Hierarchy of players in swap robust voting games and minimal winning coalitions

Monisankar Bishnu and Sonali Roy

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IOWA STATE UNIVERSITY
Department of Economics
Ames, Iowa, 50011-1070

Hierarchy of players in swap robust voting games and minimal winning coalitions

Monisankar Bishnu* Sonali Roy†

Department of Economics, Iowa State University

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Abstract

Ordinarily, the process of decision making by a committee through voting is modelled by a monotonic game the range of whose characteristic function is restricted to $\{0, 1\}$. The decision rule that governs the collective action of a voting body induces a hierarchy in the set of players in terms of the a-priori influence that the players have over the decision making process. In order to determine this hierarchy in a swap robust game, one has to either evaluate a number-based power index (e.g., the Shapley-Shubik index, the Banzhaf-Coleman index) for each player or conduct a pairwise comparison between players in order to find out whether there exists a coalition in which player i is desirable over another player j as a coalition partner. In this paper we outline a much simpler and more elegant mechanism to determine the ranking of players in terms of their a-priori power using only *minimal winning coalitions*, rather than the entire set of winning coalitions.

Keywords Simple game, Swap robust game, Desirability, Weak desirability, Lexicographic ordering.

JEL classification C71, D71

*Email: mbishnu@iastate.edu

†Email: sroy@iastate.edu

1 Introduction

The issue of voting power and its measurement concerns any collective decision making body which has to decide whether to accept or reject a bill by the process of voting. Typically, when a bill is presented before a voting body, the members either vote in favor of the bill or against it. Sometimes a voter has the option to abstain. However, as is common in most of the literature in this area, we ignore abstention so that the voters have only two choices - to either vote "*yes*" or "*no*". The class of mathematical structures that is used to model such situations is a *simple voting game* (SVG) or simply, a *simple game* which is defined by a finite set of *players* or *voters* and a monotonic family of winning coalitions. Examples of such decision making bodies include the United Nations Security Council, the Council of Ministers in the European Union, the Lok Sabha of the Republic of India, the board room of any corporate house etc. The voting procedure in each of these collectivities is governed by its own *constitution*, which lays down the decision making rule of the collectivity. This decision making rule aggregates individual votes to arrive at the collective decision of the voting body as a whole. There can be many kinds of decision rules, e.g., simple majority decision rule, some qualified majority rule, unanimity rule etc to name a few. In fact a decision making voting body can have any decision rule provided it satisfies the following intuitively appealing conditions:

1. If all players vote in favor of a bill, it should be passed.
2. If all players vote against a bill, it should be rejected.
3. Increased support for a bill cannot hurt its prospects.

A decision rule is often characterized in terms of how it distributes power among the individual players or voters. In this framework, by the voting power of an individual player under a given decision rule, we mean the extent of control that the player possesses over the decision making process *due to the decision rule alone.* In other words, it is the *constitutional power* of the voter (see Felsenthal and Machover (1998)).

However, often there are situations where we are more concerned with the *ranking* or *hierarchy* of players in terms of their influence rather than the quantification of the amount of a-priori influence that players have. Many *number-based indices* exist that give us a measure of the constitutional power that an individual voter possesses. The ones that are most mentioned in the literature are the Shapley-Shubik (SS) (Shapley and Shubik, 1954) and the Banzhaf-Coleman (BC) (Banzhaf 1965, Coleman 1971) indices (for more on power indices see Felsenthal and Machover (1998)). These indices associate a real number to each voter, thereby inducing a complete preordering in the set of voters. As an example, a voter i with a high value of the SS index ranks higher in the SS preordering than another voter j for whom the value of the SS index is lower. Thus, according to the SS preordering, voter i has more a-priori power or control over the collective action of the voting body than the voter j. Sometimes these indices may end up ranking the voters differently (see Saari and Sieberg (2001)). But if we restrict our attention to *weighted voting games*[1] (see Tomiyama (1987)) or better still *swap robust* (*linear*) voting games, which form a larger class of voting games than weighted voting games (see Diffo Lambo and Moulen (2002)), SS and BC preorderings coincide[2]. Thus, given a simple swap robust game, one way of finding out the hierarchy induced in the set of players by the decision rule is by evaluating any one of the above mentioned indices for each player. This means we would have to calculate the number of winning coalitions in which a player is decisive, for all the players.

The other method by which we can determine the ranking of players is by using the *desirability* (or *influence*) relation introduced by Isbell (1958) (also see Isbell (1956), Taylor (1995)). The desirability relation ranks voters with respect to how influential they are in the voting process, *without assigning numbers* to them. A voter whose vote is never *pivotal* or *critical* in any situation, i.e., who can never change the outcome of the voting process by changing the way he/she votes may be regarded as one who does not have any influence over the decision making process. A voter whose "*yes*" vote is necessary for the passage of the bill is a very influential voter. However, in most cases, a voter is critical in some situations, while in others he/she is not. That is, a typical voter can change the final outcome of the decision making process in some situations, while in other situations, the way he/she votes is irrelevant for the final outcome. It was established by Taylor (1995) that the desirability relation induces a complete preordering in the set of voters if and only if the simple game is swap robust. Furthermore, Diffo Lambo and Moulen (2002) have shown that this preordering coincides with the SS and BC preorderings if the simple game is linear or swap robust. Thus, given a swap robust simple game, we could determine the ranking of players by conducting a pairwise comparison whereby a player i is ranked higher than player j if there is a coalition in which i is desirable over j as a coalition partner. This would take less time than having to evaluate one of the classical indices discussed above, since we could restrict our attention to the set of *minimal winning coalitions* rather than the entire set of

[1] Weighted voting games are a class of SVGs that can be represented by a system of non-negative weights and a quota. The vote of each player carries a non-negative weight. A bill is passed if and only if the sum of the weights (of the votes) of all players who vote in favor of the bill is at least as large as the predefined quota. Voting by disciplined party groups in multiparty parliaments can be modelled as weighted voting games. All weighted voting games are swap robust (Taylor and Zwicker (1993)).

[2] Carreras and Freixas (2008) have extended the above result to all preorderings induced by regular semivalues in a larger class of simple games called *weakly linear* simple games.

winning coalitions. However, pairwise comparison may prove to be tedious, specially if the set of minimal winning coalitions is large.

The main result in this paper is as follows: a player i has more a-priori influence than player j *if and only if* the vector $M(i)$, defined by the cardinalities of the minimal winning coalitions to which i belongs, dominates the vector $M(j)$ in lexicographic ordering. Moreover, a player i has the same a-priori influence as player j if and only if the vector $M(i)$ is equal to $M(j)$. This elegant result can be used to determine the hierarchy (in terms of constitutional power) that is induced in the set of voters by the decision rule in a very straightforward manner. Since it involves only minimal winning coalitions, the method proves to be efficient as well in the sense that it can determine the hierarchy in a much shorter time. In what follows we restict our attention to linear or swap robust simple voting games only.

The rest of the paper is organized as follows. In section 2 we present the preliminaries. In section 3 we present the motivation behind the paper while we present our main result in section 4. Section 5 concludes.

2 Definitions and preliminaries

Let $N = \{1, 2, ..., n\}$ be a non-empty finite set. We refer to the elements of N as *players* or *voters*. The collection of all subsets of N is denoted by $\mathcal{P}(N)$. Any member of $\mathcal{P}(N)$ is called a *coalition*. Given any family of sets $\mathcal{F}$, we will use the notation $|\mathcal{F}|$ to denote the number of elements in the family.

The class of mathematical structures used to model voting situations is called simple voting games (SVG). Formally,

Definition 1 *A simple voting game* (SVG) G *is a pair* $(N; V)$*, where* N *is the set of voters, and* $V : \mathcal{P}(N) \longrightarrow \{0, 1\}$ *is the characteristic function satisfying the following conditions:*

[C1] $V(\phi) = 0$

[C2] $V(N) = 1$

[C3] $S \subset T \Rightarrow V(S) \leq V(T)$

The above definition formalizes the idea of a decision making committee in which decisions are made by vote. The decision making rule is embodied in the characteristic function V. A coalition $S \in \mathcal{P}(N)$ is said to be a *winning* (losing) coalition *if and only if* $V(S) = 1$ ($V(S) = 0$). We denote the set of winning coalitions of the game G by $\mathcal{W}$. **[C1]** says that if nobody votes in favor of the bill, the bill should be rejected. On the other hand, if everybody votes "*yes*", the bill should be passed ([**C2**]). [**C3**] is a monotonicity requirement which says that more support for a bill cannot hurt the prospects of the bill. A coalition $X \in \mathcal{W}$ is said to be a *minimal winning coalition* if no proper subset of it is winning, i.e., if for any $X' \subset X$, $X' \notin \mathcal{W}$. We denote the set of minimal winning coalitions in G by $\mathcal{W}^{\min}$. Because of the monotonicity requirement [**C3**] on the decision rule, one can see that the family of winning coalitions $\mathcal{W}$, can be totally characterized by the set $\mathcal{W}^{\min}$. Furthermore, since the set $\mathcal{W}^{\min}$ is determined by V, we can say that $\mathcal{W}^{\min}$ reflects the decision rule.

Given a coalition $X \in \mathcal{W}$, a player $i \in X$ is said to be a *critical defector* in X if $X \backslash \{i\} \notin \mathcal{W}$. It therefore follows that given a coalition $X \in \mathcal{W}^{\min}$, every player belonging to X is a critical defector, since it can render the coalition losing by leaving it. A player $i \in N$ is a *dummy player* if he/she does not belong to any minimal winning coalition. It is obvious that the way a dummy player votes is inconsequential to the final outcome of the voting process. Sometimes, in the context of constitutional power, dummy players may arise inadvertently. For example, the allocation of weights in the original six member Council of Ministers of the European Union made Luxembourg a dummy player under the voting rule during 1958-72 (see Felsenthal and Machover (2001)). On the other hand, a player $i \in N$ is called a *veto player* if he/she is a member of every minimal winning coalition. Thus, a veto player can prevent the adoption of a bill unilaterally (irrespective of how others vote) by voting "*no*". Examples of veto players abound in the real world. The five permanent members of the United Nations Security Council are veto players.

Next we define the *desirability* or the *influence relation* introduced by Isbell (1958).

Definition 2 *Consider an SVG and two players i, $j \in N$. Let $i \succeq_D j$ if and only if $\forall X \subseteq N \backslash \{i, j\}$, $X \cup \{j\} \in \mathcal{W} \Rightarrow X \cup \{i\} \in \mathcal{W}$. Then $\succeq_D$ is a preordering called the desirability relation. $i \succ_D j$ if and only if $i \succeq_D j$ and $j \not\succeq_D i$. $i \approx_D j$ if and only if $i \succeq_D j$ and $j \succeq_D i$.*

$\succ_D$ is called the *strict* desirability relation and $\approx_D$ is the *equi*-desirability relation. We say that a player i is *desirable over j as a coalition partner* for $X \subseteq N \backslash \{i, j\}$ if $X \cup \{i\} \in \mathcal{W}$ but $X \cup \{j\} \notin \mathcal{W}$.

Remark 1 *It easily follows from the above definition that given an SVG and two voters $i, j \in N$, if $i \succ_D j$, then there must exist a coalition $X \subseteq N \backslash \{i, j\}$ such that $X \cup \{i\} \in \mathcal{W}^{\min}$ but $X \cup \{j\} \notin \mathcal{W}^{\min}$. This is because if $i \succ_D j$, then there exists a coalition $X \subseteq N \backslash \{i, j\}$ such that $X \cup \{i\} \in \mathcal{W}$ but $X \cup \{j\} \notin \mathcal{W}$. Suppose that $X \cup \{i\} \notin \mathcal{W}^{\min}$. Then there must be a coalition $X' \subset X \cup \{i\}$, such that $X' \in \mathcal{W}^{\min}$. Furthermore, we must have $i \in X'$, otherwise* *it contradicts the fact that* $X \notin \mathcal{W}$*. Thus* X' *is a minimal winning coalition containing* i *but not* j*. However, since* $X \cup \{j\} \notin \mathcal{W}$*, we cannot have* $X' \backslash \{i\} \cup \{j\} \in \mathcal{W}^{\min}$.

Definition 3 *An* SVG *is swap robust if for any two coalitions* $X, Y \in \mathcal{W}$*, and* $i \in X \setminus Y$*,* $j \in Y \setminus X$*, either* $X \setminus \{i\} \cup \{j\} \in \mathcal{W}$ *or* $Y \setminus \{j\} \cup \{i\} \in \mathcal{W}$*.*

The concept of swap robustness was introduced by Taylor and Zwicker (1993). An SVG is swap robust if one could swap two members in two winning coalitions and get at least one winning coalition as a result. Swap robust games, also known in the literature as *complete or linear voting games,* is a big class of SVGs for which the influence relation defined above induces a complete preordering in the set of voters (see Taylor (1995)). For more on swap robust games, also see Taylor and Zwicker (1999), Carreras and Freixas (1996), Freixas and Pons (2008), Freixas and Molinero (2009).

3 The desirability relation and winning coalitions

Let us begin this section by an example. Consider a simple game G on the voter set $\{a, b, c, d, e\}$, where the decision rule results in the following set of minimal winning coalitions: $\mathcal{W}^{\min} = \{\{a, b\}, \{a, c, d\}, \{a, c, e\}, \{b, c, d, e\}\}$. It can be verified that the game is swap robust.

In order to determine the hierarchy of the players in terms of their constituional power using one of the classical indices, say the BC index, we would need to list the entire set of winning coalitions.

$W = \{\{a, b\}, \{a, c, d\}, \{a, c, e\}, \{b, c, d, e\}, \{a, b, c\}, \{a, b, d\}, \{a, b, e\}, \{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}, \{a, b, c, d, e\}\}$

Let us define the following set:

$C(i) = \{X \in \mathcal{W} : i \in X \text{ and } X \backslash \{i\} \notin \mathcal{W}\}$

That is, $C(i)$ is the set of all winning coalitions in which voter $i \in N$ is a critical defector.

The Banzhaf-Coleman index BC for a voter $i \in N$ is given by

$$BC_i = \frac{C(i)}{2^{n-1}}$$

Therefore, evaluating BC for each player in the given simple game we get, $BC_a = \frac{10}{16}$, $BC_b = \frac{6}{16}$, $BC_c = \frac{4}{16}$, $BC_d = \frac{2}{16}$, $BC_e = \frac{2}{16}$. Thus, we have the following hierarchy: $a \succ_D b \succ_D c \succ_D d \approx_D e$. According to us, the main disadvantage of using the power indices in order to determine the ranking among players lies in the fact that not only do we need to find *all* winning coalitions but also determine the number of winning coalitions in which a voter is a critical defector[3].

If on the other hand we use the desirability relation to reveal the hierarchy, we can restrict our attention to the set $\mathcal{W}^{\min}$ only. Since by replacing a by b in the coalition $\{a, c, d\}$, we fail to get a winning coalition, it means that there exists a situation in which a is more desirable as a coalition partner than b. Hence $a \succ_D b$. Since $\{a, b\} \in \mathcal{W}^{\min}$ but $\{a, c\} \notin \mathcal{W}$, we have $b \succ_D c$. Similarly, since $\{a, c, e\} \in \mathcal{W}^{\min}$ but $\{a, d, e\} \notin \mathcal{W}$, we have $c \succ_D d$. However, there does not exist a coalition in which d is more desirable than e, or where e is more desirable than d. Hence $d \approx_D e$. Thus we reveal the hierarchy as $a \succ_D b \succ_D c \succ_D d \approx_D e$. But this kind of pairwise comparison may prove cumbersome specially if the set $\mathcal{W}^{\min}$ is large.

Since, considering the set of minimal winning coalitions alone reduces a lot of work, we also restrict our attention to the set $\mathcal{W}^{\min}$. However, the question arises whether we can use the information provided by $\mathcal{W}^{\min}$ more efficiently in order to reveal the hierarchy of the players in a shorter time. In this paper we present a mechanism to determine the hierarchy that is decidedly simpler than the existing methods and is devoid of any algebraic computations.

Let us define another set:

$C(i; m) = \{X \in C(i) : |X| = m\}$, where m is an integer, $1 \leq m \leq n$.

That is, $C(i; m)$ is the set of all winning coalitions of size m, in which voter i is a critical defector.

Carreras and Freixas (2008) introduced the term *weak desirability relation* for the following:

Definition 4 *Consider an SVG and two players i, $j \in N$. Let $i \succeq_d j$ if and only if for every integer m, $1 \leq m \leq n$, $|C(i; m)| \geq |C(j; m)|$. Then $\succeq_d$ is a preordering called the weak desirability relation. $i \succ_d j$ if and only if $i \succeq_d j$ and $j \not\succeq_d i$. $i \approx_d j$ if and only if $i \succeq_d j$ and $j \succeq_d i$.*

$\succ_d$ is called the *strict* weak desirability relation and $\approx_d$ is the *weak equi*-desirability relation.

Diffo Lambo and Moulen (2002) established that the weak desirability preordering coincides with the desirability preordering if the game is swap robust. (They however did not

[3]Recently Kirsch and Langner (2009) have presented a combinatorial formula to calculate the SS and BC indices by using the set of minimal winning coalitions only.

use the term weak desirability, which was introduced later by Carreras and Freixas (2008)). Their result is formally stated without the proof as Theorem 3.1 below:

Theorem 3.1 *Consider a swap robust SVG and $i, j \in N$. Then the following two statements are equivalent:*

(*a*) $i \succeq_D j$
(*b*) *For every integer* m, $1 \leq m \leq n$, $|C(i;m)| \geq |C(j;m)|$, *that is,* $i \succeq_d j$.

Furthermore, the following two statements are also equivalent:

(*c*) $i \succ_D j$
(*d*) *For every integer* m, $1 \leq m \leq n$, $|C(i;m)| \geq |C(j;m)|$ *and there is at least one value of* m *for which* $|C(i;m)| > |C(j;m)|$.

If the above theorem keeps itself when we restrict our attention to the set of minimal winning coalitions, we might have an easier way of determining the hierarchy of players from the set $\mathcal{W}^{\min}$. Define the following sets:

- $C^{\min}(i) = \left\{X \in \mathcal{W}^{\min} : i \in X\right\}$
- $C^{\min}(i;m) = \left\{X \in C^{\min}(i) : |X| = m\right\}$, where m is an integer $1 \leq m \leq n$.

That is, $C^{\min}(i)$ is the set of all minimal winning coalitions to which voter $i \in N$ belongs. By the definition of a minimal winning coalition, i is a critical defector in each of the coalitions in $C^{\min}(i)$. Similarly, $C^{\min}(i;m)$ is the set of all minimal winning coalitions of size m of which voter i is a member. Given a swap robust game, if we can show that $i \succeq_D j$ is equivalent to saying that for every integer m, $1 \leq m \leq n$, $\left|C^{\min}(i;m)\right| \geq \left|C^{\min}(j;m)\right|$, we will have achieved our objective. However, it is not so. We find that for a swap robust SVG, and two voters i and j, $i \succeq_D j$ does not imply that for every integer m, $1 \leq m \leq n$, $\left|C^{\min}(i;m)\right| \geq \left|C^{\min}(j;m)\right|$. Furthermore, we know that if $i \succeq_D j$ then we must have $|C(i)| \geq |C(j)|$. However, $i \succeq_D j$ does not imply that $\left|C^{\min}(i)\right| \geq \left|C^{\min}(j)\right|$. If we go back to the example stated at the beginning of the section we have:

- $\left|C^{\min}(a;2)\right| = 1$, $\left|C^{\min}(a;3)\right| = 2$, $\left|C^{\min}(a;4)\right| = 0$
- $\left|C^{\min}(b;2)\right| = 1$, $\left|C^{\min}(b;3)\right| = 0$, $\left|C^{\min}(b;4)\right| = 1$
- $\left|C^{\min}(c;2)\right| = 0$, $\left|C^{\min}(c;3)\right| = 2$, $\left|C^{\min}(c;4)\right| = 1$
- $\left|C^{\min}(d;2)\right| = 0$, $\left|C^{\min}(d;3)\right| = 1$, $\left|C^{\min}(d;4)\right| = 1$

- $\left|C^{\min}(e;2)\right| = 0,\ \left|C^{\min}(e;3)\right| = 1,\ \left|C^{\min}(e;4)\right| = 1$

Thus $a \succ_D b$ but $\left|C^{\min}(a;4)\right| < \left|C^{\min}(b;4)\right|$. Similarly, $b \succ_D c$ but $\left|C^{\min}(b;3)\right| < \left|C^{\min}(c;3)\right|$. Also we have $b \succ_D c$ but $\left|C^{\min}(b)\right| < \left|C^{\min}(c)\right|$. Thus this example reveals that neither the value of $\left|C^{\min}(i)\right|$ nor that of $\left|C^{\min}(i;m)\right|$ says much about the relative position of player i in the hierarchy of players.

4 The desirability relation and minimal winning coalitions

Given an SVG, consider the set $\mathcal{W}^{\min}$ of all minimal winning coalitions. Let us partition the set as follows

$$\mathcal{W}^{\min} = \left\{\mathcal{W}^{\min}(k_1),\ \mathcal{W}^{\min}(k_2), ..., \mathcal{W}^{\min}(k_T)\right\}$$

where $\mathcal{W}^{\min}(k_t) = \{X \in \mathcal{W}^{\min} : |X| = k_t\}$. T denotes the number of different sizes of minimal winning coalitions and is a positive integer ≥ 1. k_t is an integer such that $n \geq k_t \geq 1$, for all $t \in \{1, ..., T\}$. We also have $k_1 < k_2 < ... < k_T$. Thus, k_1 denotes the size of the smallest minimal winning coalition and k_T is the size of the largest minimal winning coalition. In the example discussed in the previous section, $T = 3$, $k_1 = 2$, $k_2 = 3$ and $k_3 = 4$.

For every $i \in N$, let us consider the following vector that is defined by the cardinalities of the minimal winning coalitions to which it belongs:

$$M(i) = (c_1(i), c_2(i), ..., c_T(i)),\ \text{where } c_t(i) = \left|C^{\min}(i;k_t)\right|, 1 \leq t \leq T.$$

It is obvious that $c_t(i) \geq 0\ \forall t \in \{1, 2, ..., T\}$. Thus $M(i) \in (\mathbb{N} \cup \{0\})^T$, where $\mathbb{N}$ denotes the set of natural numbers. The t^{th} coordinate of the vector $M(i)$ is the number of minimal winning coalitions of size k_t of which voter i is a member. If $i \in N$ is a dummy player, then $M(i) = (0, 0, ..., 0)$. If $i \in N$ is a veto player, then $M(i) = (w_1, w_2, ..., w_T)$, where $w_t = \left|\mathcal{W}^{\min}(k_t)\right|$, $\forall t \in \{1, 2, ..., T\}$.

Definition 5 *Consider an SVG and any two players $i,\ j \in N$. Let $M(i)$ and $M(j) \in (\mathbb{N} \cup \{0\})^T$ be the associated vectors. Then $M(i) = M(j)$ if and only if $c_t(i) = c_t(j)$ $\forall t \in \{1, 2, ..., T\}$.*

Definition 6 *Consider an SVG and any two players $i,\ j \in N$. Let $M(i)$ and $M(j) \in (\mathbb{N} \cup \{0\})^T$ be the associated vectors. Then $M(i)$ is said to dominate $M(j)$ if and only if*

$M(i)$ lies above $M(j)$ in lexicographic ordering, that is, there exists a $\bar{t} \in \{1, 2, ..., T\}$ such that $c_{\bar{t}}(i) > c_{\bar{t}}(j)$ and $c_t(i) = c_t(j)$ $\forall t < \bar{t}$. We denote this by $M(i) >_L M(j)$.

We write $M(i) \geq_L M(j)$ if either $M(i) = M(j)$ or $M(i) >_L M(j)$.

Next we present a proposition which is crucial in proving Theorem 4.2 below.

Proposition 4.1 *Consider a swap robust SVG and any two players $i, j \in N$. Let $M(i), M(j) \in (\mathbb{N} \cup \{0\})^T$ be the associated vectors. Let h $(1 \leq h \leq T)$ be the maximum value of t such that $c_t(i) = c_t(j)$ $\forall t \in \{1, 2, ..., h\}$. Then $\forall X \in C^{\min}(i; k_t)$, such that $j \notin X$ and $\forall t \in \{1, 2, ..., h\}$, we have $X \backslash \{i\} \cup \{j\} \in C^{\min}(j; k_t)$. Also $\forall Y \in C^{\min}(j; k_t)$, such that $i \notin Y$ and $\forall t \in \{1, 2, ..., h\}$, we have $Y \backslash \{j\} \cup \{i\} \in C^{\min}(i; k_t)$.*

Proof. We will prove this by induction on t. Since the SVG is swap robust, let us assume without loss of generality that $i \succeq_D j$.

Step 1: Take a coalition $Y_1 \in C^{\min}(j; k_1)$, where k_1 is the size of the smallest minimal winning coalition. We can assume that $c_1(j) \neq 0$. Now, i may or may not belong to Y_1. If $i \in Y_1$, then we have $Y_1 \in C^{\min}(i; k_1)$. Suppose $i \notin Y_1$. Then, since $i \succeq_D j$, $Y_1 \backslash \{j\} \cup \{i\} \in \mathcal{W}$ and i is a critical defector in it. However, $Y_1 \backslash \{j\} \cup \{i\}$ may or may not be a minimal winning coalition. If $Y_1 \backslash \{j\} \cup \{i\} \in \mathcal{W}^{\min}$, then $Y_1 \backslash \{j\} \cup \{i\} \in C^{\min}(i; k_1)$. But suppose that $Y_1 \backslash \{j\} \cup \{i\} \notin \mathcal{W}^{\min}$. This means there exists a coalition $Y_1' \subset Y_1 \backslash \{j\}$ such that $Y_1' \cup \{i\} \in \mathcal{W}^{\min}$. However it is obvious that $|Y_1' \cup \{i\}| < k_1$. This contradicts the fact that k_1 is the size of the smallest minimal winning coalition. Therefore, we can construct an injective mapping

$$\varphi_1 : C^{\min}(j; k_1) \longrightarrow C^{\min}(i; k_1)$$

where

$$\varphi_1(Y) = \begin{cases} Y \backslash \{j\} \cup \{i\} \text{ if } i \notin Y \\ Y \text{ otherwise} \end{cases}.$$

This together with the fact that $c_1(i) = c_1(j)$ implies that the mapping is bijective, that is, $\forall X \in C^{\min}(i; k_1)$, such that $j \notin X$ we have $X \backslash \{i\} \cup \{j\} \in C^{\min}(j; k_1)$.

Step 2: Now consider the family $C^{\min}(j; k_2)$. Take a coalition $Y_2 \in C^{\min}(j; k_2)$. If $i \in Y_2$, then we have $Y_2 \in C^{\min}(i; k_2)$. Suppose $i \notin Y_2$. Since by assumption $i \succeq_D j$, we must have $Y_2 \backslash \{j\} \cup \{i\} \in \mathcal{W}$ and i is a critical defector in it. However, $Y_2 \backslash \{j\} \cup \{i\}$ may or may not be a minimal winning coalition. If $Y_2 \backslash \{j\} \cup \{i\} \in \mathcal{W}^{\min}$, then $Y_2 \backslash \{j\} \cup \{i\} \in C^{\min}(i; k_2)$. Now suppose that $Y_2 \backslash \{j\} \cup \{i\} \notin \mathcal{W}^{\min}$. This means there exists a coalition $Y_2' \subset Y_2 \backslash \{j\}$ such that $Y_2' \cup \{i\} \in \mathcal{W}^{\min}$. However it is obvious that $|Y_2' \cup \{i\}| < k_2$. This means $|Y_2' \cup \{i\}| = k_1$, that is, $Y_2' \cup \{i\} \in C^{\min}(i; k_1)$. But we have already shown in step 1 that $\forall X \in C^{\min}(i; k_1)$, such that $j \notin X$ we have $X \backslash \{i\} \cup \{j\} \in C^{\min}(j; k_1)$ This implies that $Y_2' \cup \{j\} \in C^{\min}(j; k_1)$

which in turn contradicts that $Y_2 \in \mathcal{W}^{\min}$. Therefore, we can again construct an injective mapping

$$\varphi_2 : C^{\min}(j; k_2) \longrightarrow C^{\min}(i; k_2)$$

where

$$\varphi_2(Y) = \begin{cases} Y \backslash \{j\} \cup \{i\} \text{ if } i \notin Y \\ Y \text{ otherwise} \end{cases}.$$

This together with the fact that $c_2(i) = c_2(j)$ gives us that $\forall X \in C^{\min}(i; k_2)$, such that $j \notin X$ we have $X \backslash \{i\} \cup \{j\} \in C^{\min}(j; k_2)$.

Step 3: Consider the family $C^{\min}(j; k_3)$. Take a coalition $Y_3 \in C^{\min}(j; k_3)$. If $i \in Y_3$, then we must have $Y_3 \in C^{\min}(i; k_3)$. If $i \notin Y_3$, we must have $Y_3 \backslash \{j\} \cup \{i\} \in \mathcal{W}$ and i must be a critical defector in it. However, $Y_3 \backslash \{j\} \cup \{i\}$ may or may not be a minimal winning coalition. If $Y_3 \backslash \{j\} \cup \{i\} \in \mathcal{W}^{\min}$, then $Y_3 \backslash \{j\} \cup \{i\} \in C^{\min}(i; k_3)$. Now suppose that $Y_3 \backslash \{j\} \cup \{i\} \notin \mathcal{W}^{\min}$. Then there exists a coalition $Y_3' \subset Y_3 \backslash \{j\}$ such that $Y_3' \cup \{i\} \in \mathcal{W}^{\min}$. However it implies that $|Y_3' \cup \{i\}| < k_3$. This means $|Y_3' \cup \{i\}|$ is equal to either k_1 or k_2, that is, either $Y_3' \cup \{i\} \in C^{\min}(i; k_1)$ or $Y_3' \cup \{i\} \in C^{\min}(i; k_2)$. But we have already shown in steps 1 and 2 above that $\forall X \in C^{\min}(i; k_t)$, such that $j \notin X$ we have $X \backslash \{i\} \cup \{j\} \in C^{\min}(j; k_t)$, $t = 1, 2$. This implies that either $Y_3' \cup \{j\} \in C^{\min}(j; k_1)$ or $Y_3' \cup \{j\} \in C^{\min}(j; k_2)$. This contradicts that $Y_3 \in \mathcal{W}^{\min}$. Therefore, we can again construct an injective mapping

$$\varphi_3 : C^{\min}(j; k_3) \longrightarrow C^{\min}(i; k_3)$$

where

$$\varphi_3(Y) = \begin{cases} Y \backslash \{j\} \cup \{i\} \text{ if } i \notin Y \\ Y \text{ otherwise} \end{cases}$$

This combined with the fact that $c_3(i) = c_3(j)$ implies that $\forall X \in C^{\min}(i; k_3)$, such that $j \notin X$ we have $X \backslash \{i\} \cup \{j\} \in C^{\min}(j; k_3)$.

Step 4: Let us now assume that the result holds for all values of $t = 1, 2, ..., h - 1$. We will now prove that the result holds for $t = h$ too. Consider a coalition $Y_h \in C^{\min}(j; k_h)$. If $i \in Y_h$, then we have $Y_h \in C^{\min}(i; k_h)$. But suppose that $i \notin Y_h$. Again by the assumption that $i \succeq_D j$, we must have $Y_h \backslash \{j\} \cup \{i\} \in \mathcal{W}$ and i must be a critical defector in it. However, $Y_h \backslash \{j\} \cup \{i\}$ may or may not be a minimal winning coalition. If $Y_h \backslash \{j\} \cup \{i\} \in \mathcal{W}^{\min}$, then $Y_h \backslash \{j\} \cup \{i\} \in C^{\min}(i; k_h)$. If $Y_h \backslash \{j\} \cup \{i\} \notin \mathcal{W}^{\min}$, it means there must exist a coalition $Y_h' \subset Y_h \backslash \{j\}$ such that $Y_h' \cup \{i\} \in \mathcal{W}^{\min}$. However it is obvious that $|Y_h' \cup \{i\}| < k_h$. Without loss of generality we suppose that $|Y_h' \cup \{i\}| = k_t$, that is, $Y_h' \cup \{i\} \in C^{\min}(i; k_t)$ for some $1 \leq t < h$, (say) t'. But since we have assumed at the beginning of step 4 that the result holds for $1 \leq t < h$, we have that $\forall X \in C^{\min}(i; k_{t'})$, such that $j \notin X$, $X \backslash \{i\} \cup \{j\} \in C^{\min}(j; k_{t'})$.

This implies that $Y_h' \cup \{j\} \in C^{\min}(j; k_{t'})$ which in turn contradicts that $Y_h \in \mathcal{W}^{\min}$. Therefore, we can again construct an injective mapping

$$\varphi_h : C^{\min}(j; k_h) \longrightarrow C^{\min}(i; k_h)$$

for $t = h$ too, where

$$\varphi_h(Y) = \begin{cases} Y \backslash \{j\} \cup \{i\} \text{ if } i \notin Y \\ Y \text{ otherwise} \end{cases}.$$

Since we also have $c_h(i) = c_h(j)$, it must be true that $\forall X \in C^{\min}(i; k_h)$, such that $j \notin X$ we have $X \backslash \{i\} \cup \{j\} \in C^{\min}(j; k_h)$. Hence the proof of the proposition. ■

Remark 2 *Note that there may be some values of t, $1 \leq t \leq h$, such that $c_t(i) = c_t(j) = 0$. This simply means that the players i and j do not belong to any minimal winning coalition of size t. We intentionally ignore such values of t. It is easy to see that the inclusion of these cases does not change the result of the proposition.*

We will now use the result of the above proposition to show that in a linear voting game, two voters i and j are equally influential if and only if the associated vectors $M(i)$ and $M(j)$ are equal to each other. Furthermore, i is strictly more influential than j if and only if $M(i)$ dominates $M(j)$ in lexicographic ordering. We ignore the case of dummies from the analysis since by definition a dummy player has *no* influence over the decision making process.

Theorem 4.2 *Let $G = (N; V)$ be a swap robust SVG and $i, j \in N$. Also let $M(i)$ and $M(j)$ be the associated vectors. Then the following two statements are equivalent:*

(a) $i \approx_D j$
(b) $M(i) = M(j)$

Furthermore, the following two statements are also equivalent:
(c) $i \succ_D j$
(d) $M(i) >_L M(j)$

Proof. It is easy to verify that $(a) \Rightarrow (b)$.

$(b) \Rightarrow (a)$: By the virtue of Proposition 4.1 we know that if h is the maximum value of t such that $c_t(i) = c_t(j)$ $\forall t \in \{1, 2, ...h\}$, then for all coalitions $X \subseteq N \backslash \{i, j\}$, $X \cup \{i\} \in C^{\min}(i; k_t) \Leftrightarrow X \cup \{j\} \in C^{\min}(j; k_t)$. If $i \succ_D j$, then there must exist a coalition $X \subseteq N \backslash \{i, j\}$ such that $X \cup \{i\} \in \mathcal{W}^{\min}$ but $X \cup \{j\} \notin \mathcal{W}^{\min}$. Since there does not exist any such coalition, we must have $i \approx_D j$.

$(c) \Rightarrow (d)$: We will prove this by contradiction. Since $(a) \Leftrightarrow (b)$, we cannot have $M(i) \neq M(j)$. Therefore suppose $M(j) >_L M(i)$. Also let h be that value of t such that $c_h(i) < c_h(j)$ and $c_t(i) = c_t(j)$, $1 \leq t < h$. Take a coalition $Y \in C^{\min}(j; k_h)$. If $i \in Y$, then we have $Y \in C^{\min}(i; k_h)$. Suppose $i \notin Y$. Then, since $i \succ_D j$, we must have $Y \backslash \{j\} \cup \{i\} \in \mathcal{W}$ and i must be a critical defector in it. However, $Y \backslash \{j\} \cup \{i\}$ may or may not be a minimal winning coalition. If $Y \backslash \{j\} \cup \{i\} \in \mathcal{W}^{\min}$, then $Y \backslash \{j\} \cup \{i\} \in C^{\min}(i; k_h)$. Since, $c_h(j) > c_h(i)$, there must exist at least one coalition $Y \in C^{\min}(j; k_h)$, $i \notin Y$ such that $Y \backslash \{j\} \cup \{i\} \notin \mathcal{W}^{\min}$. That means there exists a coalition $Y' \subset Y \backslash \{j\}$ such that $Y' \cup \{i\} \in \mathcal{W}^{\min}$. It is obvious that $|Y' \cup \{i\}| < k_h$. Therefore $Y' \cup \{i\} \in C^{\min}(i; k_t)$ for some $t \in \{1, 2 \ldots h-1\}$. But by Proposition 4.1 we know that for all coalitions $X \in C^{\min}(i; k_t)$, such that $j \notin X$ we have $X \backslash \{i\} \cup \{j\} \in C^{\min}(j; k_t)$, $1 \leq t < h$. This in turn implies that $Y' \cup \{j\} \in C^{\min}(j; k_t)$ which contradicts that $Y \in \mathcal{W}^{\min}$.

$(d) \Rightarrow (c)$: Contrary to the claim let us assume that $j \succ_D i$ (since we cannot have $i \approx_D j$). Let h be that value of t such that $c_h(i) > c_h(j)$ and $c_t(i) = c_t(j)$, $1 \leq t < h$. Using the same reasoning as above we know that there must exist at least one coalition $X \in C^{\min}(i; k_h)$, $j \notin X$ such that $X \backslash \{i\} \cup \{j\} \notin \mathcal{W}^{\min}$. That is, there exists a coalition $X' \subset X \backslash \{i\}$ such that $X' \cup \{j\} \in \mathcal{W}^{\min}$. It is obvious that $|X' \cup \{j\}| < k_h$. Therefore $X' \cup \{j\} \in C^{\min}(j; k_t)$ for some $t \in \{1, 2 \ldots h-1\}$. But by Proposition 4.1 we know that for all coalitions $Y \in C^{\min}(j; k_t)$, such that $i \notin Y$ we have $Y \backslash \{j\} \cup \{i\} \in C^{\min}(i; k_t)$, $1 \leq t < h$. This in turn implies that $X' \cup \{i\} \in C^{\min}(i; k_t)$ which contradicts that $X \in \mathcal{W}^{\min}$. ■

Now reconsider the example of section 3. $M(a) = (1, 2, 0)$, $M(b) = (1, 0, 1)$, $M(c) = (0, 2, 1)$, $M(d) = (0, 1, 1)$, $M(e) = (0, 1, 1)$. Since we can easily see that $M(a) >_L M(b) >_L M(c) >_L M(d) = M(e)$, we have $a \succ_D b \succ_D c \succ_D d \approx_D e$.

5 Conclusion

In this paper, we present an efficient method of determining the hierarchy of players who participate in a decision making process, in terms of their a-priori influence over the collective action of the voting body. This method is notably simpler than the other existing mechanisms. We show that given a swap robust voting game, a player i is more desirable or influential than another player j if the vector $M(i)$, that is defined by the cardinalities of the minimal winnig coalitions to which i belongs, lexicographically dominates the vector $M(j)$. Using this result one can very easily establish the ranking of the players without any complicated algebraic computations.

References

[1] Banzhaf, J. F., 1965. Weighted voting doesn't work: a mathematical analysis. *Rutgers Law Review* 19, 317-343.

[2] Carreras, F. and Freixas, J., 1996. Complete simple games. *Mathematical Social Sciences* 32, 139-155.

[3] Carreras, F. and Freixas, J., 2008. On ordinal equivalence of power measures given by regular semivalues. *Mathematical Social Sciences* 55, 221-234.

[4] Coleman, J. S., 1971. Control of collectivities and the power of a collectivity to act. In: Leberman, B., (Ed.), Social Choice. Gordon and Breach, New York.

[5] Diffo Lambo, L., Moulen, J., 2002. Ordinal equivalence of power notions in voting games, *Theory and Decision* 53, 313-325.

[6] Felsenthal, D. S., Machover, M., 1998. The measurement of voting power: theory and practice, problems and paradoxes, Edward Elgar, Cheltenham.

[7] Felsenthal, D. S., Machover, M., 2001. The Treaty of Nice and Qualified majority voting, *Social Choice and Welfare* 18, 431-464.

[8] Freixas, J., and Pons, M., 2008. Hierarchies achievable in simple games. *Theory and Decision*, DOI: 10.1007/s11238-008-9108-0.

[9] Freixas, J., and Molinero, X., 2009. On the existence of a minimum integer representation for weighted voting systems. *Annals of Operations Research* 166, 243-260.

[10] Isbell, J. R., 1956. A class of majority games, Quarterly Journal of Mathematics Oxford Ser. 7, 183-187.

[11] Isbell, J. R., 1958. A class of simple games, *Duke Mathematical Journa*l 25, 423-439.

[12] Kirsch, W., Langner, J., 2009. Power indices and minimal winning coalitions, *Social Choice and Welfare*, DOI: 10.1007/s00355-009-0387-3.

[13] Saari, D.G., Sieberg, K. K., 2001. Some surprising properties of power indices, *Games and Economic Behavior* 36, 241-263.

[14] Shapley L. S. and Shubik, M., 1954. A method for evaluating the distribution of power in a committee system, *American Political Science Review* 48, 787-792

[15] Taylor, A. D., Zwicker, W. S., 1993. Weighted voting, multicameral representation, and power, *Games and Economic Behavior* 5, 170-181.

[16] Taylor, A.D., 1995. Mathematics and Politics, Springer, Berlin.

[17] Taylor, A.D., Zwicker, W.S., 1999. Simple Games, Princeton University Press, Princeton, NJ.

[18] Tomiyama, Y., 1987. Simple game, voting representation and ordinal power equivalence, *International Journal on Policy and Information* 11, 67-75.