Differential games through viability theory: old and recent results

Pierre Cardaliaguet[1], Marc Quincampoix[1] and Patrick Saint-Pierre[2]

[1] Laboratoire de Mathématiques, Unité CNRS UMR6205, Université de Bretagne Occidentale, 6 avenue Le Gorgeu, 29200 Brest, FRANCE.

[2] Centre de Recherche Viabilité, Jeux, Contrôle, Université Paris IX-Dauphine, Place du Maréchal de Lattre de Tassigny, F-75775 Paris Cedex 16, FRANCE.

1 Introduction

Viability theory is interested in the study of state constrained problems for controlled systems or differential games. The terminology has been introduced by J.-P. Aubin [6] for controlled systems, but the story of the problem goes back to Poincaré, who introduced the notion of the "shadow of a set" for differential equations. The investigation of such questions for differential games has started with the pioneering works of Aubin [1], [2], [5], and has attracted since then an increasing interest. We have tried here to explain the main achievements of the theory, and to give some ideas of its new developments during the last years. We also provide a large—although not exhaustive—list of references.

For a two player differential game, the viability problem in a set K, with target C, consists, for the evader, to ensure that the state of the system reaches the target C before leaving the set of constraints K. On the contrary the pursuer tries to make the state of the system leave the set of constraints before it reaches the target. Such a problem is a typical "game of kind" in Isaacs' terminology. It is also closely connected with Krassovski and Subbotin's stable bridges [45], [68].

A first achievement of viability theory is the complete characterization of the solution of the viability game, as well as the numerical approximation of the solution ([23], [22]). It can be shown, in particular, that there is a partition of the set $K\backslash C$ starting from which one, and only one, of the players wins. The common boundary of the partition turn out to be a "semi-permeable barrier" in some Isaacs' sense [50], [24].

The second important result of viability theory is the fact that many zero-sum differential games can be expressed in terms of viability problems. These ideas are strongly inspired by Frankowska's works (see [37] for instance). As a consequence, such games have a value, and this value can be characterized in terms of minimal solution of some Hamilton-Jacobi equation (in the viscosity sense [16]). Furthermore, this value can be numerically approximated. For instance, the minimal time problem, in which the pursuer tries to capture the evader in a minimal time, can be treated in that way, without any controllability assumption on the dynamics, and even in problems with hard state constraints for the pursuer [25].

Application of these techniques are manifold. Without trying to be exhaustive, let us first cite the applications in the economic domains, in particular for dynamical economic models [9] and in finance [12], [13]. In engineering science, such a modelization has been used for the study of worst case design [33], [65], [66].

Recent extensions of the viability techniques for differential games have been obtained in several directions. The case of games with separate dynamics and hard state constraints for both players has been solved in [27] and [19] (see also [28] for the numerical approximation). This concerns typically the famous "Lion and Man" game, for which one can prove that there is a value, discontinuous in general.

Viability for stochastic control problems has attracted a lot of attention lately (see [8], [10], [11], [21], [32], [39], [43], [44], [70]). The generalization to differential game, which is not completely understood, has been investigated in [17].

Another very active research field is the domain of uncertain systems that one wants to optimize against the unknown disturbance. This leads to a Min-Max problem, which can naturally be interpreted as a differential game. The viability techniques for solving this problem have been investigated in [35], [47], [57] and [55].

Finally, one of the most challenging problems in the domain is the study of viability problems for hybrid differential games. Substantial contributions have appeared very recently (see [7], [15], [29], [30], [31], [61], [64]). In particular, the worst-case design for these games is now well understood. Note however that the question of the existence of a value in only partially solved. This very new theory has already appeared to be an extremely useful tool for modelization (see for instance [7], [20], [59], [63]).

The paper is organized in two parts. In the first part, we recall the basic theory of viability theory for differential games: the definitions of the discriminating domains and kernels, the alternative theorem, the barrier phenomenon, and the application to the minimal time problem. In the second part, we give an overview of the recent advances in the theory: games with hard state constraints, stochastic viability problems, worst case design and hybrid or impulsive games. We complete the paper by the presentation of several applications.

2 Basic viability theory for differential games

In this section, we present the main basic achievements of viability theory. The starting point is the viability game, in which one player wants that the state of the system remains in a given set, while the other player wants that the state of the system remains in that set forever. The solution of this game is clearly a set, or, more precisely, a partition of the set of constraints: in one part of this set one of the player wins, while his opponent obtains the victory in the other part.

The geometric characterization of this partition is the first goal of viability theory. It has been developed in [23, 24]. Then its numerical approximation has been the aim of intensive work: since this has been described and explained in details in the paper [25], we do not recall this subject here. Instead, we briefly show how to apply the main results on the viability game to treat a typical game of kind: the minimal time problem.

2.1 Statement of the viability problem

We investigate a differential game with dynamic described by the differential equation

$$\begin{cases} x'(t) = f(x(t), u(t), v(t)), \\ u(t) \in U, \ v(t) \in V \end{cases} \tag{1}$$

where $f : \boldsymbol{R}^N \times U \times V \to \boldsymbol{R}^N$, U and V being the control sets of the players. Throughout this section, we study the *viability game*. Beside the dynamics, this game has two data: $K \subset \boldsymbol{R}^N$ which is a closed set of constraints, and $\mathcal{E} \subset \boldsymbol{R}^N$ which is a closed evasion set. The first player—Ursula, playing with u—wants that the state of the system leaves K in finite time while avoiding $\mathcal{E}$. The goal of the second player—Victor, playing with v—is that the state of the system remains in K until reaching the evasion set $\mathcal{E}$.

In this game the main objects of investigation are the *victory domains* of the players, i.e., the set of initial positions starting from which a player can find a strategy which leads him or her to the victory.

We work here in the framework of the *nonanticipative strategies* (also called Varaiya-Roxin-Elliot-Kalton strategies). Let

$$\begin{cases} \mathcal{U} = \{u(\cdot) : [0, +\infty[\to U, \text{ measurable function } \} \\ \mathcal{V} = \{v(\cdot) : [0, +\infty[\to V, \text{ measurable function } \} \end{cases} \tag{2}$$

be the sets of time-measurable controls of respectively the first (Ursula) and the second (Victor) player.

Definition 1 (Nonanticipative strategies) *A map $\alpha : \mathcal{V} \to \mathcal{U}$ is a nonanticipative strategy (for Ursula) if it satisfies the following condition: For any $s \geq 0$, for any $v_1(\cdot)$ and $v_2(\cdot)$ belonging to $\mathcal{V}$ such that $v_1(\cdot)$ and $v_2(\cdot)$ coincide almost everywhere on $[0, s]$, the image $\alpha(v_1(\cdot))$ and $\alpha(v_2(\cdot))$ coincide almost everywhere on $[0, s]$.*
Nonanticipative strategies $\beta : \mathcal{U} \to \mathcal{V}$ (for Victor) are defined in the symmetric way.

Assume now that f is continuous and Lipschitz with respect to x. Then, for any $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, for any initial position x_0, there exists only one solution to (1). We denote this solution by $x[x_0, u(\cdot), v(\cdot)]$. In order to define the victory domains of the players, we introduce the following notation: if $C \subset \boldsymbol{R}^N$ is closed and $\varepsilon > 0$, we denote by $C + \varepsilon B$ the set

$$C + \varepsilon B = \{x \in \boldsymbol{R}^N \mid d_C(x) \leq \varepsilon\} \ ,$$

where $d_C(x)$ denotes the distance from x to C.

Definition 2 (Victory domains)
- Victor's victory domain is the set of initial positions $x_0 \in K$ for which Victor can find a nonanticipative strategy $\beta : \mathcal{U} \to \mathcal{V}$ such that for any time-measurable control $u(\cdot) \in \mathcal{U}$ played by Ursula, the solution $x[x_0, u(\cdot), \beta(u(\cdot))]$ remains in K until it reaches $\mathcal{E}$, or remains in K forever if it does not reach $\mathcal{E}$. Namely:

$$\begin{array}{l} \exists \tau \in [0, +\infty],\ \forall t \in [0, \tau),\ x[x_0, u(\cdot), \beta(u(\cdot))](t) \in K \\ \text{and, if } \tau < +\infty, \text{ then } x[x_0, u(\cdot), \beta(u(\cdot))](\tau) \in \mathcal{E}. \end{array}$$

- Ursula's victory domain is the set of initial positions $x_0 \in K$ for which Ursula can find a nonanticipative strategy $\alpha : \mathcal{V} \to \mathcal{U}$, positive ε and T such that, for any $v(\cdot) \in \mathcal{V}$ played by Victor, the solution $x[x_0, \alpha(v(\cdot)), v(\cdot)]$ leaves $K + \varepsilon B$ before reaching the set $\mathcal{E} + \varepsilon B$ and before T.
Namely

$$\begin{array}{l} \exists \tau \leq T,\ d_K(x[x_0, \alpha(v(\cdot)), v(\cdot)](\tau)) \geq \varepsilon \\ \text{and } \forall t \in [0, \tau],\ x[x_0, \alpha(v(\cdot)), v(\cdot)](t) \notin \mathcal{E} + \varepsilon B. \end{array}$$

In the definition of Victor's victory domain, the solution has to remain in the constraint until reaching the evasion set.
In the definition of Ursula's victory domain, the solution has not only to leave the constraint while avoiding the evasion set, but also to leave it "sufficiently" (say to leave $K + \varepsilon B$) while remaining "sufficiently far" from the evasion set (say at a distance not smaller than ε), and in a finite time (say not larger than T). Both ε and T have to be independent on Victor's response $v(\cdot)$.

Assumptions on f : In the sequel, we need the following assumptions

$$\begin{cases} i) & U \text{ and } V \text{ are compact} \\ ii) & f : \boldsymbol{R}^N \times U \times V \to \boldsymbol{R}^N \text{ is continuous,} \\ iii) & f \text{ is } \ell\text{-Lipschitz w.r.t. } x, \\ iv) & \forall x \in \boldsymbol{R}^N,\ \forall u \in U,\ \text{the set } \bigcup_{v \in V} \{f(x, u, v)\} \text{ is convex.} \end{cases} \tag{3}$$

A sufficient condition for assumption (3-*iv*) to be satisfied is that V is convex and f is affine with respect to v. We also assume that Isaacs' condition holds:

$$\forall (x, p) \in \boldsymbol{R}^{2N},\ \sup_{u \in U} \inf_{v \in V} < f(x, u, v), p >= \inf_{v \in V} \sup_{u \in U} < f(x, u, v), p > \tag{4}$$

2.2 Discriminating domains

We introduce in this part a class of sets defined by geometric conditions and justify this definition by characterizing it in terms of trajectories.

Definition 3 (Proximal normal) *Let D be a closed subset of $\boldsymbol{R}^N$ and $x \in D$. A vector $p \in \boldsymbol{R}^N$ is a proximal normal to D at x if $d_D(x + p) = |p|$.*
We denote by $\mathcal{NP}_D(x)$ the set of proximal normals to D at x.

This definition means that the ball centered at $x + p$ and of radius $|p|$ is tangent to D at x. Note also that $\mathcal{NP}_D(x)$ is nonempty because $0 \in \mathcal{NP}_D(x)$.

Definition 4 (Discriminating Domains) *Let $H : \boldsymbol{R}^N \times \boldsymbol{R}^N \to \boldsymbol{R}$. A closed set $D \subset X$ is a discriminating domain for H if:*

$$\forall x \in D,\ \forall p \in \mathcal{NP}_D(x),\ H(x, p) \leq 0\ .$$

We are mainly interested here in the following H:

$$H(x, p) := \begin{cases} \sup_u \inf_v < f(x, u, v), p > & \text{if } x \notin \mathcal{E} \\ \min\{\sup_u \inf_v < f(x, u, v), p >; 0\} & \text{otherwise} \end{cases} \tag{5}$$

Theorem 5 (Interpretation of discriminating domains) *We assume that f satisfies (3) and that Isaacs'condition (4) holds. Let $D \subset \boldsymbol{R}^N$ be a closed set . The following statements are equivalent:*

- *D is a discriminating domain for H defined by (5),*
- *for any initial position $x_0 \in D$, there is a nonanticipative strategy $\beta : \mathcal{U} \to \mathcal{V}$ such that the solution $x[x_0, u, \beta(u)]$ remains in D as long as it has not reached $\mathcal{E}$,*

- *for any initial position $x_0 \in D$, for any positive ε and T, for any nonanticipative strategy $\alpha : \mathcal{V} \to \mathcal{U}$, there is some control $v \in \mathcal{V}$ such that the solution $x[x_0, u, v]$ remains in $K + \varepsilon B$ on the time interval $[0, T]$ until it reaches $\mathcal{E} + \varepsilon B$.*

Remark : If we go back to our viability game, with constraints K and target $\mathcal{E}$, and if a discriminating domain D is contained in K, then D lies in Victor's victory domain, and has an empty intersection with Ursula's victory domain. Indeed, starting from D, Victor can ensure the state of the system to remain in D, hence in K.

2.3 The alternative theorem and the characterization of the victory domains

We now characterize the victory domains of the game. For this we have to introduce the notion of discriminating kernel of the constraint set K:

Theorem 6 (Discriminating kernel) *Let $H : \boldsymbol{R}^N \times \boldsymbol{R}^N \to \boldsymbol{R}$ be a lower semicontinuous map. Any closed subset K of $\boldsymbol{R}^N$ contains a largest (closed) discriminating domain for H. This set is called the discriminating kernel of K for H and is denoted by $Disc_H(K)$.*

Remarks :

- Any discriminating domain for H contained in a closed K is contained in $Disc_H(K)$. Moreover, $Disc_H(K)$ is itself a discriminating domain for H, and $Disc_H(K)$ may be empty if K does not contain any discriminating domain for H.
- One of the most interesting aspect of discriminating kernels consists in the fact that there exists algorithms for approximating these sets. We refer the reader to [25] for a survey of numerical methods.

The main achievement of viability theory for differential games is the following characterization of the victory domains:

Theorem 7 (Characterization of the victory domains) *Assume that f satisfies (3) and that Isaacs' condition (4) holds. Let the constraint set K be closed and let $\mathcal{E}$ be the evasion set. Recall that the hamiltonian H of the system is defined by (5). Then*

- *Victor's victory domain is equal to $Disc_H(K)$,*
- *Ursula's victory domain is equal to $K \backslash Disc_H(K)$.*

In particular, the victory domains of the two players form a partition of K. A similar Alternative Theorem have been obtained by Krasovskii & Subbotin in the framework of the positional strategies [45]. In fact, the discriminating domains are very close to Krasovskii & Subjoin's stable bridges, while the discriminating kernel is related with the maximal stable bridges. What singularizes the viability approach is the place taken by the geometric characterization of the victory domains, which plays an important role in the sequel.

Isaacs' semipermeable barriers

Under the assumption that the common boundary of the victory domains is smooth, Isaacs proved that it satisfies the geometric equation, now known as *Isaacs' equation*:

$$H(x, \nu_x) \;=\; 0 \qquad \text{for any } x \text{ on the boundary of the victory domain.} \tag{6}$$

where ν_x stands for the outward normal at x to Victor's victory domain. As a consequence, each player can prevent the state of the system to cross the boundary in one direction. Such a boundary is called *a semipermeable barrier*.

The boundary $\partial Disc_H(K)$ of the discriminating domain $Disc_H(K)$ actually enjoys this property in a weak sense. This phenomenon was first noticed in [50] for control problems and then extended to differential games in [24].

For simplicity we assume here that there is no evasion set: $\mathcal{E} = \emptyset$.

Proposition 8 (Geometric point of view) *Let x belong to $\partial Disc_H(K) \backslash \partial K$. Then,*

$$(i)\;\; H(x,p) \le 0,\;\; \forall p \in \mathcal{NP}_{Disc_H(K)}(x) \qquad \text{and} \qquad (ii)\;\; H(x,-p) \ge 0 \;\; \forall p \in \mathcal{NP}_{\overline{K \backslash Disc_H(K)}}(x)\,.$$

Remark : Since proximal normals are in fact *outward* normals, a proximal normal to $\overline{K \backslash Disc_H(K)}$ is in fact an inward normal to $Disc_H(K)$. Hence putting (i) and (ii) together is a weak formulation of Isaacs'equation (6).

Proposition 9 (Dynamic point of view) *Let x_0 belong to $\partial Disc_H(K)$ but not to ∂K. Then*

- *there is a nonanticipative strategy $\beta : \mathcal{U} \to \mathcal{V}$ for Victor such that*

$$x[x_0, u, \beta(u)](t) \in Disc_H(K) \qquad \forall t \geq 0,\ \forall u \in \mathcal{U}\ ,$$

- *there is a nonanticipative strategy $\alpha : \mathcal{V} \to \mathcal{U}$ for Ursula and a time $T > 0$ such that*

$$x[x_0, \alpha(v), v](t) \in \overline{K \backslash Disc_H(K)} \qquad \forall t \in [0, T],\ \forall v \in \mathcal{V}\ ,$$

Remark 10 Roughly speaking, the "solution" $x[x_0, \alpha, \beta]$ remains on $\partial Disc_H(K)$ on $[0, T]$.

Although viability for differential games are now quite well understood, there remains several important and interesting open questions on the subject:

- What kind of regularity can be expected for the boundary of $Disc_H(K)$?
- There is a natural way to define the solution $x[x_0, \alpha, \beta]$ in Remark 10. Is there a maximum principle satisfied by this solution, even in a weak sense ?
- In general, $Disc_H(K)$ is not the unique set satisfying Isaacs equation. When is there uniqueness ?

2.4 Application to the minimal time problem

Most zero-sum differential games can be expressed in terms of viability game. We just show how this principle applies for the minimal time problem.

In the minimal time problem, the dynamics of the game is still given by (1). Victor—playing with v—is now the pursuer: he aims at reaching a given target C as fast as possible. Ursula—playing with u—is now the evader: she wants to avoid the target C as long as possible. We assume here that the target $C \subset \boldsymbol{R}^N$ is closed. If $x(\cdot) : [0, +\infty) \to \boldsymbol{R}^N$ is a continuous trajectory, the first time $x(\cdot)$ reaches C is

$$\vartheta_C(x(\cdot)) = \inf\{t \geq 0 \mid x(t) \in C\}\ ,$$

with convention $\inf(\emptyset) = +\infty$.

The value functions of the game are defined as follows: the *lower value function* is given by

$$\vartheta^\flat(x_0) = \inf_\beta \sup_{u \in \mathcal{U}} \vartheta_C(x[x_0, u, \beta(u)]) \qquad \forall x_0 \notin C\ ,$$

while the *upper value function* is

$$\vartheta^\sharp(x_0) = \lim_{\epsilon \to 0^+} \left(\sup_\alpha \inf_{v \in \mathcal{V}} \vartheta_{C+\epsilon B}(x[x_0, \alpha(v), v]) \right) \qquad \forall x_0 \notin C\ .$$

Remarks :

1. In general the value functions are discontinuous.
2. The definition of the upper value is slightly more complicated than the lower one. Unfortunately, this formulation is in general necessary for the game to have a value—unless some controllability condition at the boundary of the target holds.

We now explain how to transform the minimal time problem into a viability game in $\boldsymbol{R}^{N+1}$. For this we define the *extended dynamics* $\widetilde{f} : \boldsymbol{R}^{N+1} \times U \times V \to \boldsymbol{R}^{N+1}$ by

$$\tilde{f}(\rho, x, u, v) = \{-1\} \times \{f(x, u, v)\}$$

and

$$\widetilde{K} = \boldsymbol{R}^+ \times \boldsymbol{R}^N \qquad \text{and} \qquad \widetilde{C} = \{0\} \times C\ .$$

For an initial data $\widetilde{x}_0 = (\rho_0, x_0)$, we denote by $\widetilde{x}[\widetilde{x}_0, u, v]$ the solution to

$$\begin{cases} \widetilde{x}' = \widetilde{f}(\widetilde{x}, u, v) \\ \widetilde{x}(0) = \widetilde{x}_0 \end{cases}$$

Remark : $\rho(t) = \rho_0 - t$, the first component of $\widetilde{x}(t)$, is the running cost of the problem.

The viability game associated to the minimal time problem is the following:

- Victor becomes *the evader*: he wants that $\widetilde{x}(t)$ reaches the target $\widetilde{C}$ before leaving the constraints $\widetilde{K} := \boldsymbol{R}^+ \times \boldsymbol{R}^N$.
- Ursula becomes *the pursuer*: she wants that $\widetilde{x}(t)$ leaves $\widetilde{K}$ before reaching the target $\widetilde{C}$.

As seen in the previous parts, there is an alternative theorem for this game, and we shall use it to prove the existence of a value for the minimal time problem. The link between the minimal time and the viability game is the following:

Proposition 11 *Let $\widetilde{H} : \boldsymbol{R}^{N+1} \times \boldsymbol{R}^{N+1} \to \boldsymbol{R}$ be defined by*

$$\widetilde{H}(\widetilde{x}, p_x, p_\rho) = \begin{cases} \inf_{v\in V} \sup_{u\in U} < f(x,u,v), p_x > -p_\rho & \text{if } \widetilde{x} \notin \widetilde{C} \\ \min\{0, \inf_{v\in V} \sup_{u\in U} < f(x,u,v), p_x > -p_\rho\} & \text{otherwise} \end{cases}$$

Then we have

$$\mathcal{E}pi(\vartheta^\sharp) \;=\; Disc_{\widetilde{H}}(\widetilde{K}) \;=\; \mathcal{E}pi(\vartheta^\flat)$$

where $\mathcal{E}pi(\phi) = \{(x,\rho) \in \boldsymbol{R}^{N+1} \to \boldsymbol{R} \mid \rho \geq \phi(x)\}$ denote the epigraph of a function ϕ.

Idea of the proof : Let $\widetilde{x} = \widetilde{x}[\widetilde{x}_0, u, v] = (\rho(t), x(t))$ some solution. We note that $\rho(t) = \rho_0 - t$ because $\widetilde{f} = \{-1\} \times \{f\}$. Hence $\widetilde{x}(t)$ belongs to $\widetilde{K} := \boldsymbol{R}^+ \times \boldsymbol{R}^N$ if and only if $\rho(t) = \rho_0 - t \geq 0$.

If (ρ_0, x_0) belongs to Victor's victory domain, then Victor can ensure that $\widetilde{x}(\cdot)$ reaches $\widetilde{C}$ before leaving $\widetilde{K}$. This means that $x(\cdot)$ reaches C before ρ_0. Hence

$$\rho_0 \geq \vartheta_C(x(\cdot)) \geq \vartheta^\sharp(x_0)\ .$$

This shows that Victor's victory domain is contained in the epigraph of $\vartheta^\sharp$. The other inclusions can be proved similarly, by using the different characterizations of Victor's and Ursula's victory domains. **Q.E.D.**

As a consequence, we have the following result of existence of a value for the game

Theorem 12 *If assumption (3) and Isaacs condition (4) hold, we have*

$$\vartheta^\flat(x_0) = \vartheta^\sharp(x_0) \qquad \forall x_0 \notin C\ .$$

Furthermore, both functions are the minimal nonnegative viscosity supersolutions to

$$\sup_u \inf_v < f(x,u,v), D\vartheta >= 1 \qquad \text{in } \boldsymbol{R}^N \backslash C\ .$$

Remark : The last statement is just a way to rewrite the fact that the epigraph of $\vartheta^\flat = \vartheta^\sharp$ is the largest discriminating domain contained in $\boldsymbol{R}^N \times \boldsymbol{R}_+$. An introduction to viscosity solutions with a link to differential games can be found in [16].

3 Recent results on viability theory for differential games

3.1 Differential games with state constraints

Viability theory for differential games can be extended to differential games with separate dynamics and state constraints on for both players. A typical example of such a game is the famous "Lion and Man" game, in which both lion and man have to remain in the arena. The results of this part can be found in [19, 27]. The numerical approximations of state constrained problems is the aim of the note [28].

We assume throughout this part that the game has separate dynamics: the first player, Ursula, playing with u, controls a first system

$$\begin{cases} y'(t) = g(y(t), u(t)), & u(t) \in U \\ y(0) = y_0 \in K_U \end{cases} \tag{7}$$

and has to ensure the state constraint $y(t) \in K_U$ to be fulfilled for any $t \geq 0$, where K_U is a closed subset of $\boldsymbol{R}^l$. On the other hand the second player, playing with v, controls a second system

$$\begin{cases} z'(t) = h(z(t), v(t)), & v(t) \in V \\ z(0) = z_0 \in K_V \end{cases} \tag{8}$$

and has to ensure the state constraint $z(t) \in K_V$ for any $t \geq 0$, where K_V is a closed subset of $\mathbf{R}^m$. We denote by $y[y_0, u]$ the solution to (7), by $z[z_0, v]$ the solution to (8). We set $N = l + m$.

We study again the viability game: Let $K \subset K_U \times K_V$ be a closed set of constrained and $\mathcal{E} \subset K_U \times K_V$ be a closed evasion set. Ursula wants that the state of the system (y, z) leaves K before reaching $\mathcal{E}$, while, on the contrary, Victor wants that (y, z) remains in K as long as it has not reached $\mathcal{E}$. Our aim is again to characterize the victory domains of the players.

For doing so, let us set $x = (y, z)$, $f = (g, h)$. Throughout this part, we assume that f, K_U and K_V satisfy the following regularity conditions

$$\begin{cases} (i) & U \text{ and } V \text{ are compact subsets of some finite} \\ & \text{dimensional spaces} \\ (ii) & f : \mathbf{R}^N \times U \times V \to \mathbf{R}^N \text{ is continuous and} \\ & \text{Lipschitz continuous (with Lipschitz constant } M) \\ & \text{with respect to } x \in \mathbf{R}^N; \\ (iii) & \bigcup_u f(x,u,v) \text{ and } \bigcup_v f(x,u,v) \text{ are convex for any } x. \\ (iv) & K_U = \{y \in \mathbf{R}^l,\ \phi_U(y) \leq 0\} \text{ with } \phi_U \in \mathcal{C}^2(\mathbf{R}^l;\mathbf{R}), \\ & \nabla\phi_U(y) \neq 0 \text{ if } \phi_U(y) = 0 \\ (v) & K_V = \{z \in \mathbf{R}^m,\ \phi_V(z) \leq 0\} \text{ with } \phi_V \in \mathcal{C}^2(\mathbf{R}^m;\mathbf{R}), \\ & \nabla\phi_V(z) \neq 0 \text{ if } \phi_V(z) = 0 \\ (vi) & \forall y \in \partial K_U,\ \exists u \in U \text{ such that } < \nabla\phi_U(y), g(y,u) > < 0 \\ (viii) & \forall z \in \partial K_V,\ \exists v \in V \text{ such that } < \nabla\phi_V(z), h(z,v) > < 0 \end{cases} \tag{9}$$

We note that Isaacs'assumption (4) always holds.

We now introduce the notion of *admissible controls and strategies.* For an initial position $(y_0, z_0) \in K_U \times K_V$,

$$\mathcal{U}(y_0) = \{u(\cdot) : [0, +\infty) \to U \text{ measurable} \mid y[y_0, u(\cdot)](t) \in K_U \quad \forall t \geq 0\}$$

and

$$\mathcal{V}(z_0) = \{v(\cdot) : [0, +\infty) \to V \text{ measurable} \mid z[z_0, v(\cdot)](t) \in K_V\ \forall t \geq 0\}\,.$$

Under the assumptions (9), it is well known that there are admissible controls for any initial position: namely,

$$\mathcal{U}(y_0) \neq \emptyset \quad \text{and} \quad \mathcal{V}(z_0) \neq \emptyset \qquad \forall x_0 = (y_0, z_0) \in K_U \times K_V\ .$$

For any $y \in K_U$, we set

$$U(y) = U \text{ if } y \in Int(K_U)\ , \qquad U(y) = \{u \in U \mid g(y,u) \in T_{K_U}(y)\} \text{ if } y \in \partial K_U\ ,$$

where $T_{K_U}(y)$ is the tangent half-space to the set K_U at $y \in \partial K_U$.

Remark 13 We note for later use that, under assumptions (9), the set-valued map $y \rightsquigarrow g(y, U(y))$ is lower semicontinuous with convex compact values (for definitions and properties, see [3]).

Both players now play *admissible non-anticipative strategies*: A nonanticipative strategy α for Ursula is admissible at the point $x_0 = (y_0, z_0) \in K_U \times K_V$ if $\alpha : \mathcal{V}(z_0) \to \mathcal{U}(y_0)$. Admissible nonanticipative strategies β for the second player Victor are symmetrically defined. For any point $x_0 \in K_U \times K_V$ we denote by $S_U(x_0)$ and by $S_V(x_0)$ the sets of the non-anticipative strategies for Ursula and Victor respectively.

The victory domains of the players are defined as in Definition 2, but the players now play *admissible* controls and nonanticipative strategies.

As before the characterization of the victory domains uses the notion of discriminating domains and kernels. However, because of the state constraints, the definition of the hamiltonian has to be adapted: for $x = (y, z) \in K_U \times K_V$ and $p \in \mathbf{R}^N$,

$$H(x,p) := \begin{cases} \sup_{u \in U(y)} \inf_{v \in V} < f(x,u,v), p > & \text{if } x \notin \mathcal{E} \\ \min\{\sup_{u \in U(y)} \inf_{v \in V} < f(x,u,v), p >,\ 0\} & \text{otherwise} \end{cases} \tag{10}$$

We note that H is lower semicontinuous, because of assumption (9) and Remark 13. Therefore the notion of discriminating kernel of K is well-defined.

As before, our main result is the characterization of the victory domains:

Theorem 14 (Characterization of the victory domains) *Assume that* $f = (g, h)$*,* K_U *and* K_V *satisfy (9). Let* $K \subset K_U \times K_V$ *be the closed constraint and* $\mathcal{E} \subset K_U \times K_V$ *be the closed evasion set. Let* H *be defined by (10). Then*

- *Victor's victory domain is equal to* $Disc_H(K)$*,*
- *Ursula's victory domain is equal to* $K \backslash Disc_H(K)$*.*

The main difficulty for proving the above theorem is the presence of state-constraints for both players. To overcome this difficulty, we use repetitively a technical lemma that we state here because of its usefulness for games with state constraints:

Lemma 15 ([19]) *Assume that conditions (9) are satisfied. For any* $R > 0$ *there exists* $C_0 = C_0(R) > 0$ *such that, for any* y_0*,* $y_1 \in K_U$ *with* $|y_0|$*,* $|y_1| \leq R$*, there is a nonanticipative strategy* $\sigma : \mathcal{U}(y_0) \longrightarrow \mathcal{U}(y_1)$ *with the following property: for any* $u_0(\cdot) \in \mathcal{U}(y_0)$

$$|y_0(t) - y_1(t)| + \int_{t_0}^{t} |u_0(s) - \sigma(u_0(\cdot))(s)| \; ds \; \leq \; C_0 |y_0 - y_1| e^{C_0(t-t_0)}$$

where we have set for simplicity $y_0 = y[y_0, u_0(\cdot)]$ *and* $y_1 = y[y_1, \sigma(u_0(\cdot))](t)$*.*

In other words, any admissible control u_0 for y_0 can be approximated by an admissible control $u_1 = \sigma(u_0(\cdot))$ for y_1, in a nonanticipative way.

As in the standard viability for differential games described in the first part, the alternative result for games with state constraints has several counterparts: we have applied it in [28] to prove that the minimal time problem with state constraints has a value, while [19] deals with the case of Bolza problem. Numerical approximation of such games is studied in [27].

3.2 Viability for stochastic differential games

There are some extensions of viability theory to stochastic controlled systems. For differential games, the main question is the following: let (X_t) be a controlled stochastic process driven by a s.d.e

$$dX_t = f(X_t, u_t, v_t)dt + \sigma(X_t, u_t, v_t)dW_t \tag{11}$$

and let K be a closed set of constraints. The viability game consists in finding the set of initial position $x_0 \in K$ from which Victor can ensure that $X_t \in K$ *almost surely* for any $t \geq 0$.

For controlled problems, the pioneering works are: Gautier & Thibault [39], Aubin & Da Prato [4], Buckdahn, Peng, Quincampoix & Rainer [21] and Aubin, Da Prato & Frankowska [11], to cite only few. Viability has not been thoroughly investigated for differential games. Some partial characterization results have been obtained by Bardi and Jensen in [17]. This is what we describe now.

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ be a complete stochastic basis endowed with a $k-$dimensional standard $(\mathcal{F}_t)-$Brownian motion (W_t). An admissible control for Ursula is an $(\mathcal{F}_t)-$progressively measurable process $u : [0, +\infty) \to U$. We denote by $\mathcal{U}$ the set of admissible controls. In the same way, we denote by $\mathcal{V}$ the set of admissible controls for Victor.

Throughout this part, we assume that $f : \boldsymbol{R}^N \times U \times V \to \boldsymbol{R}^N$ and $\sigma : \boldsymbol{R}^N \times U \times V \to \boldsymbol{R}^{N \times k}$ are continuous, Lipschitz continuous w.r.t. the x-variable and that U and V are compact subsets of some finite dimensional space. Under this assumption, for any initial position $x_0 \in \boldsymbol{R}^N$ and any admissible controls $u \in \mathcal{U}$ and $v \in \mathcal{V}$, equation (11) has a unique solution denoted by $X_t^{x_0,u,v}$.

Following [36], we define admissible strategies for Victor as follows: an admissible strategy for Victor is a mapping $\beta : \mathcal{U} \to \mathcal{V}$ such that, if $u_1 \equiv u_2$ a.s. on some time interval $[0, t]$, then $\beta(u_1) \equiv \beta(u_2)$ a.s. on $[0, t]$. We denote by S_V the strategies for Victor.

We now introduce the notion for second order proximal set: if $x \in \partial D$, a pair $(p, X) \in \boldsymbol{R}^N \times \boldsymbol{R}^{N \times N}$ belongs to the second order proximal set $\mathcal{NP}^2_D(x)$ to D at x if X is symmetric and if, for any $y \in D$, $p.(y - x) + \frac{1}{2}(y - x).Y(y - x) \leq o(|y - x|^2)$. We note that, if $(p, X) \in \mathcal{NP}^2_D(x)$, then p is a proximal normal to D at x.

Recall that $BUC(\boldsymbol{R}^N)$ is the set of bounded uniformly continuous maps on $\boldsymbol{R}^N$.

Theorem 16 *Let $D \subset \boldsymbol{R}^N$ be a closed set. The following statement are equivalent:*

- *for all $\ell \in BUC(\boldsymbol{R}^N)$, $\ell \geq 0$, $\ell \equiv 0$ in D, and for all $\lambda > 0$,*

$$\inf_{\beta \in S_V} \sup_{u \in \mathcal{U}} E \int_0^{+\infty} e^{-\lambda t} \ell(X_t^{x_0,u,\beta(u)}) dt = 0 \qquad \forall x_0 \in D$$

- *$\forall x \in \partial D$, $\forall (p, X) \in \mathcal{NP}_D^2(x)$,*

$$\inf_{v \in V} \sup_{u \in U} \left\{ < f(x,u,v), p > + \frac{1}{2} Tr \left(\sigma(x,u,v) \sigma^T(x,u,v) X \right) \right\} \leq 0 \,. \tag{12}$$

The first condition can be understood as a condition of approximate viability for the differential game. The second condition is purely of geometric nature.

Using the ideas of [17] it is not difficult to define the notion of discriminating kernel of a closed set K as the largest closed subset of K satisfying the geometric condition (12), and to characterize it as Victor's victory domain *for an approximate* viability problem.

One should also expect that the two above equivalent conditions in the Theorem are equivalent to the following third one (under suitable convexity assumption):

- for all $x_0 \in D$, there exists some $\beta \in S_V$ such that, for any admissible control u, we have $X_t^{x_0,u,\beta(u)} \in D$ a.s. for any $t \geq 0$.

However this has not been proved up to now.

4 Worst case design

Viability theory can also be used to control systems with imperfectly known uncertainty [55].

We consider the system

$$x'(t) = f(x(t), u(t), y(t), v(t)), \quad x(0) = e \in E_0, \tag{13}$$

where $x \in \boldsymbol{R}^n$ is the state, $u \in U$ is the control, $y \in Y$ and $v \in V(y)$ are disturbances (U, Y and $V(y)$ are given subsets of finite dimensional spaces, $E_0 \subset \boldsymbol{R}^n$). The main concern of the section is the optimal control problem where the controller wants to minimize (by choosing u) the cost

$$g(T, x(T)) \tag{14}$$

against the worst case of disturbances y and v and initial state $e \in E_0$. We distinguish two types of disturbance:

– *observable uncertainty* y, for which the current realization $y(t) \in Y$ becomes known to the controller;

– *unobservable uncertainty* $v \in V(y)$, for which the realization of $v(t) \in V(y(t))$ remains unknown;

Thus we consider a Min-Max problem or a differential game where the second player wants to maximize (by choosing e, y and v) the cost (14) while the first player—the controller—wants to minimize it (by choosing u). This specific aspect of the information available to the controller implies that the control u should be considered in a feedback form which may depend on the current and the past values of y, but not on v.

For every given control $u(\cdot)$ and observable uncertainty $y(\cdot)$, the unobservable uncertainty $v(\cdot)$ gives rise to a differential inclusion

$$x'(t) \in f(x, u(t), y(t), V(y(t))), \quad x(0) \in E_0, \tag{15}$$

whose solution is a time depending tube providing the deterministic estimation of the trajectory. Notice that a (set-valued) tube starting from E_0 is involved even in the case of precisely known initial state $x(0)$, if unobservable uncertainty is present.

Let us formulate the problem in a more precise way. Let $\mathcal{U}_{[t,\theta]}$ be the set of admissible control functions on the interval $[t, \theta]$, that is, the measurable functions with values in U. Similarly, $\mathcal{Y}_{[t,\theta]}$ denotes the set of all measurable selections of Y on $[t, \theta]$, and $\mathcal{V}_{[t,\theta]}(y(\cdot))$ denotes the set of all measurable selections of the mapping $V(y(\cdot))$ (for a given $y(\cdot)$) on the same interval. The following suppositions holds true

Condition C.

C.1. U, Y and $\bar{V}$ are compact subsets of finite dimensional vector spaces, U is convex, the mapping $y \longrightarrow V(y) \subset \bar{V}$ is compact valued and Lipschitz continuous.

C.2. The function $f : \boldsymbol{R}^n \times U \times Y \times \bar{V} \mapsto \boldsymbol{R}^n$ has the form

$$f(x,u,y,v) = f_0(x,y,v) + B(x,y)u,$$

where f_0 and B are continuous, locally Lipschitz in x uniformly with respect to the other variables. The sets $f_0(x,y,V(y))$ are convex. f has linear growth with respect to x, uniformly in u,y,v.

These assumptions will imply that for every $t \in [0,T)$, $e \in \boldsymbol{R}^n$, $u \in \mathcal{U}_{[t,T]}$, $y \in \mathcal{Y}_{[t,T]}$, and $v \in \mathcal{V}_{[t,T]}(y)$ system (13) has a unique solution on $[t,T]$ starting from e, denoted by $x[t,e;u,y,v](\cdot)$.

The optimal control for initial time t and initial compact set E is sought as a *nonanticipative strategy*, $\alpha : \mathcal{Y}_{[t,T]} \mapsto \mathcal{U}_{[t,T]}$. The guaranteed result obtained by using the strategy α for initial data (t,E) is

$$I(t,E;\alpha) := \sup\{g(T, x[t,e;\alpha(y),y,v](T)),\ e \in E,\ y \in \mathcal{Y}_{[t,T]},\ v \in \mathcal{V}_{[t,T]}(y)\}.$$

Then

$$I(t,E) := \inf_{\alpha} I(t,E;\alpha) \tag{16}$$

is the minimal guaranteed (lower) value that can be achieved starting from the set E at time t.

We denote by $X_{u,y}[t,E](\cdot)$ the reachable set, namely the set of all solutions to (15) evaluated at time t for all possible initial condition $x_0 \in E$. This is called the *solution tube* of (15) in the set $\text{comp}(\boldsymbol{R}^n)$ of all compact subsets of R^n. Moreover, for a compact set $Z \subset \boldsymbol{R}^n$ we define $G : \text{comp}(\boldsymbol{R}^n) \mapsto \boldsymbol{R}$ as

$$G(T,Z) = \sup_{z \in Z} g(T,z).$$

Then obviously definition (16) is equivalent to

$$I(t,E) = \inf_{\alpha \in \mathcal{A}_{[t,T]}} \sup_{y \in \mathcal{Y}_{[t,T]}} G(T, X_{\alpha(y),y}[t,E](T)). \tag{17}$$

In this formulation of the original problem there is no unobservable uncertainty. We passed to a problem with complete information (here y is an observable disturbance), but over the solution tubes to differential inclusion (15).

Because of the complexity of the so obtained problem we can restrict the consideration to the solution tubes of (15) in a given collection of sets $\mathcal{E}$ instead of the whole $\text{comp}(\boldsymbol{R}^n)$. Namely we are interested by tubes which gives an outer estimation of the unknown set of evolutions in a specific class of subsets (boxes ellipsoids, polyhedral sets...). By doing so we avoid to deal with the geometry of the reachable sets, which could be rather complicated. Thus we come up with the more general problem, formulated below in the case of a target that determines the termination time. We suppose in addition

C.3. The collection $\mathcal{E}$ satisfies conditions:

Condition B.1. The collection $\mathcal{E}$ consists of nonempty compact sets and is closed in the Hausdorff metric. For every compact Z there is some $E \in \mathcal{E}$ containing Z.

Condition B.2. There exists a constant $L_{\mathcal{E}}$ such that for each $\varepsilon > 0$ and each $E \in \mathcal{E}$ there exists $E' \in \mathcal{E}$ for which $E + \varepsilon B \subset E' \subset E + \varepsilon L_{\mathcal{E}} B$.

Condition B.3. For every $Z \in \text{comp}(\boldsymbol{R}^n)$ there is a unique minimal element of $\mathcal{E}$ containing Z.

The following dynamic programming principle adapts standard arguments

Proposition 17 *Under conditions* **C.1–C.3***, for every $E \in \mathcal{E}$, $t \in [0,T)$ and $s \in (t,T]$*

$$I_{\mathcal{E}}(t,E) = \inf_{\alpha \in \mathcal{A}_{[t,s]}} \sup_{y \in \mathcal{Y}_{[t,s]}} I_{\mathcal{E}}(s, E(s)),$$

$$\textit{where}\ \ E(\cdot) := X_{\alpha(y),y}[t,E](\cdot).$$

One can characterize the value function using Hamilton Jacobi equations based on Dini Epiderivatives which are dual version of proximal normals defined in section 2.2.

Definition 18 *Let $\mathcal{E}$ be a closed collection of nonempty compact sets in $\boldsymbol{R}^n$ and $J : \mathcal{E} \longrightarrow \mathbb{R}$ will be a lower semicontinuous function. Let $E \in \mathcal{E}$ be fixed and let $F : E \mapsto \text{comp}(\boldsymbol{R}^n)$ be a set-valued field on E.*

We define the lower Dini derivative of J at E in the direction of the field F as

$$D^-_{\mathcal{E}} J(E;F) := \liminf_{h,\delta \to 0+} \inf\{\frac{J(E') - J(E)}{h},\ E' \in \mathcal{E},\ (\mathcal{I} + \langle \mathcal{F})(\mathcal{E}) \subset \mathcal{E}' + \langle \delta \mathcal{B}\}.$$

For $\mathcal{E}$ and J as above we define as usually

$$\text{epi}\, J := \{(E, \nu),\ E \in \mathcal{E},\ \nu \in \mathbb{R} :\ \mathcal{J}(\mathcal{E}) \leq \nu\} \subset \mathcal{E} \times \mathbb{R},$$

which is also a closed collection of sets.

Define the extended closed collection of compact sets $\hat{}E$ in $\boldsymbol{R} \times \boldsymbol{R}^n$ as

$$\hat{}E := \{(t, E),\ t \geq 0,\ E \in \mathcal{E}\}.$$

Theorem 19 *Under conditions* **C.1-C.3***, the value function $I_{\mathcal{E}}$ is the unique minimal l.s.c. solution of the differential inequality*

$$\sup_{y \in Y} \min_{u \in U} D^{-}_{\hat{}E} J(t, E; (1, f(\cdot, u, y, V(y)))) \leq 0 \qquad \forall\, (t, E) \in \hat{}E, \tag{18}$$

with the condition

$$J(t, E) \geq G(t, E), \quad \forall\, (t, E) \in \hat{E}. \tag{19}$$

That is,

$$I_{\mathcal{E}}(t, E) = \min\{J(t, E),\ J - \textit{l.s.c. solution of } (18), (19)\}.$$

The proof of this theorem is based on a generalization of Theorem 5 but in the context of tubes with values in collections of sets [53] and to results on the regularity of such tubes [54]. In [57], the above worst case design method is applied to the case of a pursuit games with uncomplete information of the pursuer: the pursuer knows only the state of the evader at finitely many times given in advance.

5 The viability for games with impulsive dynamics

We consider a two-player differential game with separated impulsive dynamics where jumps are allowed. The first controller acts on the system

$$\begin{cases} y' &= g(y, u) \\ y^+ &= p(y^-, \mu) \end{cases} \tag{20}$$

where u and μ are respectively a continuous and a discrete control chosen by Ursula. The solution of (20) is a discontinuous trajectory $t \mapsto y(t)$.

Similarly, the second player , using controls v and ν, controls a system

$$\begin{cases} z' &= h(z, v) \\ z^+ &= q(z^-, \nu) \end{cases} \tag{21}$$

The outcome of the game is defined in the following way:

- The first player goal consists in driving the state $x := (y, z)$ into an open set Ω, while keeping it outside a closed set $\mathcal{T}$.
- The second player aim consists in driving the state in $\mathcal{T}$ while keeping it outside Ω.

We associate with a trajectory $(y(\cdot), z(\cdot))$ of (20)-(21) a payoff

$$\theta^K_{\mathcal{T}}(y(\cdot), z(\cdot)) := \inf \left\{t : (y(t), z(t)) \in \mathcal{T} \quad \text{and} \quad \forall s \leq t, \quad (y(s), z(s)) \in K\right\},$$

where $K := \boldsymbol{R}^n \setminus \Omega$.

This kind of games, in the control case has been studied in [15], [29].

We assume that

Assumption 1

$$\begin{cases} i)\ U \text{ and } M \text{ are compact convex subsets of some finite dimensional spaces;} \\ ii)\ g : \boldsymbol{R}^l \times U \mapsto \boldsymbol{R}^l, \text{ and } p : A_U \times M \mapsto \boldsymbol{R}^l \text{ are Lipschitz-continuous with respect} \\ \quad \text{to their first variable, and continuous with respect to their second variable;} \\ iii)\ g \text{ has linear growth with respect to the first variable;} \\ iv)\ A_U \text{ is compact and for all } y \in A_U, \quad p(y, M) \cap A_U = \emptyset. \end{cases}$$

The last assumption ensures that after a jump, the trajectory is continuous for some time.

Ursula's system can be characterized by a pair of set-valued functions (G, P), by changing (20) into

$$\begin{cases} y' \in G(y) := \{g(y,u) : u \in U\} \\ y^+ \in P(y^-) := \{p(y^-,\mu) : \mu \in M\} \end{cases} \tag{22}$$

The reset map P is defined only on A_U, so that the set from which jumps are allowed coincides with the domain of P, denoted by $\mathrm{Dom}\,(P)$.

Similarly Victor's system can be characterized by a pair of set-valued functions (H, Q), by changing (21) into

$$\begin{cases} z' \in H(y) := \{h(z,v) : v \in V\} \\ z^+ \in Q(y^-) := \{q(z^-,\nu) : \nu \in N\} \end{cases} \tag{23}$$

The assumptions on g and on h ensure the existence of absolutely continuous solutions defined on $[0, +\infty)$ to the differential inclusions $y' \in G(y)$ and $z' \in H(z)$. Let us denote by $S_G(y_0)$ and respectively $S_H(z_0)$ the set of such solutions.

Definition 20 (Runs and trajectories) *We call* run *of impulse system (G, P) (resp. (H, Q)) with initial condition y_0 (resp. z_0) a finite or infinite sequence $\{\tau_i, y_i, y_i(\cdot)\}_{i\in I}$ (resp. $\{\tau_i, z_i, z_i(\cdot)\}_{i\in I}$) of $(\boldsymbol{R}^+ \times \boldsymbol{R}^l \times S_G(\boldsymbol{R}^l))$ such that for all $i \in I$*

$$\begin{cases} y_i'(t) \in G(y_i(t)) \\ y_i(0) = y_i \end{cases} \quad \text{and} \quad \begin{cases} y_i(\tau_i) \in \mathrm{Dom}\,(P) \\ y_{i+1} \in P(y_i(\tau_i)) \end{cases}$$

(resp.

$$\begin{cases} z_i'(t) \in H(z_i(t)) \\ z_i(0) = z_i \end{cases} \quad \text{and} \quad \begin{cases} z_i(\tau_i) \in \mathrm{Dom}\,(Q) \\ z_{i+1} \in Q(z_i(\tau_i)) \end{cases})$$

A trajectory *of impulse system (G, P) is a function $y : \boldsymbol{R} \mapsto \boldsymbol{R}^l$ associated with a run in the following way:*

$$y(t) = \begin{cases} y_0 & \text{if } t < 0 \\ y_i(t - t_i) & \text{if } t \in [t_i, t_i + \tau_i) \end{cases} \tag{24}$$

where $t_i = \sum_{j<i} \tau_i$. We denote by $S_{G,P}(y_0)$ the set of trajectories with initial condition y_0.

We also assume that the impulse system controlled by Victor satisfies Assumption 1 and can be described similarly by a pair of set valued maps (H, Q) and we denote by $S_{H,Q}(z_0)$ the set of trajectories with initial condition z_0.

We are know in position to define strategies

Definition 21 *We call Varaiya-Roxin strategy (VR-strategy) for Ursula at initial condition $x_0 = (y_0, z_0)$ a map*

$$\mathbf{A} : S_{H,Q}(z_0) \longrightarrow S_{G,P}(y_0)$$

such that for any $\theta > 0$, and for any trajectories $z(\cdot)$ and $\tilde{z}(\cdot)$ of $S_{H,Q}(z_0)$ which coincide on $[0, \theta]$, the trajectories $y(\cdot) = \mathbf{A}(z(\cdot))$ and $\tilde{y}(\cdot) = \mathbf{A}(\tilde{z}(\cdot))$ coincide on $[0, \theta]$.
We denote by $\mathcal{A}(x_0)$ the set of VR-strategies for Ursula at x_0.

A VR-strategy for Victor at initial condition $x_0 = (y_0, z_0)$ is defined symmetrically as a map

$$\mathbf{B} : S_{G,P}(y_0) \longrightarrow S_{H,Q}(z_0)$$

such that for any $\theta > 0$, and for any trajectories $y(\cdot)$ and $\tilde{y}(\cdot)$ of $S_{G,P}(y_0)$ which coincide on $[0, \theta]$, the trajectories $z(\cdot) = \mathbf{B}(y(\cdot))$ and $\tilde{z}(\cdot) = \mathbf{B}(\tilde{y}(\cdot))$ coincide on $[0, \theta]$.
We denote by $\mathcal{B}(x_0)$ the set of VR-strategies for Victor at x_0.

We define, for all $x = (y, z) \in \boldsymbol{R}^n$ and all $D \subset \boldsymbol{R}^n$ closed, the functions

$$\mathcal{H}(x, D) := \sup_{\pi \in \mathcal{NP}_D(x)} \left\{ \sup_{u\in U} \inf_{v \in V} < f(x,u,v), \pi > \right\}, \tag{25}$$

$$\mathcal{L}_V(x, D) := \inf_{\nu \in N} \chi_D(y, q(z,\nu)), \tag{26}$$

$$\mathcal{L}_{UV}(x, D) := \sup_{\mu \in M} \inf \left\{ \chi_D(p(y,\mu), z)\ ,\ \inf_{\nu\in N} \chi_D(p(y,\mu), q(z,\nu)) \right\}, \tag{27}$$

where $f(x, u, v) = f((y, z), u, v) = (g(y, u), h(z, v))$, $\chi_D(\cdot)$ denotes the characteristic function of the set D:

$$\chi_D(x) = \begin{cases} 0 & \text{if } x \in D \\ +\infty & \text{otherwise.} \end{cases}$$

Definition 22 (Impulse Discriminating Domains) *A closed subset D of $\boldsymbol{R}^n$ is an impulse discriminating domain with target $\mathcal{T}$ if*

$$\forall x \in (D \setminus \mathcal{T}), \quad \max \{\min \{\mathcal{H}(x, D)\ ,\ \mathcal{L}_V(x, D \cup \mathcal{T})\}\ ,\ \mathcal{L}_{UV}(x, D \cup \mathcal{T})\} \leq 0. \tag{28}$$

We obtain the following generalization of Theorem 5 in the impulsive case

Theorem 23 *Assume that*

Assumption 2 *For all time $\theta > 0$, there exist a neighborhood $\mathcal{N}_U$ of* Dom (P) *and a neighborhood $\mathcal{N}_V$ of* Dom (Q) *such that*

$$\begin{array}{ll} (i)\ \forall y_0 \in \mathcal{N}_U, & \exists y(\cdot) \in S_G(y_0) \quad \text{such that } \exists\ t_0 \leq \theta, \quad y(t_0) \in \mathrm{Dom}\,(P) \\ (ii)\ \forall z_0 \in \mathcal{N}_V, & \exists z(\cdot) \in S_H(z_0) \quad \text{such that } \exists\ t_0 \leq \theta, \quad z(t_0) \in \mathrm{Dom}\,(Q)\,. \end{array}$$

Let D and $\mathcal{T}$ be closed subsets of $\boldsymbol{R}^n$. Under Assumptions 1 and 2, D is an impulse discriminating domain with target $\mathcal{T}$ if and only if for all $x_0 = (y_0, z_0) \in D$, there exists a VR-strategy $\mathbf{B}$ such that for any $y(\cdot) \in S_{G,P}(y_0)$, the trajectory $(y(\cdot), \mathbf{B}(y(\cdot)))$ stays in D as long as $\mathcal{T}$ as not been reached, namely:

$$\forall t \leq \inf\{s : (y(s), \mathbf{B}(y(\cdot))(s)) \in \mathcal{T}\}, \qquad (y(t), \mathbf{B}(y(\cdot))(t)) \in D.$$

As in section 2.4, the above theorem can be applied to the characterization and the existence of a value associated with the payoff $\theta_{\mathcal{T}}^K(y(\cdot), z(\cdot))$ (cf [30]).

Remark 24 Differential games with impulsive dynamics is a very new field and there are several important and interesting open questions on the subject:

- What are the conditions for existence of the value for hybrid differential games: the players can not only jumps but also they can "switch" between several dynamics.
- What are the natural hypothesis allowing to consider games with many instantaneous jumps (which are forbidden here by assumption 1)?

6 Examples

In this section we illustrate the different results presented in the previous section through several examples taken from recent studies in ecology, in economics and in finance. The first one deals with the management of renewable resources in the case when we don't really know what are the precise dynamic which governs the evolution of the resource. The second one is related to the evaluation of a call in the presence of barriers. Such problem can be formalized in the frame of hybrid differential games theory.

6.1 Application I: Management of Renewable resources

A full discussion of this application can be found in [14].

Verhulst and Graham

The simplest ecological model for managing resources was originated by Graham and taken up by Schaeffer, where it is assumed that the exploitation rate of the resource is proportional to the biomass and the economic activity.

Let $x \in \mathbf{R}_+$ denote the biomass of the renewable resource and $v \in \mathbf{R}_+$ the economic effort for exploiting it.

The constraints are of different types and written as follows:

1. *Ecological constraints* : $\forall\ t \geq 0,\ \ 0\ \leq\ x(t)\ \leq\ b$ where b is the carrying capacity of the resource.
2. *Economic constraints* : $\forall\ t \geq 0,\ \ cv(t) + C\ \leq\ \gamma v(t)\ x(t)$ where $C \geq 0$ is a fixed cost, $c \geq 0$ the unit cost of economic activity and $\gamma \geq 0$ the price of the resource with $\gamma b > c$.
3. *Production constraints* : $\forall\ t \geq 0,\ \ 0\ \leq\ v(t)\ \leq\ \overline{v}$, where $\overline{v}$ is maximal exploitation effort satisfying $\frac{C}{\gamma b - c} \leq \overline{v}$.

Setting $a := \dfrac{C + c\overline{v}}{\gamma\overline{v}}$, the economic constraints implies that $\forall t \geq 0,\ x(t) \in [a, b]$ and $v \in V(x) := \left[\frac{C}{\gamma x - c}, \overline{v}\right]$.

The Verhulst logistic dynamics and the Schaeffer proposal are summed up as follows:

$$\begin{cases} x'(t) \ = \ rx(t)\left(1 - \dfrac{x(t)}{b}\right) - v(t)x(t) \\ v(t) \ \in \ V(x(t)) \ := \ \left[\dfrac{C}{\gamma x(t) - c}, \overline{v}\right] \end{cases} \quad . \tag{29}$$

where the admissible economic effort depends on the very level of the resource. The viability kernel of the interval $[a, b]$ is clearly an interval of the form $[x_-, x_+]$.

The <u>*rigidity*</u> of the economic behavior is expressed through a constraint on the velocity of the economic effort v' :

$$\forall\, t \geq 0, \ \ -d \ \leq \ v'(t) \ \leq \ +d\,.$$

Taking into account the rigidity of the economic efforts leads to study what is so called the <u>*metasystem*</u> governing the state x and the regulon v which comes with the differential inclusion system

$$\begin{cases} i) \ \ x'(t) \ = \ rx(t)\left(1 - \dfrac{x(t)}{b}\right) - v(t)x(t) \\ ii) \ v'(t) \in [-d, +d] \end{cases} \tag{30}$$

and the meta-constrained set which is the graph of the set valued map V:

$$K := (\mathrm{Graph}(V)) = \{(x, v) | v \in V(x)\}\,. \tag{31}$$

One can prove that the viability kernel of K, Viab(Graph(V)), is of the form $\{(x, v) \in \mathrm{Graph}(V) | x \geq \rho^\sharp(v)\}$, where $\rho^\sharp$ is the solution to the differential equation

$$-d\frac{d\rho^\sharp}{dv} \ = \ r\left(1 - \frac{\rho^\sharp(v)}{b}\right) - u\rho^\sharp(v)$$

satisfying the initial condition $\rho^\sharp(v_-) = x_-$.

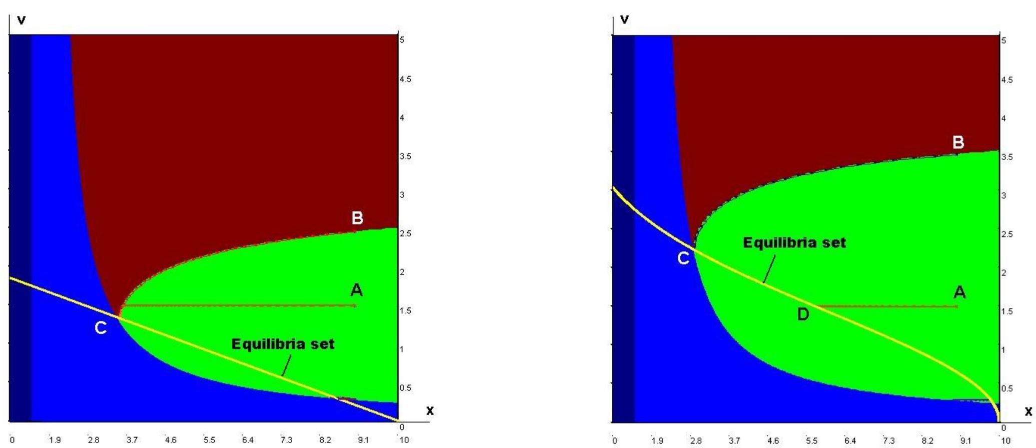

Fig. 1. Regulation Maps and heavy solutions under Verhulst-Schaeffer Verhulst-Schaeffer metasystems: $x'(t) = rx(t)\left(1 - \frac{x(t)}{b}\right) - v(t)x(t),\ v'(t) \in [-d, +d]$ on left side and the Verhulst-Inert metasystems: $x'(t) = \sqrt{\alpha}\ x(t)\sqrt{2\log\left(\frac{b}{x(t)}\right)} - v(t)x(t),\ v'(t) \in [-d, +d]$ on right side. The equilibrium sets are respectively the graphs $r\left(1 - \frac{x}{b}\right)$ and $\sqrt{\alpha}\sqrt{2\log\left(\frac{b}{x}\right)}$. Starting from A, heavy solutions minimize $|v'|$ as long as the evolution remains in the viability kernel. The velocity of the economic effort first is null then changes when it reaches the boundary of the viability kernel, at a suitable time for maintaining viability. Here this is occurs as soon as it reaches the equilibria set.

6.2 Towards Dynamical Meta Games

In fact we do not really know what are the dynamical equations governing the evolution of the resource. We could fix Malthusian feedbacks $\widetilde{u}$ in a given class $\widetilde{\mathcal{U}}$ of continuous feedbacks as parameters and study the viability kernel $\text{Viab}_{\widetilde{u}}([a,b])$ of the interval $[a,b]$ under the system

$$\begin{cases} (i) \;\; x'(t) \;=\; (\widetilde{u}(x(t)) - v(t))x(t) \\ (ii)\; v(t) \;\in\; V(x(t)) \;:=\; \left[\dfrac{C}{\gamma x(t) - c}, \overline{v}\right] \end{cases} \tag{32}$$

where the control parameter is v, but, instead of fixing feedbacks, we can study "meta-games" by setting bounds c and d on the velocities of the growth rate $u(t)$ and the exploitation effort $v(t)$, regarded as meta-controls, whereas the meta-states of the meta-game are the triples (x,u,v):

$$\begin{cases} (i) \;\;\; x'(t) \;=\; (u(t) - v(t))x(t) \\ (ii) \;\; u'(t) \;\in\; B(0,c) \\ (iii)\; v'(t) \;\in\; B(0,d) \end{cases} \tag{33}$$

subjected to the viability constraints $u(t) \in \mathbb{R}$ and $\frac{C}{\gamma x(t)-c} \;\leq\; v(t) \;\leq\; \overline{v}$.

Then computing the discriminating kernel of meta game we get the following result as shown in Figure2 where the meta-controls are the velocities $|v'(t)| \;\leq\; d$ of economic activity bounded by a constant d and the "meta-tyches" corresponding to uncertainty are the velocities of the growth rates of the renewable resources.

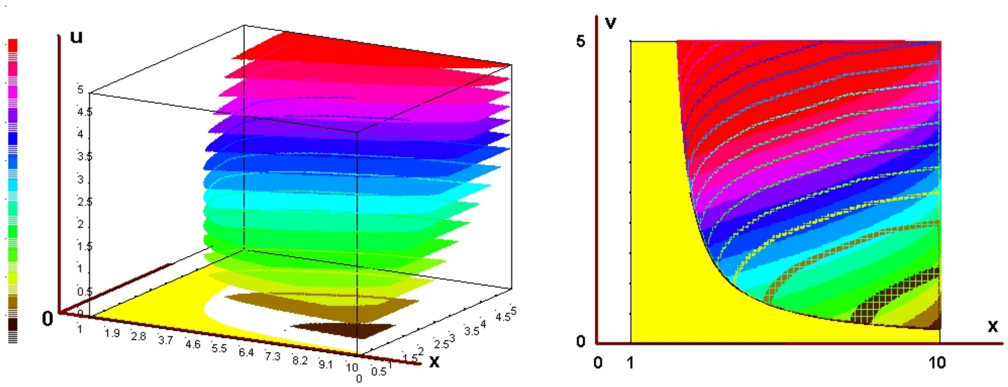

Fig. 2. Discriminating Kernel of meta game (35). The constrained set $\{(x,v,u) \in [a,b] \times [0,\overline{v}] \times [-c,+c] \mid v \;\geq\; \frac{C}{\gamma x - c}\}$ translates economic constraints.

The Palikinesis Function

Considering that the velocities of the growth rate $u(t)$ are unknown, we now aim at measuring the maximal norm of this velocity for which a given position (x,u,v) belongs to the Discriminating kernel. This can be considered as an inverse problem: If the state initial position is (x_0, v_0), what is the maximal level of the velocities of the growth rate in any future time that the system might face if the velocity of the economic activity is "well chosen". The palikinesis value is connected to the *"range of environmental tolerance over which an organism can persist"* introduced by Joel Brown when defining the Hutchinsonian Niche in his T2.

The Palikinesis function which indicates the maximal level of risk under which viability can be maintained is defined by:

$$\psi(x,u,v) := \inf_{\beta} \sup_{u(\cdot)\in\mathcal{U}} \textbf{esssup}_{t>0} \|v'(t)\| \tag{34}$$

where the infimum is over the strategies β ensuring the viability.

Proposition 25 *The hypograph of the Palikinesis function is the discriminating kernel associated with the following system:*

$$\begin{cases} (i) \quad x'(t) \ = \ (u(t)-v(t))x(t) \\ (ii) \quad u'(t) \ \in \ B(0,|z|) \\ (iii) \ v'(t) \ \in \ B(0,d) \\ (iv) \quad z'(t) \ = \ 0 \end{cases} \tag{35}$$

subjected to the viability constraints $u(t) \in \mathbb{R}$ *and* $\frac{C}{\gamma x(t)-|z|} \leq v(t) \leq \overline{v}$

6.3 Application II: Evaluation of Barrier Options

This application is an extension to hybrid games of the Guaranteed Capture Basin method developed in [49, 62, 63].

A put or a call is a contract giving the right to buy (call) or to sell (put) a quantity of an asset of price S at a given date or at any date t before a fixed date T (American put or call). The point is to determine the value W of the contract at the start. Facing the risks inherent, the seller builds up a theoretical portfolio in investing into the underlying asset by self-financing yielding the same losses and profits as the put or call.

Constraints, Payoff and "Target"

1) $\forall (t,S) \in \mathbb{R}_+ \times \mathbb{R}^2_+, \ \ W \geq \mathbf{b}(t,S) \geq 0$ describes the "floor" constraints where $\mathbf{b} : (t,S) \mapsto \mathbf{b}(t,S) = U(S_0,S_1)$ for American puts and calls, and $\mathbf{b} = 0$ for European puts and calls.
2) $t = 0, \ W \geq \mathbf{c}(0,S)$ describes the targets at maturity,
3) The payoff function is $U(S_0,S_1) = (S_1 - K)^+$.

Objective

We aim at determining the set of portfolio strategies $S \rightarrow \widetilde{\pi}(S)$ such that, whatever the variations of capital $S(\cdot)$ are, the following conditions hold true:

$$\begin{aligned} &i) \ \ \forall t \in [0,T], \ W_{\widetilde{\pi}(\cdot)}(t)) \geq \mathbf{b}(T-t, S(t)) \\ &ii) \ W_{\widetilde{\pi}(\cdot)}(T) \geq \mathbf{c}(0, S(T)) \end{aligned}$$

and, amongst them, selecting the portfolio strategy such that, for all predictable variations, the initial value of the portfolio corresponds to the cheapest capital $V(T,S(0))$ which we identify as the evaluation function of the put or call.

The Dynamical Game describing the Replicating Process

Consider a riskiness asset and an underlying risky asset of respective prices S_0 and S_1. Let $S = (S_0,S_1) \in \mathbb{R}^2$ and $\pi = (\pi_0,\pi_1)\mathbb{R}^2$ the array of which each component is the total number of assets in a portfolio of value: $W_\pi \ = \ \pi_0 S_0 + \pi_1 S_1$ The riskiness and the risky assets are governed by a deterministic and a non deterministic differential equation

$$\begin{cases} S_0'(t) = S_0(t)\gamma_0(S_0(t)) \\ S_1'(t) = S_1(t)\gamma_1(S_1(t), v(t)) \end{cases}$$

The variations of price $S(t)$ of assets at date t help find the variations $W_{\pi(\cdot)}(t)$ of capital as a function of a strategy $\pi(\cdot)$ of the replicating portfolio. Indeed, the value of the replicating portfolio is given by $W_\pi(t) := \pi_0(t)S_0(t) + \pi_1(t)S_1(t)$.

The self-financing principle of the portfolio reads

$\forall\, t \geq 0, \ \ \langle \pi'(t), S(t) \rangle \ = \ \pi_0'(t)S_0(t) + \pi_1'(t)S_1(t) \ = \ 0$

so that the value of the portfolio satisfies

$$W'(t) = \langle \pi(t), S'(t) \rangle = \pi_0(t)S_0(t)\gamma_0(S(t)) + \pi_1(t)S_1(t)\gamma_1(S_1(t), v(t))$$

which is

$W'(t) = W(t)\gamma_0(S(t)) - \pi_1(t)S_1(t)(\gamma_0(S_0(t)) - \gamma_1(S_1(t), v(t)))$.

Let $\tau(t) := T - t$, then $S(t)$ and $W(t)$ change according to:

$$\begin{cases} \tau'(t) &= -1 \\ S_0'(t) &= S_0(t)\gamma_0(S_0(t)) \\ S_1'(t) &= S_1(t)\gamma_1(S_1(t), v(t)) \\ W'(t) &= W(t)\gamma_0(S(t)) - \pi_1(t)S_1(t)(\gamma_0(S_0(t)) - \gamma_1(S_1(t), v(t))) \end{cases}$$

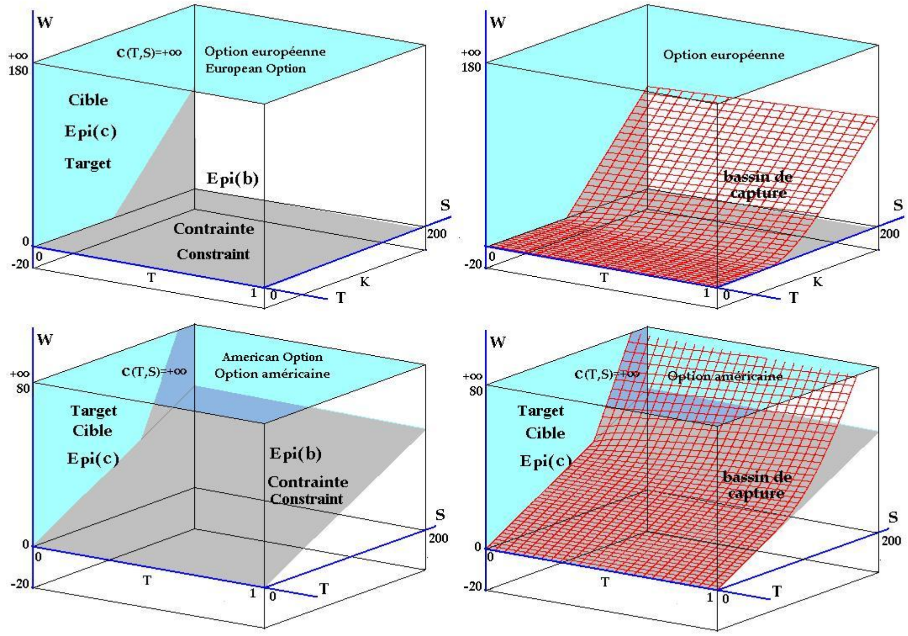

Fig. 3. Evaluating a call can be expressed as a guaranteed capture basin problem where the target is the epigraph of the payoff function at $T = 0$ and the constraint is the epigraph of the "floor" function $\mathbf{b}$ (which appear when evaluating American calls.

where $\pi(t) \in \Pi(S(t))$ and $v(t) \in Q(S(t))$.

It is out of the scope of this section to develop the numerical point of view allowing to evaluate calls or puts as shown in figure (4). Let us just point out that the discretisation process allows to recover as well the values obtained by the Cox, Ross and Rubinstein's binomial approach as the evaluation of contract in more general cases. Let us also quote some recent results linking the approach using stochastic approach and that using differential game that can be found in a forthcoming paper of Aubin & Doss.

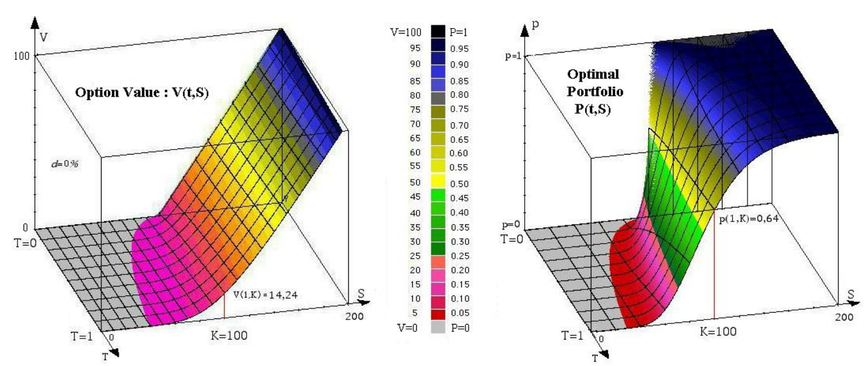

Fig. 4. Evaluation Function of an European call with uncertainty of Cox, Ross and Rubinstein type and the optimal portfolio: π_1

value of S_1	Value of the Call	π_0	π_1
80	4.53	-22.10	0.3332
90	8.61	-34.94	0.4844
100	14.24	-48.17	0.6240
110	21.04	-59.94	0.7363

The Barrier Mechanism

Barriers complicate the evaluation of the replicating portfolio. A barrier is a particular value S^* for the price $S(t)$ beyond which the contract changes. There are four types of options with barriers:

"up and in": the contract becomes effective at the first time t^* when $S(t) < S^*,\ \forall t < t^*,\ S(t^*) = S^*$
"down and in": the contract becomes effective at the first time t^* when $S(t) > S^*,\ \forall t < t^*,\ S(t^*) = S^*$
"up and out": the contract ceases at the first time t^* when $S(t) < S^*,\ \forall t < t^*,\ S(t^*) = S^*$
"down and out": the contract ceases at the first time t^* when $S(t) > S^*,\ \forall t < t^*,\ S(t^*) = S^*$.

The challenge is to evaluate today a contract which will vanish at some unknown future date.

Consider an "up and in" call and introduce a discrete variable $L \in \{0, 1\}$ which "labels " the state of the contract: effective for $L = 1$ or non effective for $L = 0$. The label L increases because we study "in" options. We consider the hybrid dynamical system

$$\begin{cases} \tau'(t) & = -1 \\ S_0'(t) & = S_0(t)\gamma_0(S_0(t)) \\ S_1'(t) & = S_1(t)\gamma_1(S_1(t), v(t)) \\ L'(t) & = 0 \\ W'(t) = & \begin{cases} W(t)\gamma_0(S(t)) - \pi_1(t)S_1(t)(\gamma_0(S_0(t)) - \gamma_1(S_1(t), v(t))) \\ \quad \text{if } \ L(t) = 1 \\ 0 \quad \text{if } \ L(t) = 0 \end{cases} \end{cases}$$

$$\begin{cases} \tau^+ & = \tau^- \\ S_0^+ & = S_0^- \\ S_1^+ & = S_1^- \\ L^+ & = \begin{cases} 1 & \text{if } \ S_1^- \geq S^* \\ L^- & \text{if } \ S_1^- < S^* \end{cases} \\ W^+ = W^- \end{cases}$$

Then applying the Discriminating Kernel Algorithm extended to hybrid systems allows to compute evaluation function of contracts embedding impulse events as presented in figure 5.

References

1. AUBIN J.-P. *Contingent Isaacs' Equations of a Differential Game*, in Differential Games and Applications, Eds. T. Başar & P. Bernhard, (1989)
2. AUBIN J.-P. *Victory and Defeat in Differential Games*, in Modeling and Control of Systems, Proceedings of the Bellman Continuum, June 1988, Ed. A. Blaquière, Lecture Notes in Control and Information Sciences, Springer-Verlag (1989)
3. AUBIN J.-P.& FRANKOWSKA H. SET-VALUED ANALYSIS. Birkhäuser. (1990)
4. AUBIN J.-P.& Da Prato G. *Stochastic viability and invariance.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 17, no. 4, 595–613. (1990)
5. AUBIN J.-P. *Differential games: a viability approach.* SIAM J. Control Optim. 28, no. 6, 1294–1320. (1991)
6. AUBIN J.-P. VIABILITY THEORY. Birkhäuser. (1991)
7. AUBIN J.-P. IMPULSE DIFFERENTIAL INCLUSIONS AND HYBRID SYSTEMS: A VIABILITY APPROACH, Lecture Notes, University of California at Berkeley (1999)
8. AUBIN J.-P. & DA PRATO G. *Stochastic Nagumo's Viability Theorem*, Stochastic Analysis and Applications, 13, 1-11 (1995)
9. AUBIN J.-P. DYNAMIC ECONOMIC THEORY: A VIABILITY APPROACH, Springer-Verlag (1997)
10. AUBIN J.-P. & DA PRATO G. *The Viability Theorem for Stochastic Differential Inclusions*, Stochastic Analysis and Applications, 16, 1-15 (1998)
11. AUBIN J.-P., DA PRATO G. & FRANKOWSKA H. *Stochastic invariance for differential inclusions.* Set-valued analysis in control theory. Set-Valued Anal. 8, no. 1-2, 181–201. (2000)
12. AUBIN J.-P., PUJAL D. & SAINT-PIERRE P. *Dynamic Management of Portfolios with Transaction Costs under Tychastic Uncertainty*, preprint (2001)
13. AUBIN J.-P. & SHI SHUZHONG, Eds STOCHASTIC AND VIABILITY APPROACHES TO DYNAMICAL UNCERTAINTY IN ECONOMICS AND FINANCE, Presses de l'Université de Beijing (2001)
14. AUBIN J.-P. & SAINT-PIERRE P. *An introduction to Viability Theory and management of renewable resources, in ch.2, Progress of Artificial Intelligence in Sustainability Science*, Kropp J. & Scheffran J. (Eds.) Nova Science Publ. Inc. (2004)

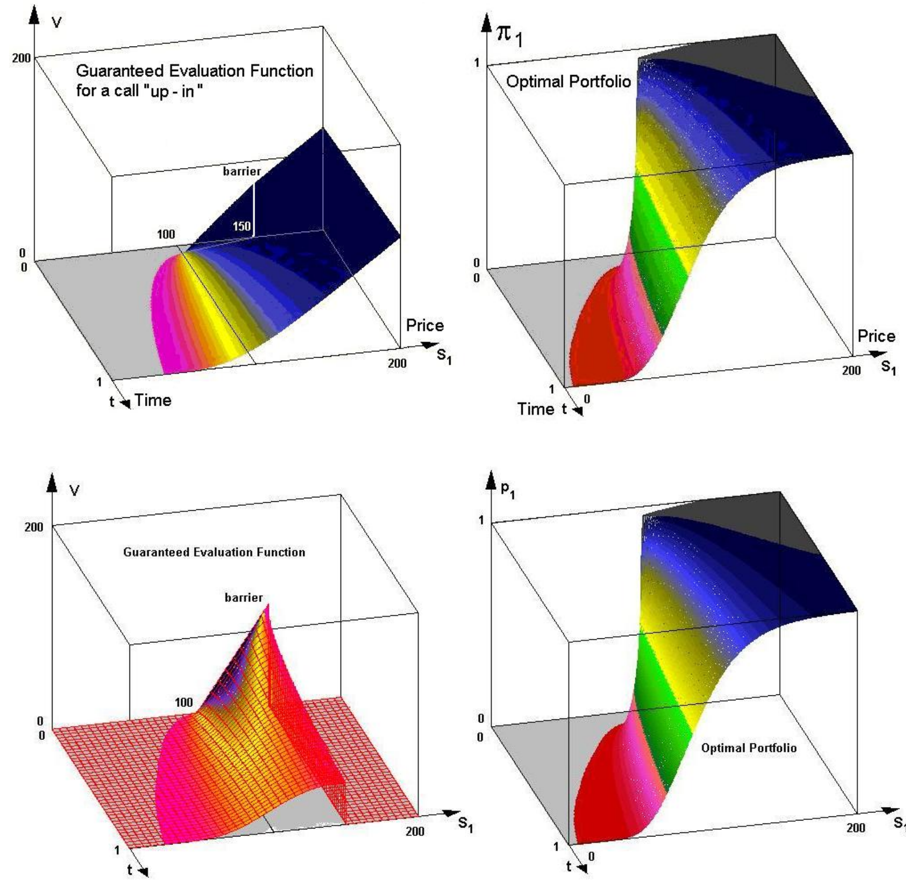

Fig. 5. Evaluation Function of an European call with barrier “up in” and “up out” with the corresponding Optimal strategy: π_1.

15. AUBIN J.-P., LYGEROS J., QUINCAMPOIX. M., SASTRY S. & SEUBE N. *Impulse differential inclusions: a viability approach to hybrid systems.* IEEE Trans. Automat. Control 47, no. 1, 2–20., (2002)
16. BARDI, M., CAPUZZO-DOLCETTA I. Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations. Systems and Control: Foundations and Applications. Boston, Birkhäuser. xvii, 570 pp. (1997)
17. BARDI, M., JENSEN R. *A geometric characterization of viable sets for controlled degenerate diffusions.* Calculus of variations, nonsmooth analysis and related topics. Set-Valued Anal. 10, no. 2-3, 129–141. (2002)
18. BARLES G. Solutions de viscosité des équations de Hamilton-Jacobi. Mathématiques & Applications (Paris). 17. Paris: Springer-Verlag. ix, 194 p. (1994)
19. BETTIOL P., CARDALIAGUET P. & QUINCAMPOIX M. Zero-sum state constrained differential games: Existence of value for Bolza problem, Preprint. (2004)
20. BONNEUIL N. & SAINT-PIERRE P. *The Hybrid Guaranteed Capture Basin Algorithm in Economics*, Preprint. (2004)
21. BUCKDAHN R., PENG S., QUINCAMPOIX M. & RAINER C. *Existence of stochastic control under state constraints.* C. R. Acad. Sci. Paris Sér. I Math. 327, no. 1, 17–22. (1998)
22. CARDALIAGUET P., QUINCAMPOIX M. & SAINT-PIERRE P. *Some Algorithms for Differential Games with two Players and one Target*, RAIRO Mathematical Modeling and Numerical Analysis, Vol. 28, N. 4, pp. 441-461. (1994)
23. CARDALIAGUET P. *A differential game with two players and one target*, SIAM J. Control and Optimization, Vol. 34, N. 4, pp. 1441-1460. (1996)
24. CARDALIAGUET P. *Non smooth semi-permeable barriers, Isaacs equation and application to a differential game with one target and two players*, Appl. Math. Opti., 36, pp. 125-146. (1997)
25. CARDALIAGUET P., QUINCAMPOIX M. & SAINT-PIERRE P. *Numerical methods for differential games*, in “Stochastic and differential games : Theory and numerical methods”, pp. 177-247. Annals of the international Society of Dynamic Games, M. Bardi, T.E.S. Raghavan, T. Parthasarathy Eds. Birkhäuser. (1999)
26. CARDALIAGUET P. & PLASKACZ S. *Invariant solutions of differential games and Hamilton-Jacobi equations for time-measurable hamiltonians*, SIAM Journal on Control and Optim. Vol. 38, N. 5, pp.1501-1520 (2000)
27. CARDALIAGUET P., QUINCAMPOIX M. & SAINT-PIERRE P. *Pursuit differential games with state constraints.*, SIAM J. Control and Optimization Vol. 39, N. 5, pp.1615-1632 (2001)

28. CARDALIAGUET P., QUINCAMPOIX M. & SAINT-PIERRE P. *Differential games with state-constraints.* ISDG2002, Vol. I, II (St. Petersburg), 179–182, St. Petersburg State Univ. Inst. Chem., St. Petersburg. (2002)
29. CRUCK E. *Target problems under state constraints for nonlinear controlled impulsive systems*, J. Math. Anal. Appl. 270 (2002), no. 2, 636–656. (2002)
30. CRÜCK E., QUINCAMPOIX, M. & SAINT-PIERRE P., *Existence of a value for Pursuit-Evasion Games with Impulsive Dynamics*, preprint. (2003)
31. CRÜCK E. & SAINT-PIERRE P. *Non linear Impulse Target Problems under State Constraint: A Numerical Analysis based on Viability Theory.* Set-Valued Analysis. Set-Valued Analysis. 12, 4, pp. 383-416. (2004)
32. DA PRATO G. & FRANKOWSKA H. *A stochastic Filippov Theorem*, Stochastic Calculus, 12, 409-426. (1994)
33. DOYEN L. & SEUBE N. *Control of uncertain systems under bounded chattering.* Dynam. Control 8 (1998), no. 2, 163–176. (1998)
34. EVANS L.C. & SOUGANIDIS P.E. *Differential games and representation formulas for solutions of Hamilton-Jacobi Equations* Indiana Univ. Math. J., 282, pp. 487-502. (1984)
35. FILIPPOVA T.F., KURZHANSKI A. B., SUGIMOTO K. & VALYI I. *Ellipsoidal calculus, singular perturbations and the state estimation problems for uncertain systems.* J. Math. Systems Estim. Control 6 (1996), no. 3, 323–338. (1996)
36. FLEMING W. & SOUGANIS P. *On the existence of value functions of two-player, zero-sum stochastic differential games.* Indiana Univ. Math. J. 38, no. 2, 293–314. (1989)
37. FRANKOWSKA H. *Lower semicontinuous solutions of Hamilton-Jacobi-Bellman equation*, SIAM J. on Control and Optimization, 31, no. 1, 257–272. (1993)
38. GAO Y., LYGEROS J. , QUINCAMPOIX M. & SEUBE N. *Approximate stabilisation of uncertain hybrid systems* Proceedings of Computation and control. 6th international workshop, HSCC 2003, Berlin: Springer. Lect. Notes Comput. Sci. 2623, pp. 203-215. (2003)
39. GAUTIER S. & THIBAULT L. *Viability for constrained stochastic differential equations.* Differential Integral Equations 6, no. 6, 1395–1414. (1993)
40. HADDAD G. *Monotone trajectories of differential inclusions with memory*, Isr. J. Math., 39, 83-100 (1981)
41. HADDAD G. *Monotone viable trajectories for functional differential inclusions*, J. Diff. Eq., 42, 1-24 (1981)
42. ISAACS R. Differential Games Wiley, New York. (1965)
43. JACHIMIAK W. *Stochastic invariance in infinite dimension*, preprint (1998)
44. KISIELEWICZ M. *Viability theorem for stochastic inclusions*, Discussiones Mathematicae, Differential Inclusions, 15, 61-74 (1995)
45. KRASOVSKII N.N. & SUBBOTIN A.I. Game-Theorical Control Problems Springer-Verlag, New-York. (1988)
46. KURZHANSKI A. B. & FILIPPOVA T.F. *On the theory of trajectory tubes—a mathematical formalism for uncertain dynamics, viability and control.* Advances in nonlinear dynamics and control: a report from Russia, 122–188, Progr. Systems Control Theory, 17, Birkhäuser Boston, Boston, MA. (1993)
47. MOITIE G., QUINCAMPOIX M. & VELIOV V. *Optimal control of uncertain discrete systems with imperfect measurements*, IEEE Trans. Automat. Control 47(11), 1909-1914 (2002)
48. PLASKACZ S. & QUINCAMPOIX M. *Value-functions for differential games and control systems with discontinuous terminal cost.* SIAM J. Control and Optim. , Vol. 39, N. 5, pp. 1485-1498 (2001)
49. PUJAL D. & SAINT-PIERRE P. *L'Algorithme du bassin de capture appliqué pour évaluer des options européennes, américaines ou exotiques*, Revue de l'Association Française de Finance, Paris. (2004)
50. QUINCAMPOIX M. *Differential inclusions and target problems*, SIAM J. Control and Optimization, 30, 324-335 (1992)
51. QUINCAMPOIX M. & SAINT-PIERRE *An algorithm for viability kernels in Hölderian case: Approximation by discrete viability kernels*, J. Math. Syst. Est. and Control, Summary : 115-120 (1995)
52. QUINCAMPOIX M. & VELIOV V. *Viability with a target: theory and applications*, in Applications of mathematics in engineering, 47-54, Heron Press (1998)
53. QUINCAMPOIX M. & VELIOV V. *Open-loop viable control under uncertain initial state information.* Set-Valued Anal. 7, no. 1, 55–87. (1999)
54. QUINCAMPOIX M. & VELIOV V. (2002) *Solution tubes to differential equations within a collection of sets.* Control and Cybernetics, 31, 3.
55. QUINCAMPOIX M. & VELIOV V. *Optimal control of uncertain systems with incomplete information for the disturbance*, Siam J. Control Opti., 43, 4, pp 1373-1399. (2005)
56. QUINCAMPOIX M. & SEUBE N. *Stabilization of uncertain Control Systems through Piecewise Constant Feedback.* J. Math. Anal. Appl. N. 196 pp.240-255. (1998)
57. RIGAL S. *A set evolution approach to the control of uncertain systems with discrete time measurement*, preprint. (2004)
58. SAINT-PIERRE P. *Approximation of the viability kernel*, Applied Mathematics & Optimisation, 29, 187-209 (1994)
59. SAINT-PIERRE P. *Evaluation of crisis, reversibility, alert management for constrained dynamical systems using impulse dynamical systems.* Numerical methods and applications, 255–263, Lecture Notes in Comput. Sci., 2542, Springer, Berlin, (2003)

60. SAINT-PIERRE P.) *Viable Capture Basin for Studying Differential and Hybrid Games*, (to appear) International Game Theory Review, World Scientific Publishing Company. (2003
61. SAINT-PIERRE P. *Hybrid Kernels and Capture Basins for Impulse Constrained Systems*, Proceedings of Hybrid Systems. (2003)
62. SAINT-PIERRE P. *The Guaranteed Hybrid Kernel Algorithm applied to evaluate barrier options in Finance*, Proceeding MTNS04. (2004)
63. SAINT-PIERRE P. *Viable capture basin for studying differential and hybrid games: application to finance.* Int. Game Theory Rev. 6, no. 1, 109–136. (2004)
64. SAINT-PIERRE P. *Approximation of capture basins for hybrid systems*, in *Proceedings of the 2001 European Control Conference*, Porto, Portugal, September 4-7 . (2001)
65. SEUBE N., MOITIE R. & LEITMANN G. *Viability analysis of an aircraft flight domain for take-off in a windshear.* Lyapunov's methods in stability and control. Math. Comput. Modelling 36, no. 6, 633–641. (2002)
66. SEUBE N., MOITIE R. & LEITMANN G. *Aircraft take-off in windshear: a viability approach.* Set-valued analysis in control theory. Set-Valued Anal. 8 (2000), no. 1-2, 163–180. (2000)
67. SEREA O.-S. *Discontinuous differential games and control systems with supremum cost.* J. Math. Anal. Appl. 270, no. 2, 519–542. (2002)
68. SUBBOTIN A. I. Generalized solutions of first-order PDEs. The dynamical optimization perspective. Translated from the Russian. Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA. (1995)
69. VELIOV V. *Sufficient conditions for viability under imperfect measurement.* Set-Valued Anal. 1 (1993), no. 3, 305–317. (1993)
70. ZABCZYK J. *Stochastic invariance and consistency of financial models*, preprint Scuola Normale di Pisa (1999)