problem of estimation error in the mean-variance model, an investor can maximize a Bayesian expected utility function defined with respect to a prior density of the unknown parameters. This approach pionnered by Zellner and Chetty (1965) and Bawa, Brown and Klein (1979) was further investigated in the literature with proven empirical success (Frost and Savarino (1986), Jorion (1985, 1986), Pastor and Stambaugh (2000)).

The Bayesian mean-variance analysis supposes that investors have full information and are certain that the specified prior density is perfectly identical to the true density. Nevertheless investors may not have perfect confidence on any prior density due to incomplete information and should manifest this by considering multiple prior densities, each with unknown plausibility. In the more general context of decision making under uncertainty, this situation referred as Knightian uncertainty (or ambiguity) describes the uncertainty to the underlying probabilities. As Ellsberg (1961) shows in several thought experiments, decision-makers are averse to Knightian uncertainty because they prefer known to unknown and ambiguous probabilities of the states of the world.[1] Moreover, Gilboa and Schmeidler (1989) demonstrate that the

[1] See also Becker and Brownson (1964) and Camerer and Weber (1992) for more designed experiments which confirm Ellsberg's intuition.

maxmin expected utility decision rule, that is, the decision rule which maximizes the minimum expected utility over the set of prior densities is compatible with utility maximization under aversion to Knightian uncertainty. Applications of the maxmin principle in the context of portfolio selection are nested to the general framework of robust optimisation (Goldfarb and Iyengar (2003), Tütücün and Koenig (2004), Garlappi, Uppal and Wang (2007)). The main principle is to maximise the investor's expected utility in the worst case scenario regarding the estimation of the unknown parameters. Our paper is related to these works and attempts to provide within a regression framework a methodology to incorporate the investors's aversion to parameter uncertainty when computing the mean-variance portfolio.[2]

The development of our methodology follows three key steps: first, we build on Britten-Jones (1999) and demonstrate that the computation of the feasible mean-variance portfolio can be recast within a regression framework by minimizing the usual quadratic risk function. Second, we introduce parameter uncertainty into the mean-variance model through a constrained regression problem. In the constrained problem, we minimize the quadratic risk function under the constraint that the true weights of the mean-variance portfolio lie within a fuzzy ellipsoidal set. This set summarizes the investor's incomplete information about the true weights of the mean-variance portfolio. Third we rely on the minimax principle (Wald (1945)) to incorporate the investor's aversion to parameter uncertainty. Formally, among all possible allocations that reflect his incomplete information given by the fuzzy ellipsoidal set, he chooses the best allocation in the worst case, that is, the allocation which achieves the smallest maximum quadratic risk.

It is worth noticing that our methodology contrasts with alternative existing approaches (Goldfarb and Iyengar (2003), Tütücün and Koenig (2004), Wang (2005), Garlappi, Uppal and Wang (2007)) in three major points: first, we make a clear connection between our methodology and the Bayesian mean-variance models. For-

[2] In the rest of the paper, the term "uncertainty" will thus refer to the "Knightian uncertainty".

mally we demonstrate following Pilz (1986) that our minimax mean-variance portfolio can be interpreted as a Bayesian mean-variance portfolio with the least favorable prior density over the fuzzy ellipsoidal set. This result that gives more insights on the ability of our methodology to model aversion to parameter uncertainty allows for a clear comparison with traditional Bayesian mean-variance models. Second, it turns out from our analytic expressions that the minimax mean-variance portfolio shrinks the weights of the feasible mean-variance portfolio towards the weights of the minimum-variance portfolio. However and importantly, the performance of the minimax mean-variance portfolio does not necessarily lie within the performances of these target portfolios. Third, closed form expressions for the standard errors of the minimax mean-variance portfolio weights are available. This is possible because of the regression method we adopt. Hence, statistical significance of the optimal portfolio weights can be easily conducted.

Two empirical applications are conducted to illustrate the relevance of the minimax mean-variance portfolio. In the first application we use excess returns on the Fama-French's 6 size and book-to-market assets and compare the accuracy and the stability of the weights of the minimax mean-variance portfolio with the weights of the feasible mean-variance portfolio. The results show that allowing for parameter uncertainty aversion radically decreases the imprecision and the instability of mean-variance portfolios weights. The second application evaluates the economic benefit of incorporating aversion to parameter uncertainty into the mean-variance model, comparing the out-of-sample performance of the minimax mean-variance portfolio with traditional Bayesian mean-variance portfolios. Using the Fama-French's 25 size and book-to-market assets, the obtained results show that relying on the least favorable prior density leads to optimized Bayesian mean-variance portfolios that exhibit higher out-of-sample performances compared to the Bayes-Stein shrinkage portfolio of Jorion (1986) and the Bayesian data-and-model approach of Pastor and Stambaugh (2000).

The rest of the paper is organized as follows. In section 2 we briefly review the standard mean-variance model of asset allocation along with the associated problem of parameter uncertainty aversion. In section 3 we introduce our methodology to model aversion to parameter uncertainty and present the minimax mean-variance portfolio. Section 4 illustrates numerically the gain arising from taking into account aversion to parameter uncertainty notably in term of portfolio weights accuracy and stability, while in Section 5 we compare our minimax mean-variance portfolio with traditional Bayesian mean-variance portfolios. Further extensions are considered in section 6 and the last section concludes the paper.

2 Mean-Variance Model and Aversion to Parameter Uncertainty

Consider an investor who faces the choice of a portfolio among the universe of k risky investable assets. At time t, let r_t be the $k \times 1$ vector of excess (over the risk-free rate) returns on the k risky assets. We denote μ_t and Σ_t respectively the mean and the covariance matrix of r_t. In the mean-variance model, the investor chooses the portfolio that maximizes the expected return for a given level of risk. It is well-known that this strategy is equivalent to select at time T the portfolio $x = (x_1, ..., x_k)'$ to maximize the next-period mean-variance utility

$$\begin{aligned} U_{T+1}(x) &= \mathbb{E}(r_{T+1})' x - \frac{\gamma}{2} x' Var(r_{T+1}) x \\ &= \mu'_{T+1} x - \frac{\gamma}{2} x' \Sigma_{T+1} x, \end{aligned} \tag{1}$$

where the coefficient γ is the degree of relative risk aversion of the investor. The solution to the above problem is

$$w = \gamma^{-1} \Sigma_{T+1}^{-1} \mu_{T+1}. \tag{2}$$

This solution, when scaled to meet the restriction that the asset weights must sum to one, leads to the so-called tangency portfolio

$$w = x (\iota' x)^{-1} = \Sigma_{T+1}^{-1} \mu_{T+1} \left(\iota' \Sigma_{T+1}^{-1} \mu_{T+1}\right)^{-1}, \tag{3}$$

with ι a vector $k \times 1$ of ones. The tangency portfolio is independent of the risk aversion parameter γ and corresponds to the portfolio of risky assets that maximizes the Sharpe ratio. Theoretically, (3) illustrates that the mean-variance optimal rule is computationally unfeasible since the true parameters characterizing the tangency portfolio are unknown. The usual plug-in solution consists in replacing the unknown parameters by corresponding estimators. Under the assumption that r_t is independent and identically distributed, and follows a multivariate normal distribution, natural estimators (maximum likelihood) for the two moments correspond respectively to the empirical mean and covariance matrix, that is

$$\widehat{\mu}_{T+1} = n^{-1} \sum_{t=T-n+1}^{t=T} r_t, \tag{4}$$

$$\widehat{\Sigma}_{T+1} = n^{-1} \sum_{t=T-n+1}^{t=T} \left(r_t - \widehat{\mu}_{T+1}\right) \left(r_t - \widehat{\mu}_{T+1}\right)', \tag{5}$$

with n the available sample size. As a result, the feasible tangency portfolio is computed by replacing the two unknown moments in (3) by their empirical counterparts, leading to the following allocation

$$\widehat{w} = \widehat{\Sigma}_{T+1}^{-1} \widehat{\mu}_{T+1} \left(\iota' \widehat{\Sigma}_{T+1}^{-1} \widehat{\mu}_{T+1}\right)^{-1}. \tag{6}$$

However, the estimator in (6) are different from the true weights in (3) and from the difference arises the problem of estimation error. The cost of ignoring this error has been widely documented in the literature. Indeed, relying on the empirical moments generally leads to instable sub-optimal mean-variance portfolios with extremely large positive and negative weights which are not meaningful economically. The Bayesian mean-variance analysis offers a convenient frame within which the problem of estimation error can be treated. Formally, Bayes estimators of the tangency portfolio weights have the following expression

$$\widehat{w}_b = \left(\widehat{\Sigma}_{T+1}^b\right)^{-1} \widehat{\mu}_{T+1}^b \left(\iota' \left(\widehat{\Sigma}_{T+1}^b\right)^{-1} \widehat{\mu}_{T+1}^b\right)^{-1}, \tag{7}$$

where $\widehat{\mu}_{T+1}^b$ and $\widehat{\Sigma}_{T+1}^b$ correspond to the first two moments of the predictive density $p\left(r_{T+1} \left| \mathcal{F}_T \right.\right)$ of asset returns, with $\mathcal{F}_T$ the set of information available. The predictive density is obtained by integrating with respect to μ_{T+1} and Σ_{T+1} the a posteriori density $p\left(r_{T+1}, \mu_{T+1}, \Sigma_{T+1} \left| \mathcal{F}_T \right.\right)$, that is

$$p\left(r_{T+1} \left| \mathcal{F}_T \right.\right) = \int_{\mu_{T+1}} \int_{\Sigma_{T+1}} p\left(r_{T+1}, \mu_{T+1}, \Sigma_{T+1} \left| \mathcal{F}_T \right.\right) d\mu_{T+1} d\Sigma_{T+1}. \tag{8}$$

The a posteriori density is derived by updating an a priori density $p\left(\mu_{T+1}, \Sigma_{T+1}\right)$ of the unknown parameters with the sampling information. The theoretical contributions in this branch of the literature differ from the choice of the prior density. Earlier Bayesian methods (Brown (1976, 1978), Klein and Bawa (1976)) rely on diffuse-prior while Bayes-Stein shrinkage (Frost and Savarino (1986), Jorion (1985, 1986)) are built with conjugate or hyperparameter priors. Priors that reflect an investor's degree of belief in a given asset pricing model lead to the so-called Bayesian data-and-model method (Pastor (2000), Pastor and Stambaugh (2000), Wang (2005)).

A central hypothesis in the Bayesian analysis is that uncertainty is mesurable and can be summarized by a single prior density. This kind of uncertainty is called risk by Knight (1921) and should be differentiated from the non mesurable uncertainty or ambiguity describing the situation where decision-makers fail to assess with accuracy the probability distribution of the relevant parameters. This distinction is at the core of our paper and we provide in the sequel an econometric methodology which treats the problem of parameter uncertainty along with the reported evidence that investors are averse to such ambiguous situations (Ellsberg (1961)).

3 Aversion to Parameter Uncertainty and the Minimax Mean-Variance Portfolio

This section is divided into two parts. In the first part we extend the results in Britten-Jones (1999) and introduce an ellipsoid-constrained regression model that solves the computation of the mean-variance portfolio when parameter uncertainty is of concern. In the second part, we rely on the minimax principle to solve the constrained regression model and show that the resulting allocation rule deals with the issue of parameter uncertainty aversion.

3.1 Incorporating Parameter Uncertainty into the Mean-Variance Model

Our starting point is the regression-based approach for the computation of the mean-variance portfolio (Britten-Jones (1999)). Formally, consider the following regression model

$$\begin{cases} Y = Xw + u, \\ E(u) = 0, \;\; E(uu') = \sigma_u^2 \end{cases}, \tag{9}$$

where X is the $n \times k$ matrix of excess returns on the k risky assets, w the $k \times 1$ vector of parameters, u the noise term, and Y the $n \times 1$ constant vector with all entries equal to δ with

$$\delta = \left(\iota' \widehat{\Sigma}_{T+1}^{-1} \widehat{\mu}_{T+1}\right)^{-1} \left(1 + \widehat{\mu}_{T+1}' \widehat{\Sigma}_{T+1}^{-1} \widehat{\mu}_{T+1}\right). \tag{10}$$

Let $D = \{\widetilde{w} : \widetilde{w} = CY\}$ be the class of homogeneous linear solutions for the regression model (9) with C a $k \times n$ unknown matrix and consider the quadratic risk function $\mathcal{R}(\widetilde{w}, w)$

$$\mathcal{R}(\widetilde{w}, w, A) = \mathbb{E}(\widetilde{w} - w)' A (\widetilde{w} - w),$$

with A a $k \times k$ positive definite matrix. The following proposition recasts the computation of the feasible tangency portfolio within a regression framework.

Proposition 1 *With the regression model (9) and the class of homogeneous linear solutions* $\widetilde{w} = CY$*, the optimal unbiased estimator of* w *under the risk function* $\mathcal{R}(\widetilde{w}, w, A)$ *is identical to the weights of the feasible tangency portfolio in (6), that is*

$$\arg\min_{\widetilde{w} \in D} \mathcal{R}(\widetilde{w}, w, A) \;\; = \;\; \left(X'X\right)^{-1} X'Y \; = \widehat{w}. \tag{11}$$

The proof is straightforward using first the equivalence between the least-squares estimation of (9) and the minimization of the quadratic risk function $\mathcal{R}(\widetilde{w}, w, A)$ (see Theorem 4.1 in Rao and Toutenburg (1999)), and second the results of Theorem 1 in Britten-Jones (1999). The proposition states that the traditional plug-in method to compute the weights of the tangency portfolio and the minimization of the quadratic risk function $\mathcal{R}(\widetilde{w}, w, A)$ lead to the same solution.

Now, the question of interest is how one can exploit this result to deal with the issue of estimation error. First let us recall that from a Bayesian perspective, the problem of estimation error in the mean-variance model is usually treated by specifying a prior density for the unknown parameters μ_{T+1} and Σ_{T+1}. The direct equivalence in our regression framework would be to form a prior density $p(w)$ directly for the unknown weights of the tangency portfolio. The resulting Bayesian mean-variance portfolio can be obtained by minimizing (with respect to $\widetilde{w} \in D$) the Bayes risk given by

$$\mathbb{E}_p \mathcal{R}(w, \widetilde{w}, A) = \int_{\mathbb{R}^k} \mathcal{R}(w, \widetilde{w}, A)\, p(dw). \tag{12}$$

As already stressed, the uncertainty that arises from the arbitrariness of the choice of the prior density $p(w)$ is at the core of this paper. An approach we take here to incorporate this uncertainty into the mean-variance model is to solve the problem (11) by searching the optimal weights over a fuzzy set which reflects the incompleteness of the investor's information, and hence his inability to form the prior density $p(w)$. The corresponding minimization problem can be written as follows

$$\begin{cases} \widehat{w}^* = \underset{\widetilde{w} \in D}{\arg\min}\ \mathcal{R}(\widetilde{w}, w, A) \\ \text{s.t.} \quad w \in \Gamma, \quad \iota' w = 1 \end{cases}, \tag{13}$$

with $\widehat{w}^*$ an estimator of the weights of the tangency portfolio and Γ the fuzzy set.[3] To construct the fuzzy set Γ, consider an investor who believes that the empirical mean $\widehat{\mu}_{T+1}$ provides a good approximation to the true expected asset returns μ_{T+1}. This investor should rely on the feasible tangency portfolio by replacing the two unknown moments in (3) by their empirical counterparts. In the converse case where the investor assumes that the uncertainty in approximating the expected asset returns

[3] The notion of fuzzy sets is introduced by Zadeh (1965) and provides an interesting alternative (in representing uncertainty) to the conventional approaches using probabilistic modelling. For comprehensive descriptions and interpretations of the notion of fuzzy sets, see for e.g. Bandemer and Gottwald (1995) or Zimmermann (2001).

by its empirical counterpart is too high to be economically valuable, he should allocate the assets by cancelling the contribution of the expected returns in (3). This task can be achieved by setting the expected returns to the constant vector. The corresponding allocation is identical to the weights of the global minimum-variance portfolio given by[4]

$$\widehat{w}_{\min} = \widehat{\Sigma}_{T+1}^{-1} \iota \left(\iota' \widehat{\Sigma}_{T+1}^{-1} \iota \right)^{-1}. \tag{14}$$

As a consequence, the distance between the true tangency portfolio weights w and the weights $\widehat{w}_{\min}$ of the global minimum-variance portfolio can serve as an indicator of the level of uncertainty in estimating the expected asset returns. This distance can be formulated in the form of the following ellipsoidal constraint

$$\Gamma = \left\{ w : \left(w - \widehat{w}_{\min} \right)' H \left(w - \widehat{w}_{\min} \right) \leq \kappa \right\}, \tag{15}$$

where H is a $k \times k$ positive definite matrix and κ a positive real number. The ellipsoid defines a region where the true asset weights are believed to lie and the volume of this region is controlled by the parameter κ. For convenience, we will use the following re-parameterization for κ

$$\kappa = n \left[\Phi^{-1} \left(1 - \eta/2 \right) \right]^2, \tag{16}$$

with $\eta \in (0, 1)$ and $\Phi^{-1}(.)$ the inverse of the standard normal cumulative distribution function. This is useful because we summarize the infinite possible values of $\kappa \in \mathbb{R}_+$ via the bounded parameter η. Note that η can be viewed as the investor's degree of skepticism about the estimation of the expected asset returns. Indeed, when the investor assigns a value for η near one (κ near zero), he has high uncertainty about the estimation of the expected asset returns, and materializes this by locating the true unknown tangency portfolio weights near the weights of the global minimum-variance portfolio. In the opposite case where η diverges from one, the investor's skepticism

[4] Note that we focus here on the uncertainty in estimating the expected asset returns since they are more affected by estimation error than the covariance matrix. For empirical evidences see Merton (1980) and Chopra and Ziemba (1993). In Section 6 we will extend our framework by considering uncertainty in estimating both the expected asset returns and the covariance matrix.

about the precision of the empirical mean $\widehat{\mu}_{T+1}$ diminishes. It is worth noting that the role of the scaling factor n in (16) is to relax the ellipsoidal constraint when larger sample sizes are available, because in these cases and independently of the investor's degree of skepticism η, the precision of the empirical mean $\widehat{\mu}_{T+1}$ becomes larger.

A simple choice for the matrix H would be the identity matrix. But this is a rather naive choice, since it measures the distance between w and $\widehat{w}_{\min}$ by placing equal importance on the deviation of each component of w from that of $\widehat{w}_{\min}$. A better alternative would be to weight the importance of the deviation by a matrix that reflects the precision of the estimated weights of the global minimum-variance portfolio. More precisely, we set the matrix H to the $k \times k$ diagonal matrix with the i-th element being the inverse of the variance of the i-th weight of the global minimum-variance portfolio, that is, $H = (h_{ij})$ with

$$h_{ij} = \begin{cases} \frac{1}{Var\left(\widehat{w}_{\min,i}\right)} & \text{if} \quad i = j \\ 0 & \text{else} \end{cases} . \tag{17}$$

The expression of the variance of the i-th weight of the global minimum-variance portfolio is given by Bodnar and Schmid (2008) and corresponds to

$$Var\left(\widehat{w}_{\min,i}\right) = \frac{\iota\widehat{\Sigma}_{T+1}^{-1}\iota'\sigma_{ii}^{(-)} - \left(\sum_{m=1}^{k}\sigma_{im}^{(-)}\right)^2}{(n-k)\left(\iota\widehat{\Sigma}_{T+1}^{-1}\iota'\right)^2}, \tag{18}$$

with $\widehat{\Sigma}_{T+1}^{-1} = \left(\sigma_{im}^{(-)}\right)$ the inverse of the sample covariance matrix.

3.2 Minimax Principle and Aversion to Parameter Uncertainty

In the last part of this section, we show how to solve the constrained minimization program (13) taking care of the investor's aversion to parameter uncertainty.[5] Precisely, we rely on the minimax principle (Wald (1945)) to solve this program.

[5] We focus in a first time on the constrained least squares problem (13) without the last restriction that the asset weights must sum to one. We will consider this restriction later.

Definition 2 *The minimax estimator* $\widehat{w}^{*} \in D = \{\widetilde{w} : \widetilde{w} = CY\}$ *solution of the constrained problem (13) is defined as follows*

$$\sup_{w \in \Gamma} \mathcal{R}(w, \widehat{w}^{*}, A) = \inf_{\widetilde{w} \in D} \sup_{w \in \Gamma} \mathcal{R}(w, \widetilde{w}, A). \tag{19}$$

An investor that allocates assets using the minimax estimator $\widehat{w}^{*}$ in (19) is averse to the uncertainty (or ambiguity) arising from having multiple prior densities. To show this, let $\mathcal{P}$ be the set of arbitrary prior distributions p on Γ and consider the Bayes risk

$$\mathbb{E}_p \mathcal{R}(w, \widetilde{w}, A) = \int_{\Gamma} \mathcal{R}(w, \widetilde{w}, A)\, p(dw). \tag{20}$$

Proposition 3 *There exists a prior density* $p_0 \in \mathcal{P}$ *such that the Bayes estimator that minimizes the Bayes risk in (20) corresponds to the minimax estimator* $\widehat{w}^{*}$ *in (19), that is*

$$\widehat{w}^{*} = \inf_{\widetilde{w} \in D} \mathbb{E}_{p_0} \mathcal{R}(w, \widetilde{w}, A),$$

with the following relation

$$\mathbb{E}_{p_0} \mathcal{R}(w, \widetilde{w}, A) \geq \mathbb{E}_p \mathcal{R}(w, \widetilde{w}, A) \quad \forall p \in \mathcal{P}.$$

See Theorem 1 in Pilz (1986) for the proof. The prior density p_0 is termed the least favorable prior density, that is the prior density which leads to the maximum Bayes risk. Our minimax estimator of the mean-variance portfolio can thus be viewed as a Bayesian mean-variance portfolio for an investor who has multiple prior densities (over the fuzzy set Γ) and chooses the one which corresponds to the least favorable (or worst-case) scenario. Following Gilboa and Schmeidler (1989), this investment strategy is evidently compatible with aversion to uncertainty.

To derive explicit solutions for the estimator $\widehat{w}^{*}$, additional structural assumptions are needed. In the case where the matrix A is of rank one, the explicit solution (Kuks and Olman (1971, 1972)) is given by

$$\widehat{w}^{*} = \left(X'X + \kappa^{-1}\sigma_u^2 H\right)^{-1} \left(X'Y + \kappa^{-1}\sigma_u^2 H \widehat{w}_{\min}\right). \tag{21}$$

An operational version of this estimator is obtained by replacing σ_u^2 by its least-squares estimator $\widehat{\sigma}_u^2$. It can be seen easily that $\widehat{w}^*$ behaves like a shrinkage estimator and makes - with respect to the parameter κ or equivalently η - a trade-off between $\widehat{w}$ the weights of the feasible mean-variance portfolio and the weights $\widehat{w}_{\min}$ of the global minimum-variance portfolio. The corresponding limiting behaviours are

$$\lim_{\eta \to 0} \widehat{w}^* = \widehat{w}, \tag{22}$$

$$\lim_{\eta \to 1} \widehat{w}^* = \widehat{w}_{\min}. \tag{23}$$

To finish, remark that $\widehat{w}^*$ is not properly an estimator of the tangency portfolio weights because its components do not sum to one except for the limiting cases $\widehat{w}$ and $\widehat{w}_{\min}$. Therefore, we follow Toutenburg (1980) defining the equality restricted analogue of the estimator $\widehat{w}^*$ in (21), that is

$$\overline{w}^* = \widehat{w}^* - D^{-1}\iota \left(\iota' D^{-1}\iota\right)^{-1} \left(\iota'\widehat{w}^* - 1\right), \tag{24}$$

with $D = X'X + \kappa^{-1}\widehat{\sigma}_u^2 H$. A salient feature of our regression approach is that the bias and the variance of $\overline{w}^*$ can be derived (Toutenburg (1980)) yielding

$$\text{bias } \overline{w}^* = \kappa^{-1}\widehat{\sigma}_u^2 \left[-I_k + D^{-1}\iota \left(\iota' D^{-1}\iota\right)^{-1} \iota'\right] D^{-1} H w,$$

$$\begin{aligned} \widehat{\sigma}_u^{-2} Var\left(\overline{w}^*\right) &= D^{-1}X'XD^{-1} \\ &\quad + D^{-1}\iota \left(\iota' D^{-1}\iota\right)^{-1} \iota' D^{-1} X'X D^{-1}\iota \left(\iota' \left(X'X\right)^{-1} \iota\right)^{-1} \\ &\quad - D^{-1}\iota \left(\iota' D^{-1}\iota\right)^{-1} \iota' D^{-1} X'X D^{-1} \\ &\quad - D^{-1} X'X D^{-1}\iota \left(\iota' D^{-1}\iota\right)^{-1} \iota' D^{-1}, \end{aligned} \tag{25}$$

The bias (resp. the variance) is a decreasing (resp. increasing) function of κ. This implies that when the investor degree of skepticism in estimating the expected returns is high ($\eta \to 1$) the parameter $\kappa \to 0$ and the bias (resp. the variance) increases (resp. decreases). Hence, allowing for uncertainty aversion has a shrinkage-like effect on the estimated portfolio weights which are biased but more precise according to the usual bias-variance trade-off for regression parameters. Notice that for the limiting case $\eta \to 0$ ($\kappa \to \infty$), we have bias $\overline{w}^* = 0$ and

$$Var\left(\overline{w}^*\right) = \widehat{\sigma}_u^2 \left[S^{-1} - S^{-1}\iota' \left(\iota S^{-1}\iota'\right)^{-1} \iota S^{-1}\right], \tag{26}$$

which correspond to the bias and the variance of the feasible tangency portfolio.

4 Stability and Accuracy of the Minimax Mean-Variance Portfolio

In this section, we evaluate the behaviour (from the viewpoint of stability and accuracy) of the minimax mean-variance portfolio, using the monthly excess returns of the Fama-French's 6 size and book-to-market assets from July 1963 to September 2009. Thus the number of assets is equal to $k = 6$ and the total number of time series observations is equal to $N = 555$ months. We set the estimation sample size to $n = 60$ months and use a "rolling-window" procedure to compute the out-of-sample mean-variance portfolio weights and returns. More precisely, the asset returns for the first $n = 60$ months are used to compute $\overline{w}^*$ and the corresponding portfolio return for the next month. This step is repeated by moving each time the estimation window (including the data for a new month and dropping the data for the earliest month) until the end of the data set is reached. Note that at the end of the procedure, we have computed $N - n$ portfolio weights and returns.

Figure 1 in Appendix reports for four different values of the investor's degree of skepticism η the boxplot of the estimated portfolio weights $\overline{w}^*$ over the out-of-sample period. The considered values are 0%, 10%, 75% and 100%. Recall that the first value $\eta = 0\%$ corresponds to the feasible tangency portfolio that ignores parameter uncertainty, while the last value $\eta = 100\%$ corresponds to the global minimum-variance portfolio. Compared to the feasible tangency portfolio in which parameter uncertainty is not of concern, the minimax mean-variance portfolios have more stable weights. Indeed, while the weights of the feasible tangency portfolio range dramatically from -1258 to 2884, the weights $\overline{w}^*$ of the new optimal portfolio range from -8 to 11 (resp. -4 to 5) for $\eta = 10\%$ (resp. $\eta = 75\%$). Therefore, aversion to parameter uncertainty leads to less extreme positions and portfolio weights that vary much less over time. This is the case, because for a given value of $\eta \neq 0$ the estimator $\overline{w}^*$ of the optimal portfolio weights shrinks the weights of the feasible tangency portfolio towards the weights of the global minimum-variance portfolio, and thus benefits from the stability of the latter portfolio. To give more evidence about the stabilization effect of the minimax mean-variance investment strategy, we report in Figure 2 (see Appendix) the boxplot of the estimated variances of the optimal portfolio weights displayed in Figure 1. The variances are computed using formula (25) and the observed patterns illustrate the reduction of estimation error due to the shrinkage effect of the estimator $\overline{w}^*$. The variances of the feasible tangency portfolio weights range from 0.23 to 2105 illustrating the fact that sample means are very noisy estimators of the true expected asset returns. In the same time, allowing for uncertainty aversion with a small amount of uncertainty equal to 10% lower the variances which lie between 0.02 and 3.2.

To illustrate the effect of aversion to parameter uncertainty on the out-of-sample performance of the mean-variance model, we compute from the $N-n$ optimal portfolios returns three statistics: the out-of-sample means, standard-deviations and sharpe ratios. Table 1 in Appendix displays these statistics for different values of the investor's degree of skepticism η. First, one can see that for the limiting case $\eta = 0\%$ (resp. $\eta = 100\%$) the statistics are identical to those of the feasible tangency portfolio (resp. the global minimum-variance portfolio). Second, the minimax mean-variance portfolio $\overline{w}^*$ that allows for parameter uncertainty aversion exhibits higher means and lower volatilities. For instance, while the out-of-sample mean (resp. standard-deviation) of the feasible tangency portfolio is equal to -0.1732 (resp. 3.9550) the same statistic when the investor's degree of skepticism corresponds to $\eta = 10\%$ is equal to 0.0175 (resp. 0.1937). Third, for a given value of $\eta \in]0, 1[$ the out-of-sample Sharpe ratio of our minimax mean-variance portfolio $\overline{w}^*$ does not lie necessarily within the sharpe ratios of the two limiting cases ($\eta = 0\%$ and $\eta = 100\%$). Indeed, for $\eta \geq 20\%$ the out-of-sample Sharpe ratios are higher than the ones of the feasible tangency and the global minimum-variance portfolios. Figure 3 in Appendix gives a complete description of the evolution of the Sharpe ratio with respect to η. The figure depicts a quadratic evolution, with an increase in Sharpe ratio up to a given value of η and a decrease over this value. Relatively to this evolution our methodology to deal with the issue of uncertainty aversion contrasts with the multi-prior approach of Garlappi, Uppal and Wang (2007). In the multi-prior approach, uncertainty is introduced in the mean-variance model by specifying confidence intervals (around the true expected asset returns) in the form of ellipsoid constraint, that is

$$G\left(\mu_{T+1}, \widehat{\mu}_{T+1}\right) = \left\{\mu_{T+1} : \frac{n\left(n-k\right)}{\left(n-1\right)k}\left(\mu_{T+1} - \widehat{\mu}_{T+1}\right)' \Sigma_{T+1}\left(\mu_{T+1} - \widehat{\mu}_{T+1}\right) \leq \epsilon\right\}, \tag{27}$$

with ϵ the level of uncertainty in estimating expected asset returns. Aversion to uncertainty is modelled by referring to the following constrained mean-variance optimization

$$\begin{cases} \overline{w} = \underset{w}{\operatorname{argmax}} \ \underset{\mu_{T+1}}{\min} \ \mu'_{T+1}w - \frac{\gamma}{2}w'\Sigma_{T+1}w, \\ \text{s.t. } \ \mu_{T+1} \in G\left(\mu_{T+1}, \widehat{\mu}_{T+1}\right) \text{ and } \iota'w = 1. \end{cases} \tag{28}$$

In this setting, the standard problem of utility maximization over the unknown portfolio weights is modified, by first minimizing the investor's utility with respect to the expected returns which are constrained to lie within the ellipsoid $G\left(\mu_{T+1}, \widehat{\mu}_{T+1}\right)$. This ellipsoid reflects the investor's a priori information about the true expected asset returns for a given level of uncertainty ϵ. Therefore, there is a close connection between the max-min problem (28) and the minimax regression approach we follow in this paper. The methodological difference arises from the fact that in our framework, the uncertainty is instead about the true unknown portfolio weights. Garlappi, Uppal and Wang (2007) show that the solution of the max-min problem (28) is a weighted average of the classical mean-variance portfolio that ignores parameter uncertainty and the global minimum-variance portfolio

$$\overline{w} = \rho \widehat{w}_{\min} + (1-\rho)\,\widehat{w} \tag{29}$$

with ρ a scalar that depends on the level of uncertainty ϵ. Thus, the performance of this asset allocation rule lies between the ones of the two reference portfolios. As already stressed, our minimax mean-variance portfolio $\overline{w}^*$ differs from $\overline{w}$ with regards to this characteristic.

5 Comparison with Uncertainty Neutral Bayesian Mean-Variance Portfolios

From proposition 3, we have shown that the minimax mean-variance portfolio corresponds to a Bayesian mean-variance portfolio under the least favorable prior density. Hence, we compare in this section the out-of-sample performances of the minimax mean-variance portfolio $\overline{w}^*$ with traditional uncertainty neutral Bayesian mean-variance portfolios.

5.1 Description of the portfolios Considered

The first mean-variance portfolio we consider is the Bayes-Stein shrinkage portfolio developed by Jorion (1986), which exploits the idea of shrinkage estimation (Stein (1955), James and Stein (1961)). Under this model, the weights of the tangency portfolio has the following expression

$$\widehat{w}_{bs} = \left(\widehat{\Sigma}^{bs}_{T+1}\right)^{-1} \widehat{\mu}^{bs}_{T+1} \left(\iota' \left(\widehat{\Sigma}^{bs}_{T+1}\right)^{-1} \widehat{\mu}^{bs}_{T+1}\right)^{-1}, \tag{30}$$

with $\widehat{\mu}^{bs}_{T+1}$ and $\widehat{\Sigma}^{bs}_{T+1}$ the shrinkage estimators of μ_{T+1} and Σ_{T+1} equal to

$$\widehat{\mu}^{bs}_{T+1} = (1-\widehat{v})\,\widehat{\mu}_{T+1} + \widehat{v}\widehat{\mu}^{g}_{T+1}\iota, \tag{31}$$

$$\widehat{\Sigma}^{bs}_{T+1} = \left(1 + \frac{1}{n+\widehat{\lambda}}\right)\overline{V}_{T+1} + \frac{\widehat{\lambda}}{n\left(n+1+\widehat{\lambda}\right)}\frac{\iota\iota'}{\iota'\overline{V}^{-1}_{T+1}\iota}, \tag{32}$$

$$\widehat{v} = \frac{k+2}{(k+2) + n\left(\widehat{\mu}_{T+1} - \widehat{\mu}^{g}_{T+1}\iota\right)'\overline{V}^{-1}_{T+1}\left(\widehat{\mu}_{T+1} - \widehat{\mu}^{g}_{T+1}\iota\right)},$$

$$\widehat{\lambda} = \frac{k+2}{\left(\widehat{\mu}_{T+1} - \widehat{\mu}^g_{T+1}\iota\right)' \overline{V}^{-1}_{T+1} \left(\widehat{\mu}_{T+1} - \widehat{\mu}^g_{T+1}\iota\right)},$$

$$\overline{V}_{T+1} = \frac{n\widehat{\Sigma}_{T+1}}{n-k-2},$$

$$\widehat{\mu}^g_{T+1} = \frac{\iota'\overline{V}^{-1}_{T+1}\widehat{\mu}_{T+1}}{\iota'\overline{V}^{-1}_{T+1}\iota}.$$

Thus, the empirical mean $\widehat{\mu}_{T+1}$ are shrunk towards the target value $\widehat{\mu}^g_{T+1}\iota$ with $\widehat{v}$ the bounded shrinkage parameter. This operation while introducing a small amount of bias in the estimation procedure of the expected asset returns significantly reduces the variance, leading to more stable out-of-sample portfolio weights (see Jorion (1986) and DeMiguel et al. (2009a, 2009b) for empirical applications).

The second mean-variance portfolio is derived from the Bayesian data-and-model approach in Pastor and Stambaugh (2000). For a brief description, suppose the existence of m benchmark portfolios related to a given asset-pricing model. The key issue in the Bayesian data-and-model approach is to make a balance between the asset-pricing model and the sampling information. In this line, the two extreme views for an investor is to believe or not to believe in the asset-pricing model. In the former case, estimators of the two unknown moments μ_{T+1} and Σ_{T+1} are computed from the asset-pricing model

$$r_t = \beta r_{f,t} + e_t, \quad \forall t = T-n+1, ..., T, \tag{33}$$

where $r_{f,t}$ is the $m \times 1$ vector of excess returns on the benchmark portfolios, β is the $k \times m$ matrix of the betas, and e_t the $k \times 1$ vector of residuals. In the latter case, empirical counterparts of the two moments are used instead, and this is equivalent to estimate the moments via the asset-pricing model (33) by allowing for a mispricing vector α. A middle approach that is more relevant than the extreme views is to update the sampling information with the prior degree of confidence about the asset-pricing model. As suggested by Pastor and Stambaugh (2000), one way of doing this is to specify a prior distribution for the asset's mispricing vector α, that is

$$p\left(\alpha\left|\Sigma_{T+1}\right.\right)=N\left(0,\frac{\sigma_{\alpha}^{2}}{s^{2}}\Sigma_{T+1}\right), \tag{34}$$

where σ_{α}^{2} is a positive parameter that controls the variance of the prior distribution of α, and s^2 a fixed parameter. Note that when $\sigma_{\alpha}^{2}=0$ the investor believes dogmatically in the asset-pricing model and for $\sigma_{\alpha}^{2}\to\infty$ the investor believes that the asset-pricing model is not useful. Wang (2005) shows that with the prior in (34), the mean of the predictive distribution of r_{T+1} has a shrinkage expression

$$\mathbb{E}\left(r_{T+1}\left|\theta\right.\right)=\theta\widehat{\mu}_{T+1}^{f}+\left(1-\theta\right)\widehat{\mu}_{T+1}, \tag{35}$$

with $\widehat{\mu}_{T+1}^{f}$ the estimator of the expected returns from the asset-pricing model, θ the shrinkage parameter that depends on σ_{α}^{2} the variance of the mispricing.[6] Hence, the weights of the tangency portfolio in the Bayesian data-and-model approach are computed by replacing μ_{T+1} and Σ_{T+1} in (3) by $\mathbb{E}\left(r_{T+1}\left|\theta\right.\right)$ and $Var\left(r_{T+1}\left|\theta\right.\right)$.

5.2 Data and Results

The comparison is conducted using $k=25$ risky assets that correspond to the 25 Fama-French size and book-to-market portfolios. The data set contains monthly excess returns (from Kenneth French's Web site) for these assets over the period July 1963-September 2009. We set the estimation sample size to $n=60$ and use the rolling-window methodology described in the last section to compute the out-of-sample portfolio weights respectively for our minimax mean-variance portfolio $\overline{w}^{*}$ and the two Bayesian mean-variance portfolios described above.

Table 2 in Appendix displays the out-of-sample means, standard deviations and Sharpe ratios for the three portfolios. For the minimax mean-variance portfolio $\overline{w}^{*}$, we report the results for different values of the investor's degree of skepticism η. To implement the Bayesian data-and-model approach we consider the Fama-French

[6] Wang (2005) also derived the formula for the second moment $Var\left(r_{T+1}\left|\mathcal{F}_{T},\theta\right.\right)$ of the predictive distribution which also has a shrinkage expression.

three-factor model and set the shrinkage parameter θ respectively to 10%, 25%, 50%, 75%, 90% and 100%. From this table, we see that the out-of-sample mean of the Bayes-Stein shrinkage portfolio is negative and equal to -0.0225. Thus, it seems, at least for the data set considered here, that the shrinkage intensity of this model is not sufficient enough to move the out-of-sample mean of the feasible tangency portfolio from a negative value (-0.0455) to a positive one. Compared to the feasible tangency portfolio that ignores parameter uncertainty, both the Bayesian data-and-model portfolios of Pastor and Stambaugh (2000) and our minimax portfolios exhibit higher means in all cases, that is for the different values of η and θ. However, the out-of-sample means are uniformly higher for the former portfolios for $\theta \geq 50\%$. To give more insights about this result, note that the Bayesian data-and-model (resp. the minimax) approach makes a compromise between the feasible tangency portfolio and the asset-pricing (resp. the minimum-variance) portfolio. Now, it turns out in the case of the data set used that the out-of-sample mean of the Fama-French three-factor portfolio ($\theta = 100\%$) is higher than the one of the minimum-variance portfolio. The converse case is obtained with regards to the out-of-sample standard-deviations, where the values respectively for the Fama-French three-factor portfolio and the minimum-variance portfolio are equal to 0.4582 and 0.0437. The minimax mean-variance portfolios benefit from this characteristic by exhibiting lower standard-deviations compared to the Bayesian data-and-model portfolios. Lastly, the out-of-sample Sharpe ratios of the minimax mean-variance portfolios are higher than the ones of the Bayesian portfolios. Summarizing our findings, we can conclude from this empirical exercise that relying on the least favorable prior density via our minimax regression approach is economically beneficial, because the corresponding portfolio leads to lower (resp. higher) out-of-sample standard-deviations (resp. Sharpe ratios).

6 Further Extensions

It is well documented in the literature that sample means are more affected by estimation error than the sample covariance matrix (Merton (1980), Chopra and Ziemba (1993)). Yet the uncertainty in estimating the covariance matrix by its empirical counterpart is not negligible for high dimensional problem, that is when the asset universe is large (Chan et al. (1999), Jagannathan and Ma (2003), Ledoit and Wolf (2003, 2004a, 2004b)). This is the case because when $k/n \to c > 0$ with k the number of assets and n the sample size, the eigenvalues of the sample covariance matrix spread out more than the true unobservable ones (Marcenko-Pastur (1967)), and the eigenvectors are not consistent (Johnstone and Lu (2004)). Therefore, unless $k/n \to 0$, errors in the sample covariance matrix can affect the mean-variance allocation through the inconsistency of eigenvalues and eigenvectors. Needless to say that investors should be averse to this type of uncertainty. Hence, our objective in this section is to extend our methodology to incorporate into the mean-variance model the investor's aversion in estimating both the expected asset returns and the covariance matrix.

If we denote w^0 the vector weights of the equally-weighted portfolio, it is easy to see that the fuzzy ellipsoidal set Λ defined as follows

$$\Lambda = \left\{ w : \left(w - w^0\right)' H \left(w - w^0\right) \leq \kappa \right\}, \tag{36}$$

can summarize the incompleteness of the investor's information. Indeed, for $\kappa \to 0$ the investor allocates his wealth equally between the assets because he believes that sampling information is completely useless. In the opposite case where $\kappa \to \infty$, he relax his belief and search the weights of the tangency portfolio in a larger space. As previously, we use for convenience a re-parameterization of the parameter κ, that is

$$\kappa = n \left[\Phi^{-1}\left(1 - \eta/2\right)\right]^2, \tag{37}$$

with η the level of uncertainty in estimating the two unknown moments μ_{T+1} and

Σ_{T+1}. For the choice of the matrix H, we use the diagonal matrix with elements the inverse of the empirical variances of the asset returns. Hence, the more risky is an asset, the less important is its contribution to the construction of the ellipsoid. Using once again the minimax principle, the optimal portfolio weights $\widetilde{w}^*$ for an uncertainty averse investor will be identical to $\overline{w}^*$ in (24) except that the weights of the target minimum-variance portfolio is replaced by the weights of the noise-free equally-weighted portfolio

$$\widetilde{w}^* = \overrightarrow{w}^* - D^{-1}\iota \left(\iota' D^{-1}\iota\right)^{-1} \left(\iota'\overrightarrow{w}^* - 1\right), \tag{38}$$

with $D = X'X + \kappa^{-1}\widehat{\sigma}_u^2 H$ and

$$\overrightarrow{w}^* = D^{-1}\left(X'Y + \kappa^{-1}\widehat{\sigma}_u^2 H w^0\right). \tag{39}$$

Thus, our methodology does not change radically. One can expect that for large dimensional problems where the minimum-variance portfolio performs poorly, shrinking (using our minimax method) the feasible tangency portfolio towards the equally-weighted portfolio would be more beneficial. This is all the more true as it is shown by DeMiguel et al. (2009a) that the equally-weighted portfolio constitutes a relevant benchmark investment rule as many others investment strategies, even sophisticated, cannot beat it.

To illustrate all these statements, we consider the excess returns for the Fama-French's 100 size and book-to-market assets over the period July 1963 - September 2009. The estimation sample size is set to $n = 120$ and a rolling-window procedure is use to compute $\overline{w}^*$ and $\widetilde{w}^*$ which are considered as competitive investment strategies. Recall that $\overline{w}^*$ (resp. $\widetilde{w}^*$) refers to the weights of the minimax mean-variance portfolio for an investor's who worries about the estimation of the expected asset returns (resp. the expected asset returns and the covariance matrix). Table 3 in Appendix reports for different values of the investor's degree of skepticism η the out-of-sample means, standard errors and Sharpe ratios of the two investment strategies. The table also displays the same statistics for the target minimum-variance and equally-weighted portfolios. As expected, the out-of-sample standard deviations (resp. the Sharpe ratios) of the equally-weighted portfolio are lower (resp. higher) than those of the minimum-variance portfolio. As a consequence, for the different values of η the minimax mean-variance allocation rule $\widetilde{w}^*$ is uniformly better than $\overline{w}^*$ at least for the data set considered here.

7 Conclusion

Standard models of asset allocation would be very easy to implement if the true parameters characterizing the distribution of asset returns are perfectly known. In practice, this is not the case and the knowledges of investors are generally too limited to form a single prior density for these parameters. The resulting Knightian uncertainty (or ambiguity) renders the usual uncertainty-neutral Bayesian solutions restrictive because investors are averse to the uncertainty arising from having multiple prior densities.

For an investor allocating assets using the mean-variance model, this article provides an econometric framework to incorporate his aversion to parameter uncertainty. The starting point of our methodology is the regression-based approach for the computation of the mean-variance portfolio in Britten-Jones (1999). Extending this approach using the minimax principle (Wald (1945)) we provide an estimator for the optimal portfolio weights that solves the mean-variance problem under aversion to parameter uncertainty. The uncertainty-averse mean-variance portfolio weights correspond to the minimax estimator for regression parameters, that is the estimator that minimizes the upper bound of the usual quadratic risk function. This upper bound is derived by searching over a fuzzy ellipsoidal set that summarizes the incompleteness of the investor's information about the unknown parameters. The new estimator has a shrinkage expression and makes a compromise between the usual plug-in estimator of the mean-variance portfolio and the global minimum-variance portfolio. An empirical application using the Fama-French 6 size and book-to-market assets shows that the minimax estimator by introducing a small amount of bias improves substantially the accuracy and the stability of the optimal portfolio weights. Moreover, the minimax mean-variance portfolio corresponds to a Bayesian mean-variance portfolio with the least favorable prior density. Using the Fama-French 25 size and book-to-market assets we show that relying on the least favorable prior density leads to better out-of-sample performances compared to traditional hyperparameter or pricing-model priors.

Appendix : Tables and figures

Figure 1: Boxplots of the out-of-sample weights of the minimax mean-variance portfolio using the Fama-French's 6 size and book-to-market assets

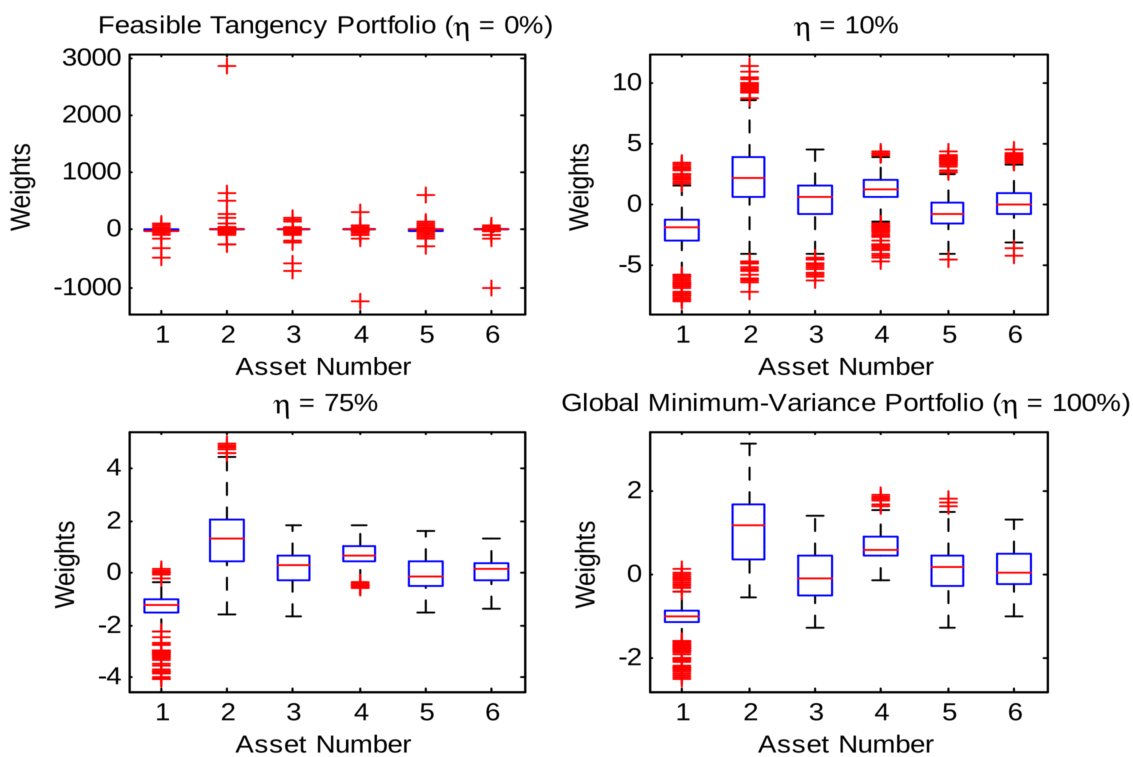

Figure 2: Boxplots of the variances of the out-of-sample weights of the minimax mean-variance portfolio using the Fama-French's 6 size and book-to-market assets

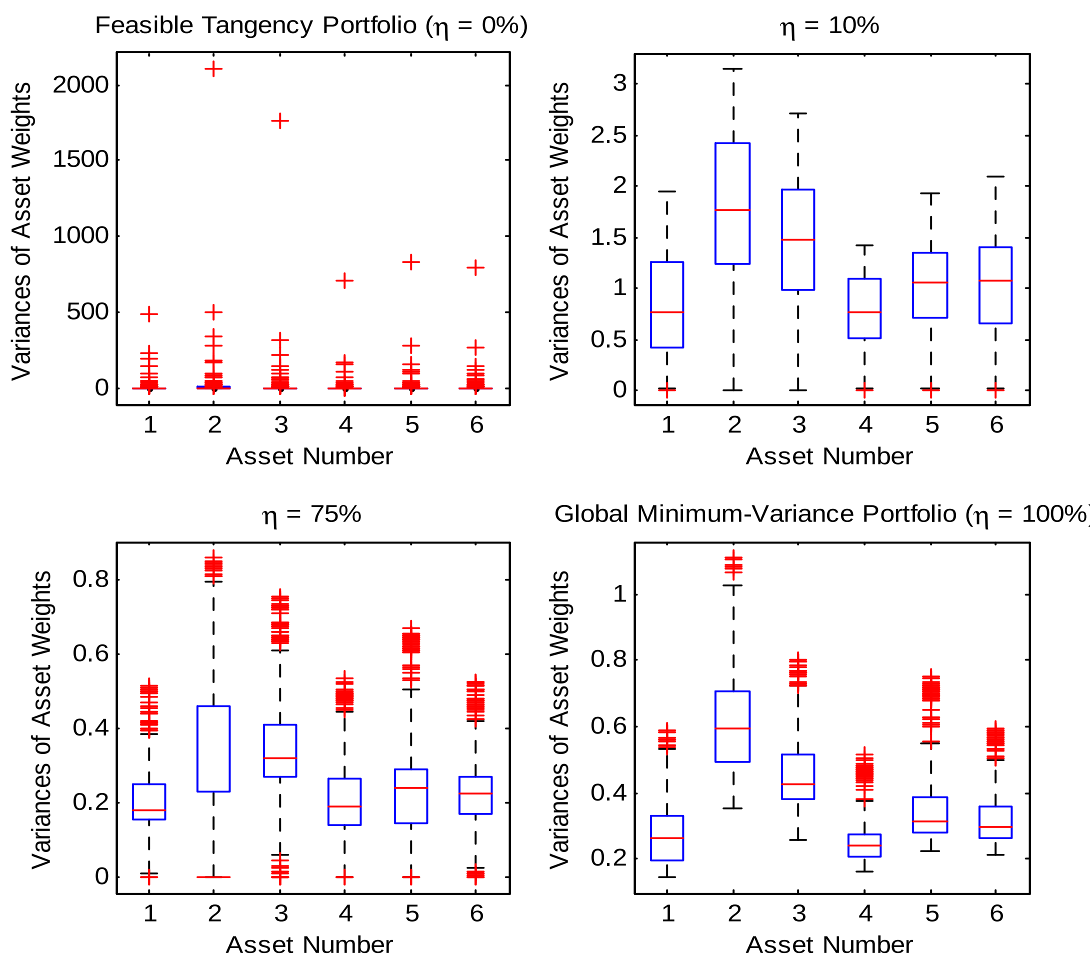

Figure 3: Evolution of the Sharpe ratio of the minimax mean-variance portfolios with respect to the investor's degree of skepticism η

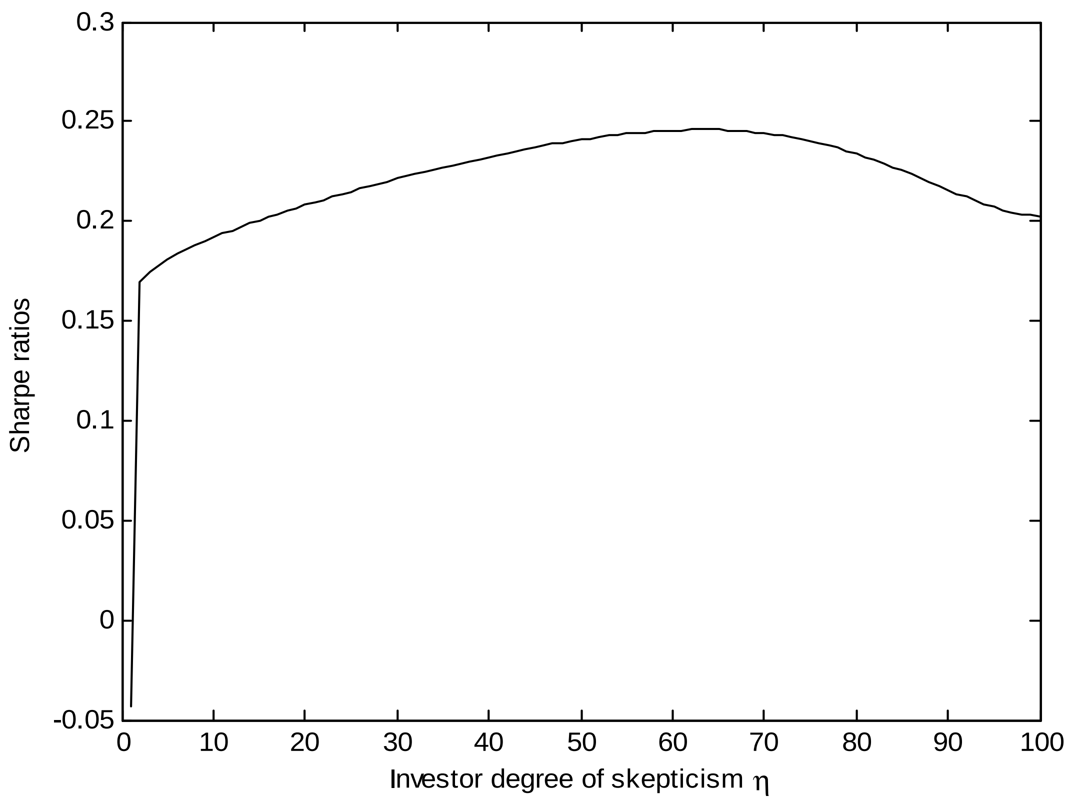

Table 1: Monthly out-of-sample statistics of the minimax Mean-Variance Portfolios with the Fama-French's 6 size and book-to-market assets

Strategy	Mean	Standard Deviation	Sharpe ratio
Feasible Mean-Variance	−0.1732	3.9550	−0.0438
Minimum-Variance	0.0081	0.0398	0.2025
		Minimax mean-variance	
$\eta = 0\%$	−0.1732	3.9550	−0.0438
$\eta = 10\%$	0.0175	0.1937	0.0903
$\eta = 20\%$	0.0166	0.0793	0.2095
$\eta = 30\%$	0.0158	0.0711	0.2225
$\eta = 40\%$	0.0150	0.0643	0.2334
$\eta = 50\%$	0.0141	0.0582	0.2416
$\eta = 60\%$	0.0130	0.0528	0.2457
$\eta = 70\%$	0.0117	0.0479	0.2436
$\eta = 80\%$	0.0102	0.0438	0.2325
$\eta = 90\%$	0.0087	0.0407	0.2139
$\eta = 99\%$	0.0081	0.0398	0.2026
$\eta = 100\%$	0.0081	0.0398	0.2025

Notes: This table provides for different values of the investor's degree of skepticism in estimating the expected asset returns, the monthly out-of-sample means, standard-deviations and Sharpe ratios of the minimax mean-variance portfolios. For comparison, the table also displays the same statistics for the feasible tangency portfolio and the global minimum-variance portfolio. The statistics are computed using a rolling-window procedure over the period July 1963 to September 2009.

Table 2: Comparison of the Minimax Mean-Variance Portfolios with Bayesian Mean-Variance Portfolios (Fama-French's 25 size and book-to-market assets)

Strategy	Mean	Standard Deviation	Sharpe ratio
Feasible Mean-Variance	−0.0455	1.3891	−0.0328
Bayes-Stein shrinkage	−0.0225	0.7434	−0.0303
Minimum-Variance	0.0083	0.0437	0.1903
		Bayesian Data-and-Model	
$\theta = 10\%$	−0.0379	1.7407	−0.0218
$\theta = 25\%$	−0.0213	3.1404	−0.0068
$\theta = 50\%$	0.0588	0.7537	0.0781
$\theta = 75\%$	0.0439	0.6018	0.0730
$\theta = 90\%$	0.0255	1.0034	0.0254
$\theta = 100\%$	0.0259	0.4582	0.0566
		Minimax Mean-Variance	
$\eta = 5\%$	0.0166	0.1449	0.1145
$\eta = 10\%$	0.0166	0.1293	0.1287
$\eta = 50\%$	0.0161	0.0815	0.1981
$\eta = 75\%$	0.0148	0.0618	0.2404
$\eta = 90\%$	0.0117	0.0484	0.2428
$\eta = 99\%$	0.0084	0.0436	0.1925

Notes: This table compares the monthly out-of-sample performances (means, standard-deviations and Sharpe ratios) of the minimax mean-variance Portfolios for different values of η the investor's degree of skepticism about the estimation of the expected returns with the Bayes-Stein shrinkage portfolio of Jorion (1986) and the Bayesian data-and-model portfolio of Pastor and Stambaugh (2000). For the latter portfolio, we display the performances for different values of θ the shrinkage parameter in equation (35). The statistics are computed using excess returns of the Fama-French's 25 size and book-to-market assets over the period July 1963 to September 2009.

Table 3: Comparison of Minimax Mean-Variance Portfolios using the Fama-French's 100 size and book-to-market assets

Strategy	Mean	Standard Deviation	Sharpe ratio
Minimum-Variance	0.0039	0.0879	0.0442
Equally-Weighted	0.0055	0.0527	0.1047
Panel A: Uncertainty about only the first moment			
$\eta = 5\%$	0.0097	0.3658	0.0265
$\eta = 10\%$	0.0093	0.3285	0.0284
$\eta = 50\%$	0.0092	0.2024	0.0454
$\eta = 75\%$	0.0108	0.1454	0.0744
$\eta = 99\%$	0.0038	0.0889	0.0426
Panel B: Uncertainty about the first two moments			
$\eta = 5\%$	0.0136	0.1478	0.0919
$\eta = 10\%$	0.0131	0.1361	0.0961
$\eta = 50\%$	0.0101	0.0966	0.1048
$\eta = 75\%$	0.0072	0.0790	0.0915
$\eta = 99\%$	0.0046	0.0561	0.0827

Notes: This table compares the monthly out-of-sample performances (means, standard-deviations and Sharpe ratios) of the two types of minimax mean-variance portfolios. The first (resp. second) type includes portfolios that take into account the investor's aversion in estimating the expected asset returns (resp. the expected asset returns and the covariance matrix). The statistics are computed using a rolling-window procedure over the period July 1963 to September 2009 with an estimation sample size equal to $n = 120$.

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