A FRAMEWORK FOR CAPM WITH HETEROGENEOUS BELIEFS

CARL CHIARELLA*, ROBERTO DIECI** AND XUE-ZHONG HE*

*School of Finance and Economics
University of Technology, Sydney
PO Box 123 Broadway
NSW 2007, Australia

**Department of Mathematics for Economics and Social Sciences
University of Bologna
Viale Q. Filopanti 5, I-40126
Bologna, Italy

Abstract. We introduce heterogeneous beliefs into the mean-variance framework of the standard CAPM, in contrast to the standard approach which assumes homogeneous beliefs. By assuming that agents form optimal portfolios based upon their heterogeneous beliefs about conditional means and covariances of the risky asset returns, we set up a framework for the CAPM that incorporates the heterogeneous beliefs when the market is in equilibrium. In this framework we first construct a consensus belief (with respect to the means and covariances of the risky asset returns) to represent the aggregate market belief when the market is in equilibrium. We then extend the analysis to a repeated one-period set-up and establish a framework for a dynamic CAPM using a market fraction model in which agents are grouped according to their beliefs. The exact relation between heterogeneous beliefs, the market equilibrium returns and the ex-ante beta-coefficients is obtained. CAPM and Heterogeneous beliefs

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Corresponding author: Xue-Zhong (Tony) He, School of Finance and Economics, University of Technology, Sydney, PO Box 123 Broadway, NSW 2007, Australia. Email: Tony.He1@uts.edu.au. Ph: (61 2) 9514 7726. Fax: (61 2) 9514 7722.

1. Introduction

The Sharpe-Lintner-Mossin (Sharpe (1964), Lintner (1965), Mossin (1966)) Capital Asset Pricing Model (CAPM) plays a central role in modern finance theory. It is founded on the paradigm of homogeneous beliefs and a rational representative agent. However, from a theoretical perspective this paradigm has been criticized on a number of grounds, in particular concerning its extreme assumptions about homogeneous beliefs, information about the economic environment, and the computational ability on the part of the rational representative economic agent.

The impact of heterogeneous beliefs among investors on the market equilibrium price has been an important focus in the CAPM literature. A number of models with investors who have heterogeneous beliefs have been previously studied[1]. A common finding in this strand of research is that heterogeneous beliefs can affect aggregate market returns. However, the question remains as to how exactly does heterogeneity affect the market risk of risky assets? In much of this earlier work, the heterogeneous beliefs reflect either differences of opinion among the investors[2] or differences in information upon which investors are trying to learn by using some Bayesian updating rule[3]. Heterogeneity has been investigated in the context of either CAPM-like mean-variance models (for instance, Lintner (1969), Miller (1977), Williams (1977) and Mayshar (1982)) or Arrow-Debreu contingent claims models (as in Varian (1985), Abel (1989, 2002) and Calvet *et al.* (2004)).

In most of the cited literature, the impact of heterogeneous beliefs is studied for the case of a portfolio of one risky asset and one risk-free asset (for example Abel (1989), Detemple and Murthy (1994), Zapatero (1998), Basak (2000) and Johnson (2004)). In those papers that consider a portfolio of many risky assets and one risk-free asset, investors are assumed to be heterogeneous in their risk preferences and expected payoffs or returns of the risky assets (such as Williams (1977) and Varian (1985)), but not in their estimates of variances and covariances. The only exception seems to have been the early contribution of Lintner (1969) in which heterogeneity in both means and variances/covariances is investigated in a mean-variance portfolio context. As suggested by the empirical study of Chan *et al.* (1999), while future variances and covariances are more easily predictable than expected future returns, the difficulties in doing so should not be understated. These authors

[1]See, for example, Lintner (1969), Williams (1977), Huang and Litzenberger (1988), Abel (1989), Detemple and Murthy (1994), Zapatero (1998) and Basak (2000).

[2]See, for example, Lintner (1969), Miller (1977), Mayshar (1982), Varian (1985), Abel (1989, 2002) and Cecchetti *et al.* (2000).

[3]Typical studies include Williams (1977), Detemple and Murthy (1994) and Zapatero (1998).

argue that "while optimization (based on historical estimates of variances and covariances) leads to a reduction in volatility, the problem of forecasting covariance poses a challenge". Therefore, a theoretical understanding of the impact of heterogeneous beliefs in variances and covariances on equilibrium prices, volatility and asset betas is very important for a proper development of asset pricing theory.

Different from the above literature, various heterogeneous agent models (HAMs) have been developed to characterize the dynamics of financial asset prices resulting from the interaction of heterogeneous agents with different attitudes towards risk and different expectations about the future evolution of asset prices. One of the key elements of this literature is the expectations feedback mechanism, see Brock and Hommes (1997, 1998). We refer the reader to Hommes (2006), LeBaron (2006) and Chiarella, Dieci and He (2009) for surveys of recent literature on HAMs. This framework has successfully explained various types of market behaviour, such as the long-term swing of market prices from the fundamental price, asset bubbles and market crashes. It also shows a potential to characterize and explain the stylized facts (for example, Chiarella, He and Hommes (2006), Gaunersdorfer and Hommes (2007) and Farmer *et al.* (2004)) and various kinds of power law behaviour (for instance Lux (2004), Alfarano *et al.* (2005) and He and Li (2007)) observed in financial markets. However, most of the HAMs analyzed in the literature involve a financial market with only one risky asset[4] and are not in the context of the CAPM. In markets with many risky assets and heterogeneous investors, the impact of heterogeneity on the market equilibrium and standard portfolio theory remains a largely unexplored issue.

This paper is largely motivated by a re-reading of Lintner's early work and recent development in the HAMs literature, in particular, our recent work Chiarella, Dieci and He (2006). Although Lintner's earlier contribution discusses how to aggregate heterogeneous beliefs, the impact of heterogeneity on the market equilibrium price, risk premia and CAPM within the mean-variance framework has not been fully explored. The main obstacle in dealing with heterogeneity is the complexity and heavy notation involved when the number of assets and the dimension of the heterogeneity increase. It might be this notational obstacle that makes the paper of Lintner hard to follow, and renders rather complicated the analysis of the impact of heterogeneity on market equilibrium prices. In this paper, we reconsider the derivation of the traditional CAPM in a discrete time setting for a portfolio of one risk-free asset and many risky assets and provide a simple framework that incorporates heterogeneous beliefs. In contrast to the standard

[4]Except for some recent contributions by Westerhoff (2004), Chiarella *et al.* (2005, 2007) and Westerhoff and Dieci (2006) showing that complex price dynamics may also result within a multi-asset market framework.

setting we consider heterogeneous agents whose expectations of asset returns are based on statistical properties of past returns and so induce expectations feedback. Different from Chiarella, Dieci and He (2006) where beliefs are formed in terms of the payoff, we assume that agents form their demands based upon heterogeneous beliefs about conditional means and covariances of the risky asset returns. The market clearing prices are determined under a Walrasian auctioneer scenario. In this framework we first construct a 'consensus' belief (with respect to the means and covariances of the risky asset returns) to represent the aggregate market belief and derive a heterogeneous CAPM which relates aggregate excess return on risky assets with aggregate excess return on the aggregate market wealth via an aggregate beta coefficient for risky assets. We then extend the analysis to a repeated one-period set up and establish a framework for a dynamic CAPM using a 'market fraction' model in which agents are grouped according to their beliefs. We obtain an exact relation between heterogeneous beliefs and the market equilibrium returns and the ex-ante beta-coefficients. The framework developed here could be used for further study of the complicated impact of heterogeneity on the market equilibrium.

The paper is organized as follows. Section 2 derives equilibrium CAPM-like relationships for asset returns in the case of heterogeneous beliefs and relates a 'consensus' belief about the expected excess return on each risky asset to a 'consensus' belief about expected market return, via aggregate beta coefficients. There follows a discussion about the wealth dynamics and the beta coefficients, and how they relate to the heterogeneous beliefs about the returns on the risky assets. Finally this section also considers explicitly the supply of the risky securities, and derives equilibrium prices, and relates the aggregate beta coefficients to the market equilibrium prices. Section 3 extends the analysis to a repeated one-period set up and obtains a dynamic, "market fraction" multi-asset framework with heterogeneous groups of agents, which generalizes earlier contributions by Brock and Hommes (1998) and Chiarella and He (2001, 2002), and highlights how the aggregate ex-ante beta coefficients may vary over time once agents' beliefs are assumed to be updated dynamically at each time step as a function of past realized returns. The framework is different from that of Chiarella, Dieci and He (2007), which uses a market maker mechanism to arrive at the market price, as here we use the Walrasian auctioneer scenario. Section 4 concludes and suggests some directions for future research.

2. The CAPM with Heterogeneous Beliefs

The present section generalizes the derivation of the CAPM relationships, as carried out for instance by Huang and Litzenberger (1988) Section 4.15, to the case of investors with heterogeneous beliefs about asset returns. Some of the ideas contained in the present section are adapted from Lintner (1969), where aggregation of individual assessments about future payoffs is performed in a mean-variance framework. However, different from Lintner (1969), the aggregation is explicitly given by constructing a consensus belief, which greatly facilitates the establishment of the CAPM with heterogeneous beliefs.

Consider an economy with many agents who invest in portfolios consisting of a riskless asset and N risky assets with $N \geq 1$. Let r_f be the risk free rate of the riskless asset and $\widetilde{r}_j$ be the rate of return of risky asset j, $j = 1, 2, ..., N$. Following the standard CAPM setup, we assume that the returns of the risky assets are multivariate (conditionally) normally distributed and the utility function $u_i(x)$ of agent i is twice differentiable, concave and strictly increasing, $i = 1, 2, ..., I$. Let W_0^i be the initial wealth of agent i and w_{ij} be the wealth proportion of agent i invested in asset j. Then the end-of-period wealth, $\widetilde{W}_i$, of agent i is given by

$$\widetilde{W}_i = W_0^i \left(1 + r_f + \sum_{j=1}^{N} w_{ij}(\widetilde{r}_j - r_f) \right). \tag{1}$$

Following Huang-Litzenberger (Section 4.15), the maximization of the expected utility of the portfolio wealth of agent i is characterized by the first order condition:

$$E_i \left[u_i'(\widetilde{W}_i) \right] E_i \left[\widetilde{r}_j - r_f \right] = -E_i \left[u_i''(\widetilde{W}_i) \right] Cov_i(\widetilde{W}_i, \widetilde{r}_j) \tag{2}$$

for any $j = 1, 2, ..., N$, where $E_i(\cdot)$ is the conditional mean and $Cov_i(\cdot, \cdot)$ is the conditional covariance of agent i, characterizing the heterogeneity of the agents in their beliefs. By defining the *global absolute risk aversion* of agent i

$$\theta_i := \frac{-E_i \left[u_i''(\widetilde{W}_i) \right]}{E_i \left[u_i'(\widetilde{W}_i) \right]},$$

condition (2) becomes

$$\theta_i^{-1} E_i \left[\widetilde{r}_j - r_f \right] = Cov_i(\widetilde{W}_i, \widetilde{r}_j), \qquad j = 1, 2, ..., N. \tag{3}$$

Note that by its definition in equation (1)

$$Cov_i(\widetilde{W}_i, \widetilde{r}_j) = W_0^i \sum_{k=1}^{N} w_{ik} Cov_i \left(\widetilde{r}_k, \widetilde{r}_j \right).$$

It follows that the conditions (3) can be rewritten with vector notation as

$$\theta_i^{-1} \left(E_i[\widetilde{\mathbf{r}}] - r_f \mathbf{1} \right) = W_0^i \mathbf{\Omega}_i \mathbf{w}_i, \tag{4}$$

where $\widetilde{\mathbf{r}} = [\widetilde{r}_1, \widetilde{r}_2, ..., \widetilde{r}_N]^\top$, $\mathbf{1} = [1, 1, ..., 1]^\top \in \mathbb{R}^N$, $\mathbf{w}_i = [w_{i1}, w_{i2}, ..., w_{iN}]^\top$, $\mathbf{\Omega}_i = [\sigma_{i,jk}]_{N \times N}$, $j, k = 1, 2, ..., N$, and $\sigma_{i,jk} := Cov_i \left(\widetilde{r}_j, \widetilde{r}_k \right)$, $i =$

$1, \cdots, I$. We assume that the $\mathbf{\Omega}_i$ ($i = 1, 2, \ldots, I$) are positive definite and thus invertible. It follows from (4) that the optimal portfolio $\mathbf{w}_i$ of agent i is given by[5]

$$\mathbf{w}_i = \frac{1}{W_0^i} \theta_i^{-1} \mathbf{\Omega}_i^{-1} \left(E_i[\widetilde{\mathbf{r}}] - r_f \mathbf{1} \right). \tag{5}$$

Let $W_{m0} = \sum_{i=1}^{I} W_0^i$ be the total wealth in the economy and $\mathbf{w}_a$ be the proportions of the total wealth in the economy invested in the risky assets. The *market is in equilibrium* when the condition

$$W_{m0}\mathbf{w}_a = \sum_{i=1}^{I} W_0^i \mathbf{w}_i \tag{6}$$

is satisfied.[6] Let $\widetilde{W}_m$ represent the random end-of-period wealth in the economy. Similarly to Huang and Litzenberger (1988, Section 4.15), we define the rate of return $\widetilde{r}_m$ on the aggregate market wealth as the one which satisfies

$$\widetilde{W}_m = \sum_{i=1}^{I} \widetilde{W}_i = W_{m0}(1 + \widetilde{r}_m). \tag{7}$$

Substituting (1) into the right hand side of the first equality of (7) and performing some algebraic manipulations we find that $\widetilde{r}_m$ can also be rewritten in terms of aggregate wealth proportions as

$$\widetilde{r}_m = r_f + \mathbf{w}_a^\top \left(\widetilde{\mathbf{r}} - r_f \mathbf{1} \right). \tag{8}$$

Then we have the following result when the market is in equilibrium.

Proposition 1. *Let* $\Theta = (\sum_{i=1}^{I} \theta_i^{-1})^{-1}$. *Define a consensus belief in the covariance matrix and the expected return vector, respectively, as*

$$\mathbf{\Omega}_a = \left(\Theta \sum_{i=1}^{I} \theta_i^{-1} \mathbf{\Omega}_i^{-1} \right)^{-1}, \tag{9}$$

$$E_a[\widetilde{\mathbf{r}}] = \Theta \mathbf{\Omega}_a \sum_{i=1}^{I} \theta_i^{-1} \mathbf{\Omega}_i^{-1} E_i[\widetilde{\mathbf{r}}]. \tag{10}$$

Then, when the market is in equilibrium,

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[5]The optimal portfolio $\mathbf{w}_i$ of agent i is only implicitly defined by (5), because in general $\theta_i = \theta_i(\mathbf{w}_i)$ will depend on $\mathbf{w}_i$. Nevertheless, at this stage we are interested in equilibrium relationships involving aggregate beliefs, which do not require $\mathbf{w}_i$ to be made explicit.

[6]The condition (6) is in monetary units, it can also be expressed as aggregate demand (in quantity terms) for risky assets equals aggregate supply (also quantity terms) on dividing throughout by the equilibrium price.

(i) *the vector of proportions* $\mathbf{w}_a$ *of the total wealth in the economy invested in the risky assets is given by*

$$\mathbf{w}_a = \frac{1}{W_{m0}} \Theta^{-1} \boldsymbol{\Omega}_a^{-1} \left(E_a[\widetilde{\mathbf{r}}] - r_f \mathbf{1} \right). \tag{11}$$

(ii) *the expected return on the aggregate market wealth*

$$E_a(\widetilde{r}_m) = r_f + \Theta W_{m0} \sigma_{a,m}^2, \tag{12}$$

where

$$\sigma_{a,m}^2 = \mathbf{w}_a^T \boldsymbol{\Omega}_a \mathbf{w}_a \tag{13}$$

is the variance of the aggregate market wealth return.

(iii) *the expected returns of the risky assets satisfy*

$$E_a[\widetilde{\mathbf{r}}] - r_f \mathbf{1} = \boldsymbol{\beta}_a (E_a(\widetilde{r}_m) - r_f), \tag{14}$$

where

$$\boldsymbol{\beta}_a = (\beta_{a,1}, \beta_{a,2}, \cdots, \beta_{a,N})^\top = \frac{1}{\sigma_{a,m}^2} \boldsymbol{\Omega}_a \mathbf{w}_a, \qquad \beta_{a,j} = \sigma_{a,jm} / \sigma_{a,m}^2. \tag{15}$$

Proof. See the Appendix □

Note that the existence of the consensus covariance matrix $\boldsymbol{\Omega}_a$ follows from the fact that, in equation (9), $\boldsymbol{\Omega}_a^{-1}$ is a convex combination of positive definite matrices $\boldsymbol{\Omega}_i^{-1}$, which implies that $\boldsymbol{\Omega}_a^{-1}$ is also positive definite, and therefore nonsingular. Note also that when the consensus belief is replaced with the objective and homogeneous belief, Proposition 1 results in the standard CAPM. When agents have heterogeneous beliefs, the consensus beliefs defined in Proposition 1 provides an explicit way to aggregate the heterogeneous beliefs, under which the standard CAPM-like relation (14) holds under the heterogeneous beliefs. Note that ΘW_{m0} can be interpreted as the *aggregate relative risk aversion* of the economy in equilibrium. In particular, when $\theta_i = \theta_0$ for $i = 1, 2, \cdots, I$, we have $\Theta = \theta_0 / I$ and $\Theta W_{m0} = \theta_0 (W_{m0}/I)$, measuring the relative risk aversion of an agent at the average wealth level. The market equilibrium condition (6) allows a non-zero supply of the riskless asset in the economy. If a zero net supply of the riskless asset is assumed when the market is in equilibrium, we then obtain the following corollary.

Corollary 1. *In market equilibrium, if the riskless asset is in zero net supply in the economy, then the risk-free rate* r_f *is given by*

$$r_f = \frac{\mathbf{1}^\top \boldsymbol{\Omega}_a^{-1} E_a[\widetilde{\boldsymbol{r}}] - \Theta W_{m0}}{\mathbf{1}^\top \boldsymbol{\Omega}_a^{-1} \mathbf{1}}. \tag{16}$$

In this case the return $\widetilde{r}_m$ *on the aggregate market wealth becomes the return on the market portfolio of the risky assets, and the variance* $\sigma_{a,m}^2$ *becomes the variance of the market portfolio of the risky assets.*

Proof. It follows from $\mathbf{1}^\top \boldsymbol{w}_a = 1$ and (11) in Proposition 1 that

$$\Theta W_{m0} = \mathbf{1}^T \boldsymbol{\Omega}_a^{-1}(E_a[\widetilde{\mathbf{r}}] - r_f \mathbf{1}). \tag{17}$$

Solving for r_f leads to the result. □

Corollary 1 shows that the equilibrium risk-free rate r_f is determined endogenously when the riskless asset is in zero net supply in the economy. It in fact depends on the aggregate relative risk aversion coefficient ΘW_{m0} and the consensus beliefs in the expected return and the variance-covariance matrix of the risky assets.

In order to understand the impact of the market wealth on the risk premia and beta coefficients of the risky assets, we provide the following result.

Corollary 2. *In market equilibrium, the expected return of the economy is given by*

$$E_a(\widetilde{r}_m) = r_f + \frac{1}{\Theta W_{m0}}(E_a[\widetilde{\mathbf{r}}] - r_f \mathbf{1})^\top \boldsymbol{\Omega}_a^{-1}(E_a[\widetilde{\mathbf{r}}] - r_f \mathbf{1}), \tag{18}$$

the variance is given by

$$\sigma_{a,m}^2 = \frac{1}{(\Theta W_{m0})^2}(E_a[\widetilde{\mathbf{r}}] - r_f \mathbf{1})^\top \boldsymbol{\Omega}_a^{-1}(E_a[\widetilde{\mathbf{r}}] - r_f \mathbf{1}), \tag{19}$$

and the beta coefficients can be rewritten as

$$\boldsymbol{\beta}_a = \frac{\Theta W_{m0}}{(E_a[\widetilde{\mathbf{r}}] - r_f \mathbf{1})^\top \boldsymbol{\Omega}_a^{-1}(E_a[\widetilde{\mathbf{r}}] - r_f \mathbf{1})}(E_a[\widetilde{\mathbf{r}}] - r_f \mathbf{1}). \tag{20}$$

Proof. In market equilibrium, equation (18) follows from (11) and the result $E_a(\widetilde{r}_m) = r_f + \boldsymbol{w}_a^\top(E_a[\widetilde{\mathbf{r}}] - r_f \mathbf{1})$ (see equation (A.6) of the Appendix); equation (19) follows from and (11) and (13); and equation (20) follows from (11), (15) and (19). □

Corollary 2 expresses the equilibrium relationships where the riskless asset is not necessarily in zero net supply. If the riskless asset is in zero net supply the equilibrium relationships turn out to be explicitly independent of the total wealth in the economy.

Corollary 3. *If the riskless asset is in zero net supply in the economy, then the expected return of the market portfolio of the risky assets is given by*

$$E_a(\widetilde{r}_m) = r_f + \frac{(E_a[\widetilde{\mathbf{r}}] - r_f \mathbf{1})^\top \boldsymbol{\Omega}_a^{-1}(E_a[\widetilde{\mathbf{r}}] - r_f \mathbf{1})}{\mathbf{1}^\top \boldsymbol{\Omega}_a^{-1}(E_a[\widetilde{\boldsymbol{r}}] - r_f \mathbf{1})}, \tag{21}$$

the variance of the market portfolio of the risky assets is given by

$$\sigma_{a,m}^2 = \frac{(E_a[\widetilde{\mathbf{r}}] - r_f \mathbf{1})^\top \boldsymbol{\Omega}_a^{-1}(E_a[\widetilde{\mathbf{r}}] - r_f \mathbf{1})}{(\mathbf{1}^\top \boldsymbol{\Omega}_a^{-1}(E_a[\widetilde{\boldsymbol{r}}] - r_f \mathbf{1}))^2}, \tag{22}$$

and the beta coefficients can be rewritten as

$$\boldsymbol{\beta}_a = \frac{\mathbf{1}^\top \boldsymbol{\Omega}_a^{-1}(E_a[\widetilde{\mathbf{r}}] - r_f\mathbf{1})}{(E_a[\widetilde{\mathbf{r}}] - r_f\mathbf{1})^\top \boldsymbol{\Omega}_a^{-1}(E_a[\widetilde{\mathbf{r}}] - r_f\mathbf{1})}(E_a[\widetilde{\mathbf{r}}] - r_f\mathbf{1}). \tag{23}$$

Proof. When the riskless asset in the economy is in zero net supply, we have that (17) holds. Using this to replace ΘW_{m0} with $\mathbf{1}^\top \boldsymbol{\Omega}_a^{-1}(E_a[\widetilde{\mathbf{r}}] - r_f\mathbf{1})$ in equations (18), (19) and (20), we obtain equations (21), (22) and (23), respectively. □

It is interesting to note that, when the risk-free rate r_f is given exogenously and the riskless asset is not in zero net supply, the expected return and variance of the economy and the beta coefficients of the risky assets with the economy[7] depend on the total wealth in the economy. However, when the riskless asset in the economy is in zero net supply, the return of the economy is given by the return of the market portfolio of the risky assets. Consequently, the expected return and variance of the market portfolio and the beta coefficients of the risky assets with the market portfolio do not depend explicitly on the wealth. This difference, generated from the restriction that the riskless asset in the economy be in zero net supply, has the potential to explain the impact of heterogeneous beliefs on the risk-free rate and risk premium puzzles. We refer the reader to He and Shi (2009) for further discussion of this issue. To obtain the equilibrium prices of the risky assets, we assume that agents have CARA exponential utility of wealth functions, so that the global absolute risk aversion of agent i, $\theta_i = -E_i\left[u_i''(\widetilde{W}_i)\right]/E_i\left[u_i'(\widetilde{W}_i)\right]$, and hence the aggregate risk aversion Θ, are constants. Let $\mathbf{z} := [z_1, z_2, ..., z_N]^\top$ be the positive supply vector (number of shares) of the risky assets in the economy and denote by $\mathbf{Z} := diag[z_1, z_2, ..., z_N]$ the $(N \times N)$ diagonal matrix whose entries are the elements of $\mathbf{z}$. Then the market equilibrium prices of the risky assets can be determined according to the following corollary.

Corollary 4. *Let $\mathbf{p}_0 = [p_{01}, p_{02}, ..., p_{0N}]^\top$ be the vector of the prices of the risky assets when the market is in equilibrium. Then*

$$\mathbf{p}_0 = \mathbf{Z}^{-1}\Theta^{-1}\boldsymbol{\Omega}_a^{-1}\left(E_a[\widetilde{\mathbf{r}}] - r_f\mathbf{1}\right) = \mathbf{Z}^{-1}\sum_{i=1}^{I}\theta_i^{-1}\boldsymbol{\Omega}_i^{-1}\left(E_i[\widetilde{\mathbf{r}}] - r_f\mathbf{1}\right) \tag{24}$$

and the beta coefficients can be written as

$$\boldsymbol{\beta}_a = \frac{W_{m0}}{\mathbf{p}_0^\top \mathbf{Z}\boldsymbol{\Omega}_a\mathbf{Z}\mathbf{p}_0}\boldsymbol{\Omega}_a\mathbf{Z}\mathbf{p}_0. \tag{25}$$

[7]Note we distinguish between beta of the economy when the riskfree asset is not in zero net supply and the beta of the market (obtained when the riskfree asset is in zero net supply).

In particular, when the riskless asset is in zero net supply in the economy,

$$\boldsymbol{\beta}_a = \frac{\mathbf{p}_0^\top \mathbf{z}}{\mathbf{p}_0^\top \mathbf{Z}\boldsymbol{\Omega}_a \mathbf{Z}\mathbf{p}_0} \boldsymbol{\Omega}_a \mathbf{Z}\mathbf{p}_0. \tag{26}$$

Proof. Given the positive supply of the risky assets in the economy, the prices of the risky assets when the market is in equilibrium satisfy the condition $W_{m0}\boldsymbol{w}_a = \boldsymbol{Z}\boldsymbol{p}_0$. Substituting $\boldsymbol{w}_a$ from (11) into the last condition, we obtain the first equality in (24), the second follows by use of (10). Replacing $E_a[\widetilde{\mathbf{r}}] - r_f\mathbf{1}$ with $\Theta\boldsymbol{\Omega}_a\boldsymbol{Z}\boldsymbol{p}_0$ in equations (20) and (23) we then obtain the expressions (25) and (26) for $\boldsymbol{\beta}_a$, respectively. □

One of the advantages of the expressions for the beta coefficients in Corollary 4 is that we can use the market information about the observed beta coefficients and market prices to estimate the market consensus covariance matrix $\boldsymbol{\Omega}_a$, which may not be observed or difficult to estimate in a heterogeneous beliefs market. The implications of this observation for empirical studies is left for future research.

3. A Dynamic Market Fraction CAPM

The present section first sets up a framework for a market fraction model with heterogeneous beliefs, which extends contributions developed by Brock and Hommes (1998), Chiarella and He (2001, 2002) and He and Li (2008) in the simple case of a single risky security to a multi-asset framework. We then extend the framework to a repeated one period dynamic CAPM model. Related, but different, studies of the CAPM with heterogeneous beliefs can be found in Böhm and Chiarella (2005) and Böhm and Wenzelburger (2005).

Assume that the I investors can be grouped into a finite number of agent-types, indexed by $h \in H$, where the agents within the same group are homogeneous in their beliefs $E_h[\widetilde{\mathbf{r}}]$ and $\boldsymbol{\Omega}_h$, as well as risk aversion coefficient θ_h. Denote I_h, $h \in H$, the number of investors in group h and $n_h := I_h/I$ the market fraction of agents of type h. We then denote by $\mathbf{s} := (1/I)\mathbf{z}$ the supply of shares per investor. Note that, instead of using the aggregate risk aversion coefficient $\Theta := \left(\sum_{i=1}^I \theta_i^{-1}\right)^{-1}$ it is convenient to define the "average" risk aversion θ_a as

$$\theta_a := \left(\sum_{h\in H} n_h \theta_h^{-1}\right)^{-1},$$

where obviously $\theta_a = I\Theta$. It follows from Proposition 1 that the aggregate beliefs about variances/covariances and expected returns can be rewritten, respectively, as

$$\boldsymbol{\Omega}_a = \theta_a^{-1}\left(\sum_{h\in H} n_h \theta_h^{-1}\boldsymbol{\Omega}_h^{-1}\right)^{-1}, \qquad E_a[\widetilde{\mathbf{r}}] = \theta_a \boldsymbol{\Omega}_a \sum_{h\in H} n_h \theta_h^{-1}\boldsymbol{\Omega}_h^{-1} E_h[\widetilde{\mathbf{r}}].$$

Finally, by defining $\mathbf{S} = diag(s_1, s_2, \cdots, s_N)$, the equilibrium prices in (21) can be rewritten as

$$\mathbf{p}_0 = \mathbf{S}^{-1}\theta_a^{-1}\mathbf{\Omega}_a^{-1}\left(E_a[\widetilde{\mathbf{r}}] - r_f\mathbf{1}\right) = \boldsymbol{S}^{-1}\sum_{h\in H} n_h\theta_h^{-1}\mathbf{\Omega}_h^{-1}(E_h[\widetilde{\mathbf{r}}] - r_f\mathbf{1}).$$

We now turn to the process of formation of heterogeneous beliefs and equilibrium prices in a dynamic setting, from time t to time $t+1$. In doing so, we take the view that agents' beliefs about the returns $\widetilde{\mathbf{r}}_{t+1}$ in the time interval $(t, t+1)$, which are formed before dividends at time t are realized and prices at time t are revealed by the market, determine the aggregate demand for each risky asset at time t, which in turns produces the equilibrium prices at time t, $\mathbf{p}_t$, via the market clearing conditions. Of course, once prices and dividends at time t are realized, the returns $\mathbf{r}_t$ become known. More precisely, we assume that agents' assessments of the end-of-period joint distribution of the returns $\widetilde{\mathbf{r}}_{t+1}$ are formed at time t before the equilibrium prices at time t are determined. These beliefs remain fixed while the market finds its equilibrium vector of current prices, $\mathbf{p}_t$. In particular, the heterogeneous beliefs (or assessments) of agents about the mean and covariance structure of $\widetilde{\mathbf{r}}_{t+1}$ are functions of the information up to time $t-1$, which can be expressed as functions of the realized returns $\mathbf{r}_{t-1}$, $\mathbf{r}_{t-2}$, ..., for any group, or belief-type $h \in H$,[8]

$$\mathbf{\Omega}_{h,t} := [Cov_{h,t}(\widetilde{r}_{j,t+1}, \widetilde{r}_{k,t+1})] = \mathbf{\Omega}_h(\mathbf{r}_{t-1}, \mathbf{r}_{t-2}, ...), \tag{27}$$

$$E_{h,t}[\widetilde{\mathbf{r}}_{t+1}] = \mathbf{f}_h(\mathbf{r}_{t-1}, \mathbf{r}_{t-2}, ...), \tag{28}$$

where obviously similar representations hold also for the aggregate beliefs $\mathbf{\Omega}_{a,t} := [Cov_{a,t}(\widetilde{r}_{j,t+1}, \widetilde{r}_{k,t+1})]$ and $E_{a,t}[\widetilde{\mathbf{r}}_{t+1}]$. The market clearing prices at time t become

$$\mathbf{p}_t = \mathbf{S}^{-1}\sum_{h\in H} n_h\theta_h^{-1}\mathbf{\Omega}_{h,t}^{-1}(E_{h,t}[\widetilde{\mathbf{r}}_{t+1}] - r_{f,t}\mathbf{1}), \tag{29}$$

or in terms of the consensus beliefs,

$$\mathbf{p}_t = \mathbf{S}^{-1}\theta_a^{-1}\mathbf{\Omega}_{a,t}^{-1}(E_{a,t}[\widetilde{\mathbf{r}}_{t+1}] - r_{f,t}\mathbf{1}), \tag{30}$$

where $r_{f,t}$ is the riskfree rate over the time period from t to $t+1$.

Next, note that the return $r_{j,t}$ on asset j, realized over the time interval $(t-1, t)$ is given by

$$r_{j,t} = \frac{p_{j,t} + d_{j,t}}{p_{j,t-1}} - 1,$$

where $d_{j,t}$ denotes the realized dividend per share of asset j, $j = 1, 2, ..., N$. We can rewrite realized returns in vector notation as

$$\mathbf{r}_t = \mathbf{P}_{t-1}^{-1}(\mathbf{p}_t + \mathbf{d}_t) - \mathbf{1}, \tag{31}$$

[8]We use $E_{h,t}(\widetilde{r}_{t+1})$ to denote the expectation of $\widetilde{r}_{t+1}$ formed at time t by the agents of group h. Similarly for the notation $Cov_{h,t}(\widetilde{r}_{j,t+1}, \widetilde{r}_{k,t+1})$.

where $\mathbf{d}_t := [d_{1,t}, d_{2,t}, ..., d_{N,t}]^\top$, and $\mathbf{P}_t := diag(p_{1,t}, p_{2,t}, ..., p_{N,t})$. Equation (31), via the market equilibrium prices (29) and the beliefs updating equations (27) and (28), gives the return $\mathbf{r}_t$ as a function of $\mathbf{r}_{t-1}$, $\mathbf{r}_{t-2}, ...$ and of the realized dividends $\mathbf{d}_t$, which are assumed to follow an exogenous random process in general. Thus the dynamics of prices and returns are determined by both the endogenous dependence of returns on past returns in (31) and the exogenous stochastic dividend process.

We summarize below the dynamical system that describes the market fraction multi-asset model in terms of returns, where the market clearing prices are used as auxiliary variables

Proposition 2. *For the market fraction model, the equilibrium return vector of the risky assets is given by*

$$\mathbf{r}_t = \mathbf{F}(\mathbf{r}_{t-1}, \mathbf{r}_{t-2}, ...; \widetilde{\mathbf{d}}_t) = \mathbf{P}_{t-1}^{-1}(\mathbf{p}_t + \widetilde{\mathbf{d}}_t) - \mathbf{1},$$

where

$$\mathbf{p}_t = \mathbf{S}^{-1} \sum_{h \in H} n_h \theta_h^{-1} \mathbf{\Omega}_{h,t}^{-1}(E_{h,t}[\widetilde{\mathbf{r}}_{t+1}] - r_{f,t}\mathbf{1}),$$

$\mathbf{P}_t := diag(p_{1,t}, p_{2,t}, ..., p_{N,t})$, $\mathbf{\Omega}_{h,t} = \mathbf{\Omega}_h(\mathbf{r}_{t-1}, \mathbf{r}_{t-2}, ...)$, *and* $E_{h,t}(\widetilde{\mathbf{r}}_{t+1}) = \mathbf{f}_h(\mathbf{r}_{t-1}, \mathbf{r}_{t-2}, ...)$. *Moreover, at the beginning of each time interval* $(t, t+1)$ *the expected returns under the aggregate beliefs (based on information up to time* $t-1$*) satisfy a CAPM-like equation of the type*

$$E_{a,t}[\widetilde{\mathbf{r}}_{t+1}] - r_{f,t}\mathbf{1} = \boldsymbol{\beta}_{a,t}(E_{a,t}(\widetilde{r}_{m,t+1}) - r_{f,t}),$$

where $\widetilde{r}_{m,t+1}$ *is the rate of the return of the aggregate market wealth over the time period from* t *to* $t+1$ *defined by* $\widetilde{W}_{m,t+1} = \widetilde{W}_{m,t}(1 + \widetilde{r}_{m,t+1})$, *and* $\widetilde{W}_{m,t}$ *is the aggregate wealth in the economy at time* t. *Under the dynamical consensus belief, the rate of return on the aggregate market wealth is given by*

$$\widetilde{r}_{m,t+1} = r_{f,t} + \frac{1}{\Theta W_{m,t}}(E_{a,t}[\widetilde{\mathbf{r}}_{t+1}] - r_{f,t}\mathbf{1})^\top \mathbf{\Omega}_{a,t}^{-1}(\widetilde{\mathbf{r}}_{t+1} - r_{f,t}\mathbf{1}).$$

and the "aggregate" beta coefficients are given by

$$\boldsymbol{\beta}_{a,t} = \frac{\Theta W_{m,t}}{(E_{a,t}[\widetilde{\mathbf{r}}_{t+1}] - r_{f,t}\mathbf{1})^\top \mathbf{\Omega}_{a,t}^{-1}(E_{a,t}[\widetilde{\mathbf{r}}_{t+1}] - r_{f,t}\mathbf{1})}(E_{a,t}[\widetilde{\mathbf{r}}_{t+1}] - r_{f,t}\mathbf{1}).$$

As in the discussion of the static framework in Section 2, in the case of zero net supply of the riskless asset the relationships in Proposition 2 become independent of the total wealth in the economy. Thus we can state

Proposition 3. *If the riskless asset is in zero net supply over the time period in the economy, then the equilibrium risk-free rate is given by*

$$r_{f,t} = \frac{\mathbf{1}^\top \mathbf{\Omega}_{a,t}^{-1} E_{a,t}[\widetilde{\boldsymbol{r}}_{t+1}] - \Theta W_{m,t}}{\mathbf{1}^\top \mathbf{\Omega}_{a,t}^{-1}\mathbf{1}}.$$

Consequently,

$$\widetilde{r}_{m,t+1} = \frac{(E_{a,t}[\widetilde{\mathbf{r}}_{t+1}] - r_{f,t}\mathbf{1})^\top \mathbf{\Omega}_{a,t}^{-1} \widetilde{\mathbf{r}}_{t+1}}{(E_{a,t}[\widetilde{\mathbf{r}}_{t+1}] - r_{f,t}\mathbf{1})^\top \mathbf{\Omega}_{a,t}^{-1}\mathbf{1}}$$

and

$$\boldsymbol{\beta}_{a,t} = \frac{(E_{a,t}[\widetilde{\mathbf{r}}_{t+1}] - r_{f,t}\mathbf{1})^\top \mathbf{\Omega}_{a,t}^{-1}\mathbf{1}}{(E_{a,t}[\widetilde{\mathbf{r}}_{t+1}] - r_{f,t}\mathbf{1})^\top \mathbf{\Omega}_{a,t}^{-1}(E_{a,t}[\widetilde{\mathbf{r}}_{t+1}] - r_{f,t}\mathbf{1})}(E_{a,t}[\widetilde{\mathbf{r}}_{t+1}] - r_{f,t}\mathbf{1}).$$

Note that the "aggregate" betas are time varying due to time varying beliefs about both the first and second moments of the returns distribution.

4. Discussion

Unlike the traditional paradigm of the representative agent and rational expectations, recent literature has directed a great deal of attention to a new paradigm of heterogeneity and bounded rationality. The new paradigm provides a platform for analysing the complicated market behaviour that comes from the interaction of heterogeneous, boundedly rational and adaptive agents and for explaining empirical anomalies which are a challenge for the traditional paradigm. It becomes clear that heterogeneity and bounded rationality play very important roles in our understanding of economic behaviour, in particular, their impact on the financial market. It is widely recognized that heterogeneity can have a significant impact on asset pricing. As one of the fundamental asset pricing equilibrium models, the CAPM plays a very important role in modern finance and economics. However, the framework of the traditional paradigm makes it difficult to examine the impact of heterogeneity and bounded rationality on asset pricing. This paper provides a framework for the analysis of CAPM within the new paradigm.

The main obstacle in dealing with heterogeneity is the complexity and heavy notation involved when the number of assets and the dimension of the heterogeneity increase. Within the mean-variance framework with heterogeneous beliefs, this paper overcomes this obstacle by constructing a consensus belief explicitly in order to characterize the market aggregation of the heterogeneous beliefs. Based on the consensus belief, we are able to set up a general framework for the CAPM to incorporate heterogeneous beliefs. We also extend the framework to a repeated one-period dynamical market fraction model. Within this framework, we are able to characterize exactly the relationships between market belief in equilibrium and heterogeneous beliefs, between the market risk premium of each risky asset and its beta coefficient, and derive the dynamics of beta coefficients and market equilibrium prices.

The framework provided in this paper can be used to examine the impact of various types of heterogeneity and bounded rationality on market prices and risk. For example, we may use the framework to explore the following questions: how do the optimistic or pessimistic views of agents and their confidence about their views influence the risk-free rate, equity premium and market price of the risk? which belief or investment strategy will have significant impact on the market equilibrium price? Recent HAMs literature that considers portfolios of one riskless asset and one risky asset demonstrates that bounded rational behaviour of heterogeneous agents can cause the market to be more complicated and less efficient than the standard paradigm allows for, generating many of the stylized facts and observed market anomalies. Within the framework of the dynamic CAPM with multiple risky assets, we can examine if the traditional diversification effect still holds. We can also study how learning and adaptive behaviour of heterogeneous agents contribute to the survivability of agents and market volatility. In particular, it would be interesting to know if the framework for the dynamic CAPM can be used to explain empirical evidence on the time variation of beta, which measures the time varying risk of risky assets. We believe that the framework established in this paper can be used to tackle such questions and issues, all of which we leave to future research.

Appendix

Proof of Proposition 1. With the optimal portfolio $\mathbf{w}_i$ defined by (5), we sum (5) across i and obtain

$$\sum_{i=1}^{I} W_0^i \mathbf{w}_i = \sum_{i=1}^{I} \theta_i^{-1} \mathbf{\Omega}_i^{-1} \left(E_i[\widetilde{\mathbf{r}}] - r_f \mathbf{1} \right). \tag{A.1}$$

In market equilibrium, it follows from (A.1) that the proportions of the market wealth invested in the risky assets are given by

$$\mathbf{w}_a = \frac{1}{W_{m0}} \sum_{i=1}^{I} W_0^i \mathbf{w}_i = \frac{1}{W_{m0}} \sum_{i=1}^{I} \theta_i^{-1} \mathbf{\Omega}_i^{-1} \left(E_i[\widetilde{\mathbf{r}}] - r_f \mathbf{1} \right). \tag{A.2}$$

Using the ‘consensus’ belief about the variance and covariance matrix of returns, $\mathbf{\Omega}_a$, defined in (9) of Proposition (1), we have

$$\mathbf{\Omega}_a^{-1} = \Theta \sum_{i=1}^{I} \theta_i^{-1} \mathbf{\Omega}_i^{-1}, \tag{A.3}$$

where we recall that $\Theta := \left(\sum_{i=1}^{I} \theta_i^{-1} \right)^{-1}$. Then it follows from (A.2) (A.3) and the ‘consensus’ belief about the market aggregate return,

$E_a[\widetilde{\mathbf{r}}]$, defined in (10) of Proposition 1 that

$$\begin{aligned}\mathbf{w}_a &= \frac{1}{W_{m0}}\left(\sum_{i=1}^{I}\theta_i^{-1}\mathbf{\Omega}_i^{-1}E_i[\widetilde{\mathbf{r}}] - \Theta^{-1}\mathbf{\Omega}_a^{-1}r_f\mathbf{1}\right)\\ &= \frac{1}{\Theta W_{m0}}\mathbf{\Omega}_a^{-1}\left(\Theta\mathbf{\Omega}_a\sum_{i=1}^{I}\theta_i^{-1}\mathbf{\Omega}_i^{-1}E_i[\widetilde{\mathbf{r}}] - r_f\mathbf{1}\right)\\ &= \frac{1}{\Theta W_{m0}}\mathbf{\Omega}_a^{-1}\left(E_a[\widetilde{\mathbf{r}}] - r_f\mathbf{1}\right),\end{aligned}$$

from which

$$\mathbf{\Omega}_a\mathbf{w}_a = \frac{1}{\Theta W_{m0}}\left(E_a[\widetilde{\mathbf{r}}] - r_f\mathbf{1}\right). \tag{A.4}$$

Then, with the consensus belief, the variance of the market return $\sigma_{a,m}^2 = \mathbf{w}_a^\top\mathbf{\Omega}_a\mathbf{w}_a$ is given by

$$\sigma_{a,m}^2 = \frac{1}{\Theta W_{m0}}\mathbf{w}_a^\top\left(E_a[\widetilde{\mathbf{r}}] - r_f\mathbf{1}\right), \tag{A.5}$$

and from (8) the expected market return is given by

$$E_a(\widetilde{r}_m) = r_f + \mathbf{w}_a^\top\left(E_a\left[\widetilde{\mathbf{r}}\right] - r_f\mathbf{1}\right). \tag{A.6}$$

Both (A.6) and (A.5) imply that

$$E_a(\widetilde{r}_m) - r_f = \Theta W_{m0}\sigma_{a,m}^2 > 0, \tag{A.7}$$

that is, the aggregate expected market risk premium is proportional to the aggregate relative risk aversion of the economy and the market risk.

It follows from (A.4) and (A.7) that

$$E_a\left[\widetilde{\mathbf{r}}\right] - r_f\mathbf{1} = \frac{E_a(\widetilde{r}_m) - r_f}{\sigma_{a,m}^2}\mathbf{\Omega}_a\mathbf{w}_a \tag{A.8}$$

The entries of $\mathbf{\Omega}_a\mathbf{w}_a$ represent the aggregate covariances between the return on each risky asset and the return on the aggregate market wealth,

$$\mathbf{\Omega}_a\mathbf{w}_a = [\sigma_{a,jm}], \qquad \sigma_{a,jm} := Cov_a(\widetilde{r}_j, \widetilde{r}_m), \qquad j = 1, 2, ..., N$$

so that (A.8) can be rewritten componentwise as

$$E_a(\widetilde{r}_j) - r_f = \frac{\sigma_{a,jm}}{\sigma_{a,m}^2}[E_a(\widetilde{r}_m) - r_f], \qquad j = 1, 2, ..., N, \tag{A.9}$$

where $\sigma_{a,jm}/\sigma_{a,m}^2 = \beta_{a,j}$ represents the aggregate beta coefficient of the j-th risky asset. Equation (A.9) is the traditional CAPM relation generalized to the case of heterogeneous beliefs. The vector $\boldsymbol{\beta}_a := [\beta_{a,1}, \beta_{a,2}, ..., \beta_{a,N}]^\top$ of the aggregate beta coefficients in (A.8) is thus given by

$$\boldsymbol{\beta}_a = \frac{1}{\sigma_{a,m}^2}\mathbf{\Omega}_a\mathbf{w}_a. \tag{A.10}$$

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