Confidence in preferences*

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Abstract

Indeterminate preferences have long been a tricky subject for choice theory. One reason for which preferences may be less than fully determinate is the lack of confidence in one's preferences. In this paper, a representation of confidence in preferences is proposed. It is used to develop an account of the role which confidence which rests on the following intuition: the more important the decision to be taken, the more confidence is required in the preferences needed to take it. An axiomatisation of this choice rule is proposed. This theory provides a natural account of when an agent should defer a decision; namely, when the importance of the decision exceeds his confidence in the relevant preferences. Possible applications of the notion of confidence in preferences to social choice are briefly explored.

Keywords: Incomplete preference; Revealed preference; Confidence in preferences; Deferral of decisions; Importance of decisions; Social choice

JEL classification: D01, D71.

Under the standard economic model, a rational agent's preferences can be represented by a complete order on the alternatives; but this has been famously and repeatedly challenged. Preferences may be fuzzy, imprecise or vague (Aumann, 1962; Salles,

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1998). Preferences may be incomplete because the agent has not yet settled on the preferences which he deems appropriate, perhaps due to unresolved conflict (Levi, 1986). Or still, preferences may be incomplete because the agent does not see that some options will ever be comparable: after all, there is no reason to always expect them to be (Sen, 1997). From both a descriptive or a normative point of view, the assumption of completeness or determinacy of preferences is highly questionable.

We consider here the case of choice under certainty; the agent will be assumed to know consequences of choosing each of the alternatives, and there will be no question of beliefs or probabilities over the relevant "states". The only relevant attitude is the agent's preferences (which, as standard, are taken to be subjective). If an agent settles on a preference for one alternative over another (or decides on determinate indifference between the alternatives), we will sometimes say that he has emitted a *value judgement*: a judgement about the relative value of the alternatives for him. In situations of choice under certainty, the agent's choices are standardly taken to be guided entirely by his preferences, or, to put the same point in other terms, by his value judgements. Inversely, his preferences are traditionally taken to be derivable from his choices.

Many of the challenges to the standard model evoked above relate to the fact that agents do not always endorse clear, categorical value judgements on every pair of alternatives. One intuitive reason for this, which has been hardly emphasised though tacitly evoked at times in the literature, is that people often have differing degrees of *confidence* in their value judgements. Sometimes, they are not sure which of the alternatives is best (by their own lights). Consider moral dilemmas: an agent might be confident that it is better to sacrifice the life of one to save the lives of a hundred than not to; although he thinks that it is better to sacrifice the life of one to save the lives of five others than not to, he may be less confident in this value judgement; finally, he may be totally unsure about whether it is better to sacrifice the life of a gifted musician for that of a talented economist or not. The goal of this paper is to get a grip on the intuitive notion of *confidence in one's preferences*.

To achieve this goal, we first propose a representation of confidence in preferences (Section 1.1 and an account of its role in choice (Section 1.2). Although they may turn out to be descriptively valid, the focus is normative: admitting that it is rational to have different levels of confidence in one's preferences, the goal is say something about what sorts of confidence one can allow oneself to have and on the role confidence should play in choice. In Section 2, an axiomatisation of the notion of choice on the basis of confidence in preferences will be given. Under the proposal, confidence will be related to two aspects of choice situations, which though apparently relevant in many cases, have received little attention in the choice-theoretic literature to date: the importance of a decision to be taken, and the question of when and whether to defer the decision. In Section 3, we will discuss these two issues in detail. In Section 4, we turn to the possible application of the notion of confidence in social choice, attempting a preliminary investigation into the question, and proposing a social choice rule which takes into account the voters' confidence in their preferences. Proofs are relegated to the Appendix.

1 Preference, confidence and choice

1.1 Representing confidence in preferences

Let X be a finite set of alternatives, with at least three members. Henceforth, we use the generic terms x, y and so on to refer to elements of X, the generic terms S, T and so on to refer to subsets of X. A weak ordering on a set is a complete, reflexive, transitive relation on that set. The standard model represents preferences by weak orderings on the set of alternatives X. Let $\mathcal{P}$ be the set of weak orderings on X; we use the generic terms R, R_i, and so on to refer to elements of $\mathcal{P}$ and the generic term $\mathcal{R}$ to refer to subsets of $\mathcal{P}$. The generated strict ordering and indifference relation are defined as standard.

Weak orderings represent determinate preferences: for each pair of alternatives, either the agent strictly prefers one to the other or is determinately indifferent. The most common way of representing the preferences of someone who has not made up his mind about whether one alternative is preferred to another or judges them to be incomparable is by weakening the completeness assumption (Sen, 1970, 1997). Reflexive, transitive relations which do not necessarily satisfy completeness are called quasi-orderings. If Q is a quasi-ordering, then there are alternatives x and y such that neither xQy nor yQx; these are cases where the agent does not have any determinate preference – including determinate indifference – between the alternatives x and y. In other words, he does not endorse any value judgement concerning the comparison between x and y.

This is however not the only way to represent an agent who does not have determinate preferences over all pairs of alternatives. Another possibility is to use *sets of* *weak orderings*.[1] For a set of weak orderings $\mathcal{R}$, there may be alternatives z and w such that zRw for all $R \in \mathcal{R}$; in this case, the agent has a determinate weak preference for z over w. By contrast, there may be alternatives x and y such that neither xRy for all $R \in \mathcal{R}$ nor yRx for all $R \in \mathcal{R}$. This represents an agent who does not have any determinate preference over x and y; he endorses no value judgement concerning the comparison between these alternatives.

The representation by sets of weak orderings is strictly more expressive than the representation by a quasi-ordering in the following sense: to each set of weak orderings is associated a unique quasi-ordering which represents the same preferences, but to each quasi-ordering one may associate more than one corresponding set of weak orderings. As regards the first point, given a set of weak orderings $\mathcal{R}$, define the quasi-ordering Q as follows: for all alternatives x, y, xQy if and only if xRy for all $R \in \mathcal{R}$. It is straightforward to see that Q is a quasi-ordering and that Q and $\mathcal{R}$ represent the same preferences: the agent has weak preference, strict preference, indifference or indeterminacy according to one if and only if he does according to the other. By contrast, Figure 1 shows two different sets of weak orderings, both of which correspond to the empty quasi-ordering (for all x, y, neither xQy nor yQx); this illustrates the fact that there may be no unique set corresponding to a given quasi-ordering. One can regain uniqueness by adding a constraint on the set of orderings. We say that a set of weak orderings $\mathcal{R}$ is *tight* if, for any weak ordering R, $R \in \mathcal{R}$ if, for all alternatives x and y, xRy if $xR'y$ for all $R' \in \mathcal{R}$. This condition basically says that, if an ordering R agrees with what all orderings in $\mathcal{R}$ have in common, then R is in $\mathcal{R}$. It is straightforward to check that to each quasi-ordering Q one can associate a unique tight set of weak orderings; namely, the set containing all weak orderings R such that xRy if xQy, for all alternatives x and y.[2]

Both of these representations can be interpreted as representations of the agent's confidence in his preferences: he is confident in his preference for x over y if xQy or if xRy for all $R \in \mathcal{R}$; he has no preference concerning x and y in which he is confident if neither xQy nor yQx, or it is not the case that xRy for some $R \in \mathcal{R}$ and it is not

[1] This method is related to that proposed in Levi (1986).

[2] The "expressivity" of the notion of quasi-ordering is more appropriate for choice theory, since the information given by a choice function (under the appropriate axiomatisation) is only sufficient to pick out a unique quasi-ordering. To pick out a unique set of weak orderings, more "Boolean" information about preferences is required (for example: a is preferred to b *if* a is preferred to c). To stay closer to the traditional framework of choice theory, throughout this paper we work with the expressiveness corresponding to quasi-ordering; correspondingly, everything done with sets of weak orderings will be unique only up to tightness of the sets.

Figure 1: Two sets of ordering corresponding to the same quasi-ordering

a	d		a	b	d	c
b	c		b	a	c	d
c	b		c	d	b	a
d	a		d	c	a	b

the case that yRx for other $R \in \mathcal{R}$. As a representation of the agent's confidence in his preferences, these proposals have an evident defect: they are binary. Either the agent is completely confident in a value judgement concerning two alternatives, or he is completely unsure about any value judgement concerning the two alternatives in question.

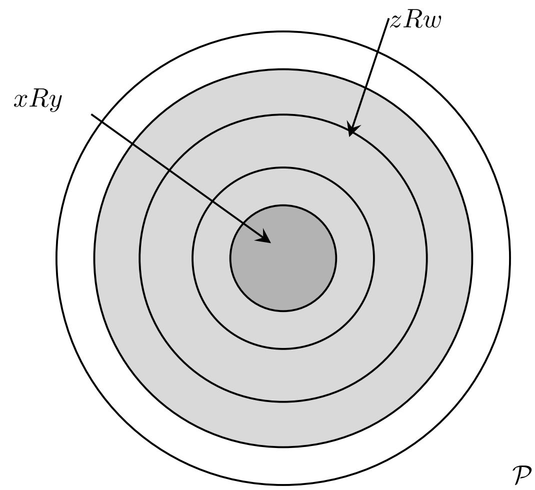
In reality, it seems that one can, rationally, have different degrees of confidence in one's preferences or value judgements. Take the example of moral dilemmas. An agent may be pretty confident that it is better to sacrifice the life of one to save the lives of a thousand than to let the thousand perish. He also thinks that it is better to sacrifice the life of one to save the lives of a ten than not to; but he is less confident in the latter value judgement. And he is more confident in that judgement than in the following judgement which he still, perhaps cautiously, endorses: that it is better to sacrifice the life of one "ordinary" person for the lives of ten petty criminals than not to. There thus appear to be degrees of confidence in one's value judgements or preferences; a model of confidence in preferences should be able to account for this.

One way to do this would be to introduce fuzzy preference relations, which do admit "degrees" of the preference. These are generally interpreted as representing the imprecision or ambiguity of the preference relation (Salles, 1998), and it is unclear what the relationship between imprecision and confidence is (see for example Keefe and Smith (1996)). We shall explore another way, which explicitly builds on the second representation presented above: instead of representing preferences by a set of weak orderings, we use a *nested family* of weak orderings.[3]

Let Ξ be a nested family of subsets of $\mathcal{P}$. Ξ represents confidence in preferences in the following way. If there is a set of weak orderings $\mathcal{R} \in \Xi$ such that xRy for all $R \in \mathcal{R}$, then the agent (weakly) prefers x to y. But he may not be very confident in

[3]That is, a set of sets of weak orderings such that, for each pair of distinct sets, one is strictly contained in the other.

Figure 2: Implausibility on the set of orderings



this value judgement: his confidence in the judgement is captured by the size of the biggest set $\mathcal{R}'$ in Ξ such that xRy for all $R \in \mathcal{R}'$. So he is at least as confident that z is better than w (by his lights) than that x is better than y if for every set of weak orderings $\mathcal{R} \in \Xi$ such that xRy for all $R \in \mathcal{R}$, zRw for all $R \in \mathcal{R}$; and he is more confident in the former value judgement if there is a set $\mathcal{R}' \in \Xi$ such that zRw for all $R \in \mathcal{R}'$ but there are some $R' \in \mathcal{R}'$ such that it is not the case that $xR'y$.

Figure 2 illustrates the idea diagrammatically. The plane is the set of weak orderings: the points are weak orderings, so for each point and for every pair of alternatives, the alternatives are ordered one way or another according to the weak ordering corresponding to that point. The (filled) circles represent the sets in the nested family of sets representing confidence in preference; the fact that a value judgement holds in a circle means that it holds for all points (weak orderings) in that circle. Finally, the fact that a value judgement holds in a bigger circle than another represents the fact that confidence in the former is higher than confidence in the latter.

It is evident from the diagram that to any nested family of sets of weak orderings in $\mathcal{P}$ there corresponds a unique weak ordering *on* the set $\mathcal{P}$ of weak orderings on X. For weak orderings R and R' on X, R is lower than R' according to the weak ordering on $\mathcal{P}$ if the smallest set in the nested family containing R' contains the smallest set in the nested family containing R'.[4] Intuitively, the order represents how *implausible* the weak orderings are as candidates for the "right" notion of preference (by the agent's lights): the higher a weak ordering is on the order, the "farther out" it is on the diagram in Figure 2, and the less the agent he feels that he has to consider it as a "right" reflection of his preferences. Implausibility is a sort of dual notion to confidence: the agent is more confident in a value judgement if it holds for all weak orderings up to a higher level of implausibility, and conversely, a highly implausible weak ordering will only be taken into account if the agent demands a high level of confidence. This leads to the following representation of confidence in preferences.

Definition 1.1. An *implausibility order* $\leqslant$ is a weak ordering on $\mathcal{P}$. $\Xi_{\leqslant} = \{\{R' | \ R' \leqslant R\} | R \in \mathcal{P}\}$ is the nested family of subsets of $\mathcal{P}$ associated with $\leqslant$.

The implausibility order $\leqslant$ is said to be *centred* if there exists a single element R with $R \leqslant R'$ for all $R' \in \mathcal{P}$. This element is called the *centre*.

Henceforth, we use $\mathcal{I}$ to denote the set of implausibility orders on $\mathcal{P}$.

The rest of this paper will develop theories of choice based on this representation of confidence in preferences. Note that the representation does impose some non-trivial conditions on the concept. In particular, it implies that for a given level of confidence, the preferences in which the agent is at least that confident are transitive and reflexive (this follows from the points made above). This is a reasonable: if one is confident to a certain degree than x is better than y, and one is confident to that degree that y is better than z, then one is confidence to at least that degree that x is better than z.

Centred implausibility orders have a single weak ordering as the least implausible ordering on the set of alternatives. (Equivalently, the nested family of sets contains a singleton set.) This represents the agent as having a "best guess" as to which value judgement is "right" for any pair of alternatives, though he may be very unconfident in this judgement in many cases (as represented by the rest of the implausibility order). We do not wish to take any specific position on whether this is a reasonable normative constraint on rational agents, or on whether it is descriptively reasonable. The centering property of implausibility orderings will generally not be a requirement for most of the results presented here.

[4]Formally: for Ξ a nested family of subsets of $\mathcal{P}$, define $\leqslant$ as follows: for any $R, R' \in \mathcal{P}$, $R \leqslant R'$ if, for all $\mathcal{R} \in \Xi$, if $R' \in \mathcal{R}$ then $R \in \mathcal{R}$. And for any weak order $\leqslant$ on $\mathcal{P}$, define the nested family of subsets Ξ to be that family containing all and only $\{R' | \ R' \leqslant R\}$ for all $R \in \mathcal{R}$. It is straightforward to see that this is a bijection from nested families of subsets of $\mathcal{P}$ to weak orderings on $\mathcal{P}$.

Finally, by analogy with the property of tightness of sets of weak orderings we say that an implausibility order $\leqslant$ is *tight* if all the sets in $\Xi_{\leqslant}$ are tight.[5]

1.2 Confidence and choice

A representation of the agent's confidence in his preferences is of little use on its own; an account of the role of confidence in choice is also required. In this section, we outline the principal ideas and notions involved in this account; in Section 2, we axiomatise the notion of rationalisability proposed here, and in Section 3 we discuss in more detail some of the central notions.

The basic intuition is simple: the more important the decision to be taken, the more confident one should be in the value judgements required to take that decision. If a choice between x and y is to be made, but the choice is not particularly important, one can chose x on the basis that on one's appraisal x is better than y, even though one is not very confident about this value judgement. But if the choice is very important, then one needs to be a lot surer of the value judgements underlying one's decision to take it, or certainly to take it responsibly. This intuition is intended to be normative – it is intended to say something about how people should decide on the basis of value judgements in which they may be more or less confident – although a full defence would go beyond this paper. It may also describe the way that people actually do make decisions in several cases, though it would require experimental work to determine to what extent this is indeed the case.

To formalise this intuition, a first requirement is a notion of importance of a choice. We thus assume that there exists a set I of possible importance levels, and that this set is equipped with a linear ordering (that is, an antisymmetric weak ordering) $\preccurlyeq$: $i \preccurlyeq j$ means that the importance level j is "higher" than the level i.

The importance levels are related to two factors in a choice problem. First of all, they are related to the degree of confidence required in a value judgement for it to play a role in the choice, via the maxim that the more important a decision, the more confident one needs to be in a value judgement for it to play a role in the choice.

So to each level of importance can be associated the value judgements in which the agent has enough confidence to use for choices of this importance. Since, as discussed above, a set of such value judgements can be represented by the appropriate set of weak orderings, the relationship between importance level and confidence can be naturally

[5] This can be formulated just in terms of the order itself as follows: $\leqslant$ is *tight* if, for any $R, R' \in \mathcal{P}$, if $\bigcap_{R_i \leqslant R'} R_i \subseteq R$, then $R \leqslant R'$.

represented by a function which associates to each importance level a set in the nested family of sets $\Xi_{\leqslant}$. Moreover, when the importance rises, the appropriate amount of confidence rises, so the set of value judgements in which there is sufficient confidence becomes smaller; in the representation, this corresponds to the fact that the set of weak orderings corresponding to higher importance level contains the set corresponding to lower importance level. Technically, this can be captured by a function $D : I \to \mathcal{P}$ such that (i) for all $i \in I$ and all $R, R' \in \mathcal{P}$ with $R' \leqslant R$, if $R \in D(i)$, then $R' \in D(i)$, and (ii) $D(i) \subseteq D(j)$ if $i \preccurlyeq j$.

Such a function captures the agent's attitude to choosing in the absence of confidence: for two agents with the same implausibility order but different D, the one with smaller $D(i)$ requires less confidence in a value judgement to use it in a decision of importance level i than the agent with higher $D(i)$. This is a subjective factor, the agent's taste for choosing in important decisions on the basis of limited confidence; or, to put it in another way, his cautiousness when it comes to choosing on the basis of value judgements in which he has limited confidence. The function will be called the *cautiousness coefficient*.

Secondly, the importance levels are supposed to capture an aspect of the choice situation or decision the agent is faced with. Some decisions are more important than others; to the former are associated importance levels which are higher (according to the order $\preccurlyeq$) than the importance levels associated to the latter. So, to each choice situation will be associated not only a set of available alternatives (sometimes called the *menu*) but also an importance level. The pair (S, i) represents the choice offered among the elements in S, with importance i. We come back to this representation of choice situations and the notion of importance level in Section 3.1.

This only leaves the definition of choice functions. Under the standard definition, a choice function c is a function from the set of non-empty subsets of X (which we denote by $\wp(X) \setminus \emptyset$) to the set of subsets of X (denoted $\wp(X)$) such that (i) for every non-empty $S \subseteq X$, $c(S) \subseteq S$; and (ii) for every non-empty $S \subseteq X$, $c(S)$ is non-empty. According to the maxim proposed above, an agent should chose based on value judgements which he is confident enough in given the importance of the decision; this implies that there may be decisions of such importance that he does not have sufficient confidence in the relevant value judgements to make a choice. We thus weaken the second condition and allow the choice function to yield empty choice sets. We define a *choice** *function* to be a function $c : \wp(X) \setminus \emptyset \to \wp(X)$ such that $c(S) \subseteq S$ for every non-empty $S \subseteq X$. $c(S)$ is called the *choice set*, and if $x \in c(S)$ then x is said to be *admissible*. For a detailed consideration and defence of this notion of choice function, see Section 3.2.

The object of study are variants of choice* functions which account for importance. An *importance-indexed choice* function* is a function $c : (\wp(X)\setminus\emptyset)\times I \to \wp(X)$ such that $c(S,i) \subseteq S$ for every non-empty $S \subseteq X$ and every $i \in I$.

Having introduced this new sort of choice function, a corresponding notion of rationalisability is required. The idea is simple: for each choice situation, the importance level picks out, via the cautiousness coefficient D, a set of weak orderings which represent all the value judgements which the agent is confident enough in to use in his choice. Then he chooses on the basis of this set of weak orderings in a specified way. We thus propose a notion of rationalisability of a choice* function by a set of weak orderings, which is then extended to a notion of rationalisability of an importance-indexed choice* function by an implausibility order.

Definition 1.2. For any $S \in X$ and $\mathcal{R} \subseteq \mathcal{P}$, let $\sup(S,\mathcal{R}) = \{x \in S|\ xRy$ for all $y \in S$ and all $R \in \mathcal{R}\}$.

A choice* function c is *rationalisable by a set of weak orderings* if there exists $\mathcal{R} \subseteq \mathcal{P}$ such that, for all non-empty $S \subseteq X$, $c(S) = \sup(S,\mathcal{R})$.

An importance-indexed choice* function c is *rationalisable by an implausibility order* if and only if there exists an implausibility order $\leqslant$ and an cautiousness coefficient D such that, for all non-empty $S \subseteq X$ and $i \in I$, $c(S,i) = \sup(S, D(i))$.

$\sup(S,\mathcal{R})$ contains those elements of S which are at least as good as all the other elements of S according to all the weak orderings in $\mathcal{R}$. Rationalisability by a set of weak orderings $\mathcal{R}$ says that an element is in the choice set if and only if it is at least as good as all other elements on the menu according to all the weak orderings in $\mathcal{R}$. Rationalisability by an implausibility order basically says that, for every importance level i, an element is in the choice set if it is at least as good as all the other alternatives according to all orderings in the set corresponding to that importance level, $D(i)$.

The notion of rationalisability by a set of weak orderings proposed above has received little attention in the choice-theoretic literature. Much more popular is the notion according to which the choice set contains those elements which are best according to at least one ordering, rather than according to all orderings; in other words, where the choice set is the *union* of the sets of best elements according to each of the weak orderings, rather than the *intersection* (Moulin, 1985).[6] The intersection notion proposed

[6]Translated in terms of quasi-orderings, the notion of rationalisibility proposed here picks out the set of

above is of course stronger than the union notion, but it is traditionally seen as problematic, because, unlike the union notion, it does not always yield non-empty choice sets. However, this property, though it may be unwanted if one is interested in rationalising choices by a single ordering or by a single set of orderings, is less problematic for implausibility orders. All that the emptiness of the choice set indicates is that there are degrees of confidence such that the agent is not confident of any particular choice to that degree. This does not imply that he cannot make a choice – he could always chose, but relying on preference judgements which he endorses, but of which he is not very confident. We shall return to this issue in detail in Section 3.2.

2 Representation

In this section we give necessary and sufficient conditions for rationalisability by an implausibility order. To this end, consider the following properties of importance-indexed choice* functions c.

For all $x, y \in X$, $S, T \subseteq X$ and $i, i' \in I$,

α^*	If $x \in S \subseteq T$ and $x \in c(T, i)$, then $x \in c(S, i)$
π^*	If $x \in S$, $y \in S \cap T$, $y \in c(T, i)$ and $x \in c(S, i)$, then $x \in c(S \cup T, i)$
Consistency	If $x \in c(S, i)$ and $i \succcurlyeq i'$, then $x \in c(S, i')$
Centering	For all $S \subseteq X$, there exists $j \in I$ such that $c(S, j)$ is non-empty

We have the following result.

Theorem 1. *An importance-indexed choice* function is rationalisable by an implausibility order if and only if it satisfies α^*, π^* and Consistency. Moreover, it is rationalisable by a centred order if and only if it satisfies Centering. In both cases, there is a unique coarsest tight rationalising implausibility order and cautiousness coefficient.*[7]

The proof is to be found in the Appendix. It relies heavily on a representation result for choice* functions, which involves the following two properties.

optimal elements, to use Sen's (1997) terminology, whereas the union notion picks out the set of *maximal* ones. Just as noted in the text, maximal elements of quasi-orderings always exist, whereas this is not the case for optimal elements.

[7] Recall that an implausibility order $\leqslant$ is coarser than $\leqslant'$ if, for any $R, R' \in \mathcal{P}$, $R \leqslant' R'$ implies that $R \leqslant R'$, but $R <' R'$ does not necessarily imply that $R < R'$. For a definition of tightness, and a discussion of its relevance here, see Section 1.1 and in particular footnote 2.

α if $x \in S \subseteq T$ and $x \in c(T)$, then $x \in c(S)$

π if $x \in S$, $y \in S \cap T$, $y \in c(T)$ and $x \in c(S)$, then $x \in c(S \cup T)$

Theorem 2. *A choice* function c is rationalisable by a set of orderings if and only if it satisfies α and π. Moreover, in this case, there is a unique tight rationalising set of orderings. Finally, if c always takes non-empty values, then the rationalising set of orderings is a singleton.*

Evidently, the properties α^* and π^* in the representation of importance-indexed choice* functions are just the importance-indexed versions of α and π. They state that α and π hold on sets of alternatives when the importance level is the same. As concerns the properties α and π themselves, the former is Sen's α (also called Chernoff's property) and requires no further discussion.

By contrast, to our knowledge, there has been little study of choice* functions (which, recall, may yield empty choice sets); accordingly, the property π and the Theorem 2 are new. To illustrate, π says that if x is a best candidate for a position from a European university and y is a best candidate from an American university, and if y is also affiliated to a European university, then x is a best candidate from among European and American universities. π^* says that the same consequence holds, given that the choices all have the same importance level.

It follows from the final clause in Theorem 2 that, on choice functions, π is equivalent to Sen's β. However, in the absence of the non-emptiness condition, π is strictly stronger than β. On the one hand, π implies β: for $x, y \in c(S)$, $S \subseteq T$ and $y \in c(T)$, π applies to x, y, $S = S \cap T$ and T, yielding that $x \in c(S \cup T) = c(T)$ as required. On the other hand, here is an example where β is satisfied but π is not: $X = \{x, y, z\}$, $c(\{x, y\}) = \{x\}$, $c(\{y, z\}) = \{y\}$, $c(\{x, z\}) = \{x\}$ and $c(\{x, y, z\}) = \{\}$. It follows from the theorem above that β is too weak to guarantee rationalisation of choice* functions by sets of weak orderings; π is the appropriate property for choice* functions.

Consistency is probably the property which differs most from traditional choice-theoretic properties which can be found in the literature. For good reason: it concerns the comparison between choices at different levels of importance. It says that any option which is admissible when the importance is high will continue to be admissible when the menu remains the same but the importance level drops. In other words, as the importance decreases, more alternatives become admissible – and so may be chosen – but no previously admissible alternatives cease to become admissible. Of course, as is standard in choice theory, the fact that an alternative is admissible does not mean that it will *actually* be chosen. So this property is compatible with (concrete) cases where the option actually chosen when the decision is important is not that which is chosen when it becomes less important: it only demands that the alternative could (rationally) have been chosen in the less important situation, or in other words that it is still admissible.

The final property, Centering, states that one can always make a choice from any menu, provided the importance level is low enough. In many cases, this might seem reasonable: although one is not confident enough in one's relevant value judgements to pick out an option when the decision is important, one has no trouble selecting some "best guesses" when little rests on the decision. This property is only required for the implausibility order to be centered (Theorem 1); as noted in Section 1.1, we do not wish to take any position here on whether the centredness of the implausibility order, and correspondingly the Centering property on importance-indexed choice* functions, is normatively advisable or descriptively acceptable in general.

Note finally that this representation, and Theorem 1, is a strict generalisation of the standard theory of choice and the axiomatisation by Sen's properties α and β. If $c(S, i) = c(S, j)$ for all importance levels i and j and all non-empty subsets S, then the properties above equivalent to the conjunction of α and β. The cautiousness coefficient sends all the importance levels to the same, singleton, set of weak orderings, so the representation collapses into the traditional representation by a weak ordering. As the cautiousness coefficient indicates, this captures the case of an agent who is insensitive to his confidence in his preferences and to the importance of the decision.

3 Discussion

In this section, we discuss in more detail two of the least standard elements of the proposal outlined above: the notion of importance level and the permissibility of empty choice sets.

3.1 The importance level

A major element of the current proposal is the extension of the ordinary representation of choice situation from a set of available alternatives (the menu) to a set of alternatives and an importance level. The latter is exogenous, insofar as it is not derived from the menu, but taken as given along with it.[8] Of course, this extra structure may make some

[8]In choice theory, little structure is assumed on the alternatives. If more structure is assumed, it becomes possible to define an equivalent of the importance level in terms of the set of alternatives on offer; see Hill (2009) for an example of how this may be done in the case of decision under uncertainty.

readers uncomfortable.

The supplementary assumptions on which this representation of choice situations relies are as follows: (i) to each choice which the agent is faced with, one can associate a set of elements from X and an importance level from I; and (ii) any pair consisting of a subset of X and an importance level from I represents a choice which the agent could conceivably be faced with.

Both of these assumptions are just versions of assumptions which are involved in the traditional representation of choice situations as subsets of a set of alternatives X. On the one hand, this representation supposes that the element x when it belongs to the menu $\{x, y\}$ is in a relevant sense "the same" as the element x in $\{x, y, z\}$. This corresponds to the first assumption above, (i), which we call for this reason *identification*. On the other hand, the representation permits that all sets of elements of X represent choice situations in which the agent might conceivably find himself; this is the second aspect, (ii), which we call *richness*.[9] In practice, the choice of the set of alternatives X is at the modeller's discretion, and he has to find a balance between these two "structural" assumptions, which, though necessary in some form or other for every theory of choice, are often in tension. Consider, for instance, some of the examples Sen raises against the most natural notion of identification among alternatives (1993; 1997), such as the example of the choice between tea and going home, and the extension by the offer of cocaine.[10] As Sen notes, one could reply to such examples by refining the set of alternatives to distinguish between the option of tea with cocaine not being on the menu and the option of tea with cocaine also being on the menu. However, this defence of identification leaves richness in a sorry state, for it demands that one can find situations in which the agent has the choice between some rather strange alternatives, such as between having tea with cocaine also being on the menu and going home with cocaine not being on the menu.

In the light of this it is not necessarily unreasonable to impose extra structure on the representation of the choice situation: as we shall see below, this sometimes allows an improvement in identification whilst limiting the damage done by richness. Of course, to the extent that such extra structure may not be easily discernable in all decision

[9]Of course, only weaker versions of this are needed, but they all require at least that for any two elements there exists menus containing them both and representing a conceivable choice situation, and this is all that is needed for the points made below to be relevant.

[10]When offered the choice between having tea with an acquaintance and going home, the agent chooses the tea, whereas when the choice is between tea with the acquaintance, cocaine with the acquaintance and going home, he chooses home; these choices violate the property α.

situations, it may not be appropriate for all examples; however, this does not imply that they are no cases where it is a relevant, and indeed useful, compromise between identification and richness. Here are some examples where a modelling of the sort proposed above seems reasonable:

- a governing body is considering policies for encouraging recycling in the population. It seems reasonable to say that in general the “same” policies are available (for example, advertising, fines, bonuses, nudging etc.), but that the importance of the decision differs according to whether the governing body is the head of a household or an office, local government, regional government, national government or an international body.
- a young academic is to present his work to a public of peers. The occasion could be an in-house closed seminar, an open seminar, an international conference, an occasion where only people who know his work are present, an occasion where potential employers could be in the audience and so on (the academic profile of the audience is the same in all cases). It seems that the “same” options are available concerning how to present his material, but the importance differs between the different cases.
- consider a classic moral dilemma where you have the choice between killing one person, thus saving ten, or refusing to kill the one, thus sacrificing the ten. There is a sense in which this is the “same” choice as that between killing ten people or letting a hundred die, and as that between killing hundred people and letting a thousand die, and so on;[11] but the gravity of the choices differs among these cases.

In all of these cases, there is certainly a sense in which the same options are available, but the importance of the choices to be taken differs. They are thus cases to which the representation of choice situations proposed above can be applied. To show that the importance level is a factor which needs to be taken into account, it suffices to establish that the admissible choices may differ depending on the importance level. This certainly seems to be true. Although the academic may try out a less standard organisation of his presentation or incorporation of material he is less sure about on a

[11] If you prefer, replace this example with the choice between killing 0.0001% (respectively, 0.001%, 0.01% and so on) of the human population, and letting 0.001% (respectively, 0.01%, 0.1% and so on) die, or even another scaling between the cases, if deemed necessary.

"friendly" audience, when the importance of the event is higher he would more likely revert to an organisation and a choice of material he is more confident in. And although it may be acceptable to try new methods of encouraging recycling on a local level, on a global level one needs to be much more confident to revert to them. Indeed, one sometimes hears people say that, although a policy "worked" when tried out on a local level, more reflection is needed before deciding whether to apply it nationally. Such assertions seem to rely on the tacit assumption that the national decision is more important than the local one, and so requires more deliberation. In fact, there are quite a few cases where people cite the importance of a decision as a relevant factor in the choice made. To take an example from moral theory, Rawls (1971, p169) explicitly raises the question of the importance of the agreement made under the "veil of ignorance" as a point in favour of his principles of justice; he thus admits that importance (of the choice under the veil of ignorance as opposed to a choice taken outwith the veil, for example) may be relevant for choices one takes.[12]

In many of these cases, one might have the impression that the choice is the same, but that the *context* differs. To take the first example, the same decision has to be taken about recycling, but in a household, local, regional, national or global context. This intuition can be captured by modelling the context by a function (call it γ) which associates to every menu an importance level: this is the importance attached to the choice among these alternatives in this context. The choice situations will thus be represented by pairs consisting of a menu (the alternatives on offer) and a context function (the context of the choice). This representation of choice is visibly equivalent to that proposed, and indeed, a notion of rationalisability for choice functions on pairs consisting of a set of alternatives and a context can be proposed and axiomatised as above (replacing appearances of i by $\gamma(S)$).

We take examples such as those given above to indicate that the representation of choice situations proposed in Section 1.2 may be relevant in several cases. Nevertheless, it is worthwhile noting that the result obtained in Section 2 remains valid even if the choice situation is represented in the traditional way, as a set of alternatives. Were one to represent the choice situations in the examples above in the standard way, then, as already noted, one would have to revert to "finer" alternatives. A natural choice would be to replace the set of alternatives X by the set $X \times I$ of pairs consisting of an alternative and an importance level. (x, i) is the alternative of choosing x in a choice of

[12]Thanks to Thibault Gajdos for suggesting this example.

importance i. The importance-indexed choice* function generates a choice* function which is defined on a subset of the menus generated by this set of alternatives: namely, on those menus consisting of elements with the same importance level. In that sense, Theorem 1 can alternatively be thought of as an axiomatisation of a rationalisation of a partially-defined choice* function on more or less standard menus. The menus on which the function is not defined are those with mixed importance levels: examples include the choice between choosing advertising to promote recycling on a national level (for example, for the whole of France), and using "nudging" techniques on a local level (for the city of Caen). As noted above, it is not always easy to make sense of such choices; indeed, the fact that a (fully defined) choice function on this set of alternatives requires choices to be made on such menus is an example of the problems which too fine an identification can pose in terms of the required richness.

Of course, the representation proposed here does not require any choices to be made on such menus, and information gained from the choices on which it is defined does not imply any particular choices on these peculiar menus. Nevertheless, if desired, it is possible to extend the notion of rationalisation proposed above to such menus: to take as example one of several possibilities, one could choose those alternatives which are best for all preference orderings singled out by the highest importance level among the alternatives on the menu.[13] Representation theorems for such notions of rationalisability can be proved, by making appropriate modifications to Theorem 1 above. Depending on one's view on these sorts of mixed-importance menus, one might be more or less attracted by these theorems.

Before closing the discussion of importance levels, let us make a remark concerning the assumption that the importance levels can be ordered by a linear order ($\preccurlyeq$). Basically, this boils down to assuming that the order of "higher importance" is transitive and complete. Although the former assumption is very intuitive, and although the latter is natural in many cases, there may be cases where the latter assumption, completeness, does not seem to be satisfied. To take the second example given above, it may not be possible to determine whether the talk given as an invitee to a seminar in one department (where, say, the person in question intends to apply for a position) is of higher, lower or equal importance than the talk given as invitee in another department (which the person in question also intends to apply to); that is, it might not be possible to rank one importance level relative to the other. There is a natural generalisation of Theorem 1 which can deal with such cases. All that is required is a relaxation of the

[13] Formally: $(x, i) \in c(S)$ if and only if $x \in \sup(S, D(\sup_{(y,j)\in S} j))$.

assumption that implausibility orders are complete: that is, that every pair of weak orderings on X can be ranked according to implausibility. If the order on the importance levels $\preccurlyeq$ is transitive but not complete, then the properties α^*, π^* and Consistency are necessary and sufficient for a rationalisation of the sort given in Definition 1.2, where the implausibility order is transitive, reflexive but not necessarily complete. The other clauses of Theorem 1 continue to hold.

3.2 Choice* functions

It has long been recognised that indifference and indeterminacy of preferences are difficult to distinguish on the basis of choice; accordingly, the problem of "deducing" preference from choice is particularly thorny in cases where preferences may be indeterminate. Recently proposed solutions have involved weakening the Weak Axiom of Revealed Preference (Eliaz and Ok, 2006), looking at sequential choice (Mandler, 2009) or invoking choices over opportunity sets and supposing preference for flexibility (Danan, 2003). The method employed in this paper is different, and very simple: it uses choice functions which may yield the empty set on some menus (which we have called choice* functions). But how are the cases where the choice* function yields the empty set to be interpreted?

The simplest answer is that the agent refuses to make a decision. In practice, this may come out in many ways. For example, he might admit that he is not sure what to do. More interestingly, there may be cases where he can *defer* the decision to whoever would next have to take it (including, perhaps, himself at a later time); this is what he would do when the choice set is empty. Deferral of decisions seems a natural option for identifying cases of incompleteness or indeterminacy of preferences, or lack of confidence in value judgements. Certainly, there seem to be several non-trivial examples where deferral, or something like it, is an option:

- a secretary takes the responsibility of making many decisions on behalf of her boss without consulting him. However, there are decisions which she could be called upon to make but which she would not accept to make in the absence of her boss, or at the least without his confirmation that her proposed decision is suitable. This is a case where she does not actually make a choice from the options available, but "defers" the decision to her boss.
- in the English law system, a judge may state in his verdict that he found the case very difficult and would grant that the case is fit for appeal. (Under English law,

a party who wishes to appeal has to ask the judge to declare the case fit for appeal at the end of the hearing.) In essence, the judge is emitting a judgement on the case, as he must, but admitting that the case should conceivably be reconsidered by others; this is the closest thing to a deferral under the obligation to express a choice or judgement.

- a person is faced with a moral dilemma, and he is unsure about the correct option. He decides to delay taking the decision, in order to consult friends, advisers, mentors and so on on the moral issues involved; the final decision will be taken on a more rounded and thought-through set of preferences. In a sense, he is deferring the decision to his future – hopefully more morally confident – self.

The most detailed study of questions relating to deferral in the economic literature is doubtless in the "preference for flexibility" tradition, following on from the ground-breaking work of Kreps (1979) (see Danan (2003) for an application of related ideas to incomplete preferences). Researchers in this tradition consider how the agent restricts or keeps open the future choices which he will be faced with. However, in most of these cases, the proposed model is strategic: first of all, the agent can choose which options to leave open for himself and which ones to rule out; moreover, the agent makes this choice on the basis of his beliefs about what he will prefer at the time when he will finally make his choice. The examples above do not seem to be strategic in either of senses: first of all, the agent cannot and does not restrict the options open to the person who will eventually decide (the secretary does not dictate what decisions are "allowed" to her boss); secondly, the agent does not decide to defer on the basis of his or her beliefs about his future preferences (if one defers the decision in the moral dilemma, one does not consider what one might end up judging to be best). Moreover, by contrast with the literature just cited, the choice of deferral does not imply that one will get to make the choice in the future: in the first two examples, the agents are deferring to someone else (respectively, the secretary's boss and the court of appeal). In fact, it is arguable that in some of these cases, the agent's own preferences play at most a very slight role. The agent does not have any personal interest in the outcome of the deferred choice; if he defers, it is not because he would like a particular choice to be taken, but because he would like the "right" choice (whatever that be) to be taken. To the knowledge of the author, such cases have received little attention in the economic literature, though the question of deferral is relevant in many cases, and a theory of when one should defer or not does seem to be required.

It is an advantage of the proposal made in the preceding sections that it can be thought of as providing a theory of when to defer. The empty choice set can be interpreted as indicating that the agent would like to defer, or that he would defer if possible. As a theory of deferral, it is eminently reasonable: it says that one should defer if one's confidence in the choice of any alternative does not match up to the importance of the decision.

As just noted, deferral is not to be understood as leaving the choice to someone whose decisions are predictable (or about which the agent has probabilities), but it is left entirely unspecified as to what the final decision may be. One might complain that, if deferral is seen as an option, then it should be incorporated into the menu offered to the agent. Indeed, this can be done, and yields a representation visibly similar to that proposed in Section 1.2.

Let us use the symbol † to represent the option of deferral; when † is present in the menu, the option of deferral is available, when it is absent, deferral is not available. The current proposal can be formulated entirely in terms of importance-indexed choice functions (ie. functions always yielding non-empty choice sets) on the set of alternatives $X \cup \{\dagger\}$ (where X is as above).

Now deferral is an alternative which has a special status with respect to the others. For one, the question of identification (see Section 3.1) is particularly complicated: whereas the alternatives are supposed to be defined at such a level of fineness that x chosen from menu S can be treated as the same x as that chosen from T, it is unclear whether there is any sense in which deferral when the choice is from menu S can ever judiciously be thought of as the "same thing" as deferring from the choice on menu T. In the face of this, one could introduce a set of different new alternatives "deferring from S", "deferring from T" and so on, with all the disadvantages in terms of richness that were discussed above. Alternatively, one could admit just one new alternative, †, but give it a distinguished role in the definition of rationalisability and in the axiomatisation. Since deferral is an special option, the axioms on choice will have to reflect some of its distinctive properties.

As regards rationalisability, the theory proposed above, under the interpretation of an empty choice set as deferral, immediately implies a notion of rationalisability for menus containing the deferral option †, namely: for all $S \subseteq X \cup \{\dagger\}$ such that $\dagger \in S$ and all $i \in I$, if $\sup(S, D(i))$ is non-empty, then $c(S, i) = \sup(S, D(i))$, and if not, then $c(S, i) = \dagger$. This renders explicit the idea that one does not defer if there are options which are optimal according to all the weak orderings in the relevant set and that one does defer (and not possibly do something else) if not. It remains to define the value of the choice function when deferral is not available. Of course, the notion of rationalisability proposed in Definition 1.2 does not deal with this case, but we have already mentionned an intuition about what one should do: choose an option that one is most confident in choosing. This yields the following definition of rationalisability of importance-indexed choice functions on sets of alternatives including an explicit deferral option.

Definition 3.1. An importance-indexed choice function c on a set of alternatives including an explicit deferral option, $X \cup \{\dagger\}$, is *rationalisable by an implausibility order* if and only if there exists an implausibility order $\leqslant$ and an cautiousness function D such that, for all non-empty $S \subseteq X \cup \{\dagger\}$ and $i \in I$, and all $x \in X$,

$$\begin{array}{lll} x \in c(S,i) & \text{if} & x \in \sup(S, D(i)) \\ & \text{or} & \dagger \notin S \text{ and } x \in \sup(S, D(j)) \text{ for all } j \text{ s.t. } \sup(S, D(j)) \neq \emptyset \\ \dagger \in c(S,i) & \text{if} & \dagger \in S \text{ and } \sup(S, D(i)) = \emptyset \end{array}$$

The first clause says that x is in the choice set if either it is admissible by the lights of the previous notion of rationalisability (Definition 1.2) or deferral is not available and x is admissible by the lights of the previous notion of rationalisability for all levels of importance where the choice set yielded by that notion is non-empty. The second clause says that one chooses to defer if the option is available and no options are admissible by the lights of the previous notion of rationalisability.

It should not be surprising that this notion of rationalisability can be axiomatised along similar lines to the axiomatisation proposed in Section 2. In fact, let the properties $\alpha^\dagger$, $\pi^\dagger$ and Consistency† be identical to the properties α^*, π^* and Consistency in Section 2, except that they apply to all $S, T \subseteq X \cup \{\dagger\}$, and consider the following new property and modification of Centering:

Deferral	If $\dagger \in c(S,i)$, then $c(S,i) \cap X = \emptyset$
Centering†	For all $S \subseteq X \cup \{\dagger\}$, there exists $j \in I$ such that $c(S,j) \neq \{\dagger\}$

Deferral just states that if one defers no alternative in X is admissible. Centering† states that for any menu there is an importance level for which one does not defer. These properties are necessary and sufficient for the rationalisability of importance-indexed choice functions where there is an explicit deferral option.

Theorem 3. *An importance-indexed choice function on a set of alternatives including an explicit deferral option is rationalisable by a centered implausibility order if and only if it satisfies $\alpha^\dagger$, $\pi^\dagger$, Consistency†, Centering† and Deferral. Moreover, there is a unique coarsest tight rationalising implausibility order and cautiousness coefficient.*

We conclude that the interpretation of empty choice sets as deferral is not only natural in many cases, but is entirely consistent with the traditional choice-theoretic methodology, via the addition of a special option for deferral into the menus.

4 Social choice and confidence

With an eye to illustrating the interest of the notion of confidence in preferences proposed here, let us briefly consider an application to social choice. This discussion is not intended to be a complete coverage of the potential importance of the notion of confidence for social choice, but rather a preliminary exploration of some of the possibilities.

The basic idea is that, if agents differ not only in their preferences but in their confidence in their preferences, then the latter factor and not solely the former can and often should be taken into account in the determination of the society's preferences. This makes sense: an agent's confidence in a preference (for x over y, for example) reflects how "sure" he is that he is "right" (by his own lights). Hence in aggregating the agents' preferences, it is not unreasonable to give those preferences of which an agent is more confident more bearing than those of which he is less confident. There are of course several ways in which this can be done; here we will consider only one.

As regards the setup, the set of alternatives, weak orderings and so on are as specified in Section 1.1. The set of members of the society (or voters) will be numbered, so the set of voters (which is not necessarily fixed) will be some $V \subseteq \mathbb{N}$. Voters give not just their preferences but also their confidence in their preferences, which, as argued above, can be represented by an implausibility order. So a *profile* is a function $w : V \to \mathcal{I}$. The task is to determine a social preference ordering on the alternatives on the basis of each possible profile. Given that the agents' preferences are not necessarily fully determinate, we allow that the social preference ordering may be indeterminate; as noted in Section 1.1, this can be captured by representing it either by a quasi-ordering or by a set of weak orderings; here we use the latter option. The objects of study are thus functions which associate to each profile a subset of $\mathcal{P}$. We shall call such functions *confidence-adjusted social choice functions* (CASC), and denote them using the generic term f.

Following on from the intuition stated above, a natural CASC would be one which selects social preferences which, on aggregate, the members of the society are most confident in. Under one way of spelling out this idea, the CASC would aim to "minimise" the total implausibility of the social preferences (by the lights of the members of the society); this is like maximising the "total confidence" of society in the social value judgements. This is by no means the only way to go; we shall briefly discuss another option below.

Every weak ordering in $\mathcal{P}$ has a place in the implausibility order of each of the members of the society; this place can be "counted" by associating to the weak ordering its "rank" on the implausibility order $\leqslant$. Formally, the "rank" of a weak ordering R under an implausibility order $\leqslant$, $n_{\leqslant}(R)$, is defined as follows: $n_{\leqslant}(R) = \sup_{R'<R}(n_{\leqslant}(R')) + 1$, where the maximum over an empty set is taken to be -1. So the orderings at the bottom of the implausibility order (the "most plausible" ones) are of rank 0, those one rung up are of rank 1 and so on. The rank of a weak ordering can be thought of as a measure of the "distance" which the ordering is from plausibility, according to the implausibility order in question. (Figure 2 in Section 1.1 makes this metaphor more vivid.)

A simple CASC which translates the idea that the social preference should be that which minimises its total implausibility is the "additive rank-based" CASC.

Definition 4.1. The *additive rank-based CASC f* is defined as follows: for any profile w, for any $R \in \mathcal{P}$,

$$(1) \qquad R \in f(w) \text{ iff } \sum_{v \in V} n_{w(v)}(R) \leqslant \sum_{v \in V} n_{w(v)}(R') \text{ for all } R' \in \mathcal{P}$$

The additive rank-based confidence-adjusted social choice function picks out the set of weak orderings whose total "implausibility", as summed over all the voters, is minimal (not greater than the total implausibility of any other orderings). In this sense, it could be thought of maximising the total confidence in the social value judgements. Of course, although only a function yielding a set of weak orderings on alternatives has been defined, this can be easily extended to a definition of a function yielding an "social" implausibility order (that is, an order on the set of weak orderings).

Observe that the additive rank-based confidence-adjusted social choice function is none other than the Borda rule, applied to orderings over alternatives rather than to alternatives themselves. Thanks to this, we immediately have, borrowing a result from Young (1974), the following axiomatisation of this rule.[14]

Following Young, we say that a CASC f is *neutral* if, for σ a permutation of the set of orderings $\mathcal{P}$, and $\hat{\sigma}$ the induced permutation of profiles, $f(\hat{\sigma}(w)) = \sigma(f(w))$ for all profiles w. It is *consistent* if, for any w, w' profiles for disjoint voter sets V and V', then $f(w) \cap f(w') \neq \emptyset$ implies that $f(w) \cap f(w') = f(w + w')$ (where $w + w'$ is the profile on $V \cup V'$ which agrees with w on V and w' on V'). It is *faithful* if, for w a profile for one voter, $f(w)$ contains only the center of the voter's implausibility measure. Finally, a CASC f has the *cancellation property* if, whenever w is a profile such that, for any $R, R' \in \mathcal{P}$, the number of voters with $R < R'$ equals the number of voters with $R' < R$, then $f(w) = \mathcal{P}$. The following holds.

Theorem 4. *A CASC f is neutral, consistent, faithful and has the cancellation property if and only if it is the additive rank-based rule.*

As indicated, the conditions involved here are versions of standard conditions in the literature, and the reader is referred to the relevant papers (especially Young (1974)) for further discussion.

The purpose of these considerations is only to give a flavour of possible applications of the notion of confidence to social choice. There are several directions which one could develop; let us just mention two.

First of all, the additive rank-based social choice rule is by far the only one, and others can be found and axiomatised in the similar way to that proposed, by exploiting the relation to voting theory. In fact, both the "additive" and the "rank-based" parts could be altered. For example, it is very probable that an axiomatisation for a "maxmin rank-based" confidence-adjusted social choice function – which yields the set of those preference orders whose worst confidence ranking across voters is highest – can be obtained by using recent results on maxmin rules in voting theory (for example, Congar and Merlin (2009)). Or, to take another example, one might be able to develop and axiomatise an "additive importance-based" confidence-adjusted social choice function – where the total is taken not of the ranks of the weak orderings under the implausibility, but of the least importance levels which are associated to sets containing the weak orderings. Each suggestion appears to bring with it different issues, which may or may not be new. For example, the discussion of the relationship between the CASC

[14]Note that, though his theorem is stated for linear orderings, Young notes in the conclusion that it applies to weak orderings as well. Naturally, the case of weak orderings is the one which is relevant here.

proposed above and a minmax version may well mimic several classic debates in social theory, in particular the debate between utilitarianism and egalitarianism. By contrast, the comparison of rank-based and importance-based rules may well turn on the question of whether the agents' tolerances of choice in the absence of confidence (cautiousness coefficients) should be taken into account in the social preferences (as would be the case under the importance-based rule) or not.

Secondly, the sort of aggregation discussed above is "ordering-wise": it works with the order on the set of weak orderings $\mathcal{P}$. A further direction to explore is "judgement-wise" aggregation. As hinted in Section 1, an implausibility order represents whether the agent is more, less or equally confident in one value judgement (say, that alternative x is better than y by his lights) than in another (say, that alternative x' is better than y'). Under "judgement-wise" aggregation, one would not aggregate the rankings of the weak orderings under the implausibility order, but, say, the rankings of the value judgements on the order on value judgements generated by the implausibility order. This sort of aggregation may be interesting because the axioms would be expressed solely in terms of confidence in value judgements, and not in terms of orders on sets of weak orderings. (For example, a "judgement-wise" version of the cancellation property would be: whenever w is a profile such that, for all alternatives x, y, x' and y', the number of voters who are more confident in the judgement that x is better than y than in the value judgement that x' is better than y' is equal to the number of voters who are more confident in the latter judgement than in the former one, the social choice is the set of all weak orderings.) Of course, there is a large literature on judgement aggregation which is relevant here (in particular Dietrich and List (2009)). A particularly interesting question is the relation between judgement-wise and ordering-wise choice rules: is it the case, for example, that the set of value judgements endorsed by the result of an additive rank-based confidence-adjusted choice rule are those in which the total confidence is highest, as calculated by looking at the rankings of the judgements? This is, to our knowledge, an open question.

5 Conclusion

People sometimes do not have preferences which are as determinate as the standard model would have us believe. Often, this is because people are not confident enough in some of the preferences they can be said to have. Of course, this may have implications for choice: people should not choose on the basis of preferences in which they are not sufficiently confident, if they can possibly avoid it.

This paper has made a start at bringing confidence in preferences into the field of choice theory. First of all, a representation of an agent's confidence in his preferences was developped, a notion of rationalisability of choice in terms of confidence in preference was proposed, and an axiomatisation of this notion was offered. The notion of rationalisability involves two main concepts which, to the knowledge of the author, have received relatively little attention in choice theory. Firstly, there is the concept of the importance of a choice, with the accompanying idea that the more important the choice, the more confident one needs to be in a preference to use it in one's choice. Secondly, there is the question of whether the agent can refuse to take a decision, or opt to defer, with the idea that this would be the appropriate course of action were the choice too important for the confidence he has in the relevant preferences. These notions, and their applications here, were discussed in detail.

Finally, in an attempt to indicate the relevance of the notion of confidence, a possible application to social choice was considered. A simple confidence-adjusted social choice function was proposed, based on the idea that the social preferences should be those in which the members of the society are, on aggregate, most confident. A simple axiomatisation was proposed for this rule, and directions for future research were mentionned.

Confidence in preferences has been given short shrift in choice theory to date. The author is confident that this should change.

Appendix

Proof of Theorem 2. Define the set of orderings $\mathcal{R}$ as follows: $R_i \in \mathcal{R}$ iff, for all $x, y \in X$, if $x \in c(\{x, y\})$, then xR_iy. First note that this set is well-defined. In particular π implies the necessary transitivity: if $x \in c(\{x, y\})$ and $y \in c(\{y, z\})$, then by π, $x \in c(\{x, z\})$. Note also that this set is tight: if R' agrees with the$R$ in $\mathcal{R}$ wherever they all agree, then $R' \in \mathcal{R}$.

It needs to be shown that this set of orderings generates c; consider $x \in S \subseteq X$.

Suppose $x \in c(S)$. Then, by α, $x \in c(\{x, y\})$ for all $y \in S$. So, xR_iy for all $y \in S$ and $R_i \in \mathcal{R}$, as required.

Suppose now that xR_iy for all $y \in S$ and $R_i \in \mathcal{R}$. Take an arbitrary enumeration of the elements of $S \setminus \{x\}$. We argue by induction that $x \in c(\{x, y_1, \dots, y_n\})$ for all n. By hypothesis and definition of $\mathcal{R}$, $x \in c(x, y_1)$. Suppose that $x \in$

$c(\{x, y_1, \ldots, y_{n-1}\})$; by hypothesis and definition of $\mathcal{R}$, $x \in c(x, y_n)$; so by π, with $x = y$, $S = \{x, y_1, \ldots, y_{n-1}\}$ and $T = \{x, y_n\}$, $x \in c(\{x, y_1, \ldots, y_n\})$. Hence $x \in c(S)$, as required.

Since, if c never takes as value the empty set, for all $x, y \in X$, either x or y (or both) belong to $c(\{x, y\})$. There is thus only one relation R such that for all $x, y \in X$, xRy iff $x \in c(\{x, y\})$: so the $\mathcal{R}$ constructed above is a singleton.

□

Proof of Theorem 1. The "only if" direction is straightforward to check. We consider here the "if" direction.

For any $i \in I$, note that $c(\bullet, i)$ is a function from sets of alternatives to sets of alternatives; it is a choice* function because the image may be empty. We will note this function c_i in what follows.

α^* and π^* imply that, for every $i \in I$, c_i satisfies α and π. Theorem 2 implies that for each $i \in I$, c_i is rationalisable by a unique tight set of weak orderings $\mathcal{R}_i$. Moreover, by Consistency and Lemma 1 (below), if $i \preccurlyeq i'$, then $\mathcal{R}_{i'} \subseteq \mathcal{R}_i$. Define $\leqslant$ as follows: $R \leqslant R'$ iff for all i such that $R' \in \mathcal{R}_i$, $R \in \mathcal{R}_i$. It is straightforward to check that this is complete, transitive and reflexive; ie. that it is an implausibility order. Define D by: $D(i) = \mathcal{R}_i$.

The representation of c by $\leqslant$ and D follows immediately from the construction. Also, by construction, $\leqslant$ is tight, and any coarser tight relation would fail to rationalise c; the uniqueness of D follows by construction. Consider finally the clause regarding centering. Centering implies that for every $S \subseteq X$, there exists $i \in I$ such that $c_i(S)$ is non-empty; by Consistency, there exists $i^* \in I$ such that, for all $S \subseteq X$, $c_{i^*}(S)$ is non-empty. By the final clause in Theorem 2, c_{i^*} is a singleton. This is the center of $\leqslant$.

□

Lemma 1. *Let choice* functions c_1 and c_2 be rationalised by tight sets of orderings $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively. If $x \in c_1(S)$ implies that $x \in c_2(S)$ for every $x \in S$ and every $S \subseteq X$, then $\mathcal{R}_1 \supseteq \mathcal{R}_2$.*

Proof. By construction of the rationalising sets of orderings in Theorem 2. The construction implies that $R \in \mathcal{R}_i$ if and only if for all $x, y \in X$, if $x \in c_i(\{x, y\})$ then xRy (for $i = \{1, 2\}$). However, for all $R \in \mathcal{R}_2$ and all $x, y \in X$, if $x \in c_1(\{x, y\})$ then, by hypothesis, $x \in c_2(\{x, y\})$, and so xRy; from which it follows that $R \in \mathcal{R}_1$, as required. □

Proof of Theorem 3. Define the implausibility order as in the proof of Theorem 1, using the part of c defined on menus containing †. It follows from the reasoning in the proof of that theorem that, for all $x \in X$ and all $S \subseteq X$, $x \in c(S \cup \{\dagger\}, i)$ iff $x \in \sup(S, D(i))$. By Deferral, if $\sup(S, D(i))$ is non-empty then $c(S \cup \{\dagger\}, i) = \sup(S, D(i))$; by the fact that the choice function always yields non-empty sets, it follows that if $\sup(S, D(i))$ is empty then $c(S \cup \{\dagger\}, i) = \{\dagger\}$. Moreover, if $\sup(S, D(i))$ is non-empty, then by $\alpha^{\dagger}$, $c(S, i) = c(S \cup \{\dagger\}, i) = \sup(S, D(i))$. Finally, it can be seen that if $\sup(S, D(i))$ is empty, then $x \in c(S, i)$ if and only if $x \in \sup(S, D(j))$ for all j such that $\sup(S, D(j))$ is non-empty. For if not, then there is a $j \in I$ such that $\sup(S, D(j))$ is non-empty but does not contain $x \in c(S, i)$. So $x \notin c(S, j)$ but $y \in c(S, j)$ for some y. Since $\sup(S, D(j))$ is non-empty and $\sup(S, D(i))$ is empty, $c(S \cup \{\dagger\}, j)$ is not contained in $c(S \cup \{\dagger\}, i)$, so, by Consistency, it is not the case that $i \preccurlyeq j$. But, given $j \preccurlyeq i$, Consistency implies that $x \in c(S, j)$ contrary to the assumption.

Uniqueness follows from construction, as in the proof of Theorem 1.

□

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