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Strong composition down. Characterizations of new and classical bankruptcy rules.

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STRONG COMPOSITION DOWN. CHARACTERIZATIONS OF NEW AND CLASSICAL BANKRUPTCY RULES*

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Abstract

This paper is devoted to the study of claims problems. We identify the family of rules that satisfy *strong composition down* (robustness with respect to reevaluations of the estate) and *consistency* (robustness with respect to changes in the set of agents) together. Such a family is the *fixed path rules*, which is a generalization of the *weighted constrained equal awards rules*. In addition, once *strong composition down* and *consistency* are combined with *homogeneity* only the *weighted constrained equal awards* rules survive. We also prove that the *constrained equal awards rule* is the only rule that satisfies *strong composition down*, *consistency* and *equal treatment of equals* together.

Keywords: strong composition down, fixed path rules, constrained equal awards rule, weighted constrained equal awards rules.

JEL Classification: D63, D70

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1 Introduction

How to adjudicate conflicting demands is a very old question, which was firstly model by O'Neill (1982). This class of claims problems refers to all situations in which a given quantity of a commodity has to be distributed among some agents when the available resource falls short of the total demand. The canonical illustration is the allotment of liquidation value of bankrupt firm among its creditors. The reader is referred to Thomson (2003) for a wide exposition of the literature.

Any *claims problem* is determined by three elements, a set of *agents*, an available amount of resource, called *estate*, and a vector of demands or *claims*. A *rule* is a way of distributing the available estate among the agents according to their claims. In this work, we follow the axiomatic approach, justifying the rules in terms of the properties they satisfy. These properties usually refer to notions of equity and stability.

One of the most widely studied rules is the so-called *constrained equal awards rule* (Maimonides, 12th Century). It proposes that all individuals should be treated uniformly, subject to no one receives more that her claim. This implies that, for any claims vector, all the agents who do not get their claim receive equal amounts, independently of how small or large the estate is. A generalization of this idea underlies the *weighted constrained equal awards rules.* The objective is to favor agents who are perceived as more deserving. For a given vector of positive weights, and for any claims vector, for the corresponding *weighted constrained equal awards rule*, all the agents who do not get their claim receive amounts that, when divided by their respective weights, are equal, independently of how small or large the estate is. We consider here a further extension: *the fixed path rules.*[1] The rules in the fixed-path family respect the spirit of the two aforementioned rules. But, unlike them, for any claims vector, all the agents that do not get their claim receive an amount that does depend on how small or large the estate is.

Among the procedural properties normally required, *composition down* (Moulin (2000)) emerges as a useful requirement. Imagine that, when estimating the value of the estate, we were too optimistic, and the actual value is smaller than expected. Now, two alternatives are open. Either we solve the new problem. Or we consider a problem in which the estate is the small one, and the claims are the allocations obtained with the overestimated estate. Composition down requires the final allocation to be independent of the chosen alternative. In this property it is implicitly assumed that, either all agents unanimously demand the original claims, or all agents unanimously demand the awards for the overestimated estate. We propose here to revise such an assumption, and to consider the possibility that some agents demand their original claims while the others demand their adjusted claims. Again, two alternatives appear: either to solve the problem under the new claims vector, or directly. If the final allocation is always independent of the chosen alternative, we say that our rule satisfies *strong composition down.*

[1]These rules were firstly introduced by Moulin (1999) in the context of resource allocation with single-peaked preferences. We adapt them to claims problem keeping the same name.

This new property states that agents will not benefit from insisting on their initial claims when others accept the reduction given by the tentative awards corresponding to the overestimated estate.

We also consider a property that provides robustness with respect to changes in the set of agents. *Consistency* refers to a situation in which a tentative distribution of the estate has been made, and an agent leaves after accepting her award. It states that the reduced problem should be solved in such a way that all remaining agents are allotted exactly the same amount as they were originally.

Our main result says that the only solutions satisfying strong composition down and consistency together are the fixed path rules. Moulin (2000) characterizes the rules fulfilling composition down, *composition up*, *homogeneity*, and consistency. This family of rules (called $\mathcal{M}$ family) is extremely wide. Interestingly, once we strengthen composition down to strong composition down in Moulin's assumptions we separate the weighted constrained equal awards rules from all the others in the family $\mathcal{M}$.

The rest of the paper is structured as follows: In Section 2 we set up the model and we present the fixed path family. In Section 3 we introduce strong composition down and we present our main result. In Section 4 we explore other properties the fixed path rules fulfill, and we provide alternative characterizations of the weighted constrained equal awards and the constrained equal awards rules. In Section 5 we conclude with some final comments and remarks. The proofs are relegated to an appendix.

2 Statement of the model. The fixed-path rules

Let $\mathbb{N}$ be the set of all potential agents. Let $\mathcal{N}$ denote the family of all finite subsets of $\mathbb{N}$. Let $\gamma \in \mathbb{R}$ be an upper bound on the agents' demand.[2] In a claims problem, or simply a **problem**, a fixed amount $E \in \mathbb{R}_{++}$, called **estate**, has to be distributed among a group of **agents** $N \in \mathcal{N}$ according to their **claims** (represented by $c = (c_i)_{i \in N} \in [0, \gamma]^N$), when E is not enough to fully satisfy all the claims. Therefore, a problem is a triple $e = (N, E, c)$ where $\sum_{i \in N} c_i \geq E$ and $c_i \leq \gamma$ for all $i \in N$. We denote by $\mathbb{C}^N$ the class of claims problems with fixed population N, and by $\mathbb{C}$ the class of all claims problems, namely

$$\mathbb{C}^N = \left\{ e = (N, E, c) \in \{N\} \times \mathbb{R}_{++} \times [0, \gamma]^N : \sum_{i \in N} c_i \geq E \right\}$$

and

$$\mathbb{C} = \bigcup_{N \in \mathcal{N}} \mathbb{C}^N.$$

[2]This upper bound is not usual in the claims problems literature. But it makes the presentation simple and the proofs much more illustrative. However, all the results and considerations in this paper remain unchanged without this assumption.

An **awards vector** for $e \in \mathbb{C}$ is a division of the estate among the agents, that is, it is a list $\boldsymbol{x} \in \mathbb{R}^N_+$ such that: (a) Each agent receives a non-negative amount which is not larger than her claim (for each $i \in N$, $0 \leq x_i \leq c_i$); and (b) the estate is exactly distributed ($\sum_{i \in N} x_i = E$). Let $\boldsymbol{X(e)}$ be the set of all awards vectors for $e \in \mathbb{C}$. A **rule** is a way of selecting awards vectors, that is, it is a function, $\boldsymbol{S} : \mathbb{C} \longrightarrow \bigcup_{e \in \mathbb{C}} X(e)$, that selects, for each problem $e \in \mathbb{C}$, a unique awards vector $S(e) \in X(e)$.

Let S be a rule and let c be a fixed claims vector, $p_S(c)$ is the path followed by $S(E, c)$ as the estate E varies from 0 to C. The path $\boldsymbol{p_S(c)}$ is called **path of awards of $\boldsymbol{S}$ for $\boldsymbol{c}$**. Any rule can be defined by the collection of its paths of awards for the different claims vectors.

The following are two of the most prominent rules in the literature. Each of them corresponds to different ideas of fairness in the distribution of the estate. The first one follows the Aristotelian notion of justice, and proposes a distribution of the estate proportional to the claims.

Proportional rule, **p**: For each $e \in \mathbb{C}$, it selects the unique awards vector $p_i(e) = \lambda \cdot c_i$ for some $\lambda \in \mathbb{R}_+$ such that $\sum_{i \in N} \lambda \cdot c_i = E$.

The second comes from Maimonides (12th Century). It says that agents should be treated equally, independently of their differences in claims. Thus, the so-called *constrained equal awards rule* proposes equality in gains, adjusting, if necessary, to ensure that no agent receives more than her claim.

Constrained equal awards rule, **cea**: For each $e \in \mathbb{C}$, it selects the unique awards vector $cea_i(e) = \min\{c_i, \lambda\}$ for some $\lambda \in \mathbb{R}_+$ such that $\sum_{i \in N} \min\{c_i, \lambda\} = E$.

Now we consider a family of rules, the so-called *weighted constrained equal awards rules*. As their name suggests, they are a generalization of the constrained equal awards rule. In the cea rule, agents' claims are fully comparable. But it may happen that differences in agents' needs ask for some adjustments. This can be done by mean of a vector of weights. For each $i \in \mathbb{N}$, let $\alpha_i \in \mathbb{R}_{++}$ be claimant i's weight, and $\alpha = (\alpha_i)_{i \in \mathbb{N}}$ the vector of weights. These weights reflect how deserving each agent is.

Weighted constrained equal awards rule with weights $\boldsymbol{\alpha = (\alpha_i)_{i \in \mathbb{N}}}$, $\mathbf{cea}^{\boldsymbol{\alpha}}$: For each $e \in \mathbb{C}$, it selects the unique awards vector $cea^\alpha_i(e) = \min\{c_i, \alpha_i \lambda\}$ for some $\lambda \in \mathbb{R}_+$ such that $\sum_{i \in N} \min\{c_i, \alpha_i \cdot \lambda\} = E$.

It is quite obvious that the constrained equal awards rule is the particular weighted constrained equal awards rule in which all the agents have the same weight.

Figure 1 illustrates the three aforementioned rules by showing the paths of awards for several claim vectors.

[Insert Figure 1 about here.]

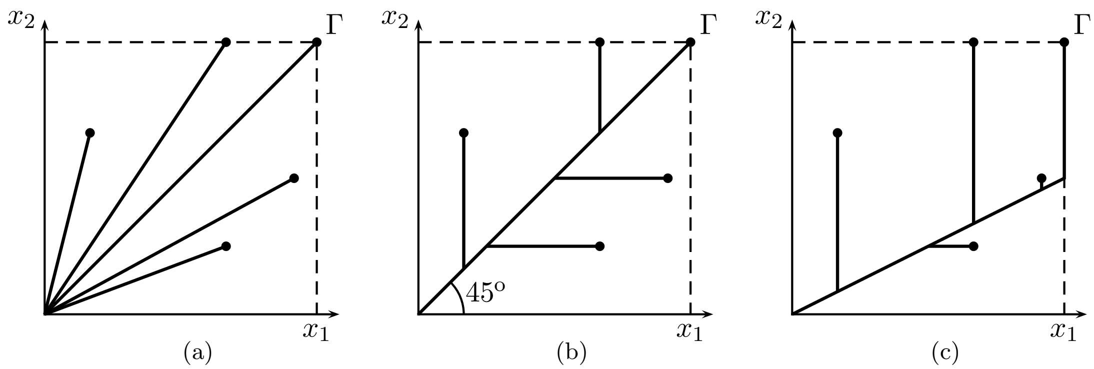

Figure 1: Path of awards for different claims vectors in two-agent problems. (a) Proportional rule. (b) Constrained equal awards rule. (c) Weighted constrained equal awards rule for $\alpha = (2, 1)$. Γ denotes the claim vector whose components are all equal to γ, $\Gamma = (\gamma, \ldots, \gamma)$.

Next we present the *fixed path rules*. These solutions were firstly introduced by Moulin (1999) in the context of resource allocation with single-peaked preferences. We adapt them to claims problem. Let us focus for a while on the two-agent framework, $N = \{i, j\}$. The typical path of awards of a fixed path rule is described as follows. There is a main ray from the origin to Γ (that will be the *fixed path*), and vertical and horizontal rays emanating from the fixed path and ending at the claims vector. Therefore, as Figure 1 shows, the *cea* and $cea^{(2,1)}$ rules are particular cases of fixed path rules. The proportional rule is not.

To provide a formal description of the fixed path rules we introduce some auxiliary notions. For each agent set $N \in \mathcal{N}$, an ***N*-path** is a mapping $\boldsymbol{p^N} : [0, n\gamma] \longrightarrow \mathbb{R}^N$ such that (a) p^N is monotonic (for each $z, z' \in [0, n\gamma]$, if $z \leq z'$ then $p^N(z) \leq p^N(z')$); and (b) for each $z \in [0, n\gamma]$, $\sum_{i \in N} p_i^N(z) = z$. Let $\boldsymbol{\pi(p^N)}$ be the curve on $\mathbb{R}^N$ that p^N draws when z varies from 0 to $n\gamma$. A *fixed path* specifies an N-path for each set $N \in \mathcal{N}$, with the requirement that the N-paths must be projectionally consistent.

Fixed path, $\boldsymbol{p}$. It is a collection $\{p^N\}_{N \in \mathcal{N}}$ such that if $N \subseteq N'$ then $\pi(p^{N'})_N = \pi(p^N)$.[3]

Associated to each fixed path we defined a *fixed path rule*. The collection of those rules is the *fixed path family*.

Fixed path rule for $\boldsymbol{p}$, $\boldsymbol{B^p}$: For each $e = (N, E, c) \in \mathbb{C}$, it selects the unique awards vector $B^p(e) = \min\{c, \lambda^p\}$ for some $\lambda^p \in p^N$ such that $\sum_{i \in N} \min\{c_i, \lambda_i^p\} = E$.[4]

[3]This notation means the following. Notice that $\pi(p^{N'})$ and $\pi(p^N)$ are two curves on two different spaces, $\mathbb{R}^{N'}$ and $\mathbb{R}^{N'}$, respectively. By $\pi(p^{N'})_N$ we denote the projection of $\pi(p^{N'})$ on the space $\mathbb{R}^N$.

[4]It is worth noting that in the case of the weighted and non-weighted constrained equal awards rules the

The fixed path family can be interpreted as follows. For each rule S and for each problem $e = (\{i,j\}, E, c)$, we define the **awards rate $r^S(e)$** as the portion of ith agent's award enjoyed by j when none of them is fully granted, i.e., $r^S(e) = \frac{S_j(e)}{S_i(e)}$ when $S_i(e) \leq c_i$, $S_j(e) \leq c_j$, and $S_j(e) \leq S_i(e)$. For the three aforementioned rules, the awards rates are the followings:

$$r^p(e) = \frac{c_j}{c_i}, \qquad r^{cea}(e) = 1, \qquad r^{cea^\alpha}(e) = \frac{\alpha_j}{\alpha_i}$$

As we can observe, the awards rate of the cea^α rule (and hence the cea rule) is constant, independently of both the claims and the estate. For the fixed path family, the awards rate does not depend on the claims but it may depend on the estate. Which is the advantage? Notice that when r^S is constant it represents how deserving agent i is in relation to agent j. But, even if some agents are perceived as being more deserving than others, we may desire that such a degree of merit vary with the size of the resource to allot. Let x_i and x_j be the awards for agents i and j when none of them is fully satisfied. For the constrained equal awards rule, x_i is always equal to x_j, regardless of the estate. For a weighted constrained equal awards rule, x_i is always equal to $\frac{\alpha_i}{\alpha_j} x_j$, regardless of the estate. For the fixed path rules, the relation between x_i and x_j may depend on the estate. For example, we may have that $x_i = x_j^2$. Hence, when the amount to divide is small, agents may receive "almost" equal awards while when the amount to divide is large their awards may differ significantly.

[Insert Figure 2 about here.]

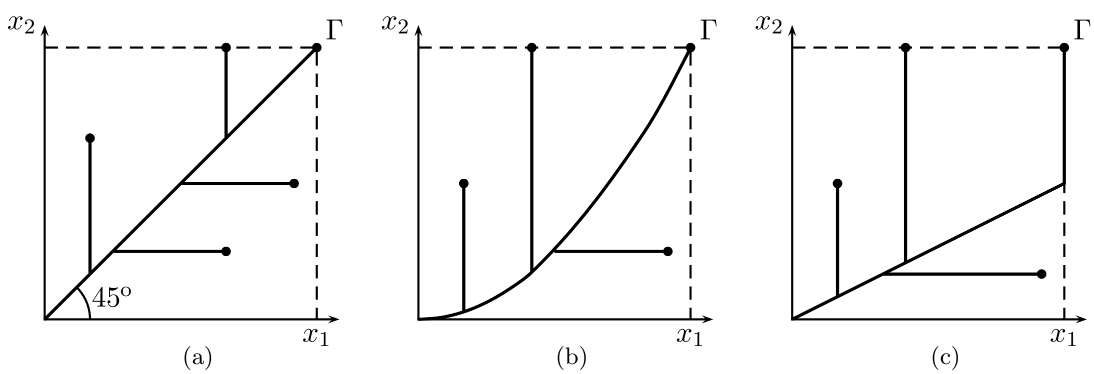

Figure 2: Illustration of the path of awards of three rules in the fixed path family. Case (a) is the constrained equal awards rule, where $p^N(z) = \left(\frac{z}{2}, \frac{z}{2}\right)$. Case (c) is a weighted constrained equal awards rule, where $p^N(z) = \left(\frac{2}{3}z, \frac{1}{3}z\right)$ if $z \leq \frac{3\gamma}{2}$, and $p^N(z) = (\gamma, z - \gamma)$ otherwise. All the three rules have in common a fixed path from $(0,0)$ to $\Gamma = (\gamma, \gamma)$.

parameter λ was an scalar. In this definition λ^p denotes an n-dimensional point in N-path p^N.

Example 2.1. Let p be a fixed path that for the two-agent case $p^{\{i,j\}}(z) = \left(\frac{z}{2}, \frac{z^2}{4\gamma}\right)$ with $z \in [0, 2\gamma]$ (Case b in Figure 2). The next table shows how the *cea*, $cea^{(2,1)}$, and B^p rules apply.

		Rules ($\gamma = 100$)		
c	E	cea	$cea^{(2,1)}$	B^p
$(30, 9)$	6	$(3, 3)$	$(4, 2)$	$(5.67, 0.32)$
$(30, 9)$	24	$(15, 9)$	$(16, 8)$	$(20, 4)$
$(30, 9)$	28	$(19, 9)$	$(19, 9)$	$(22.80, 5.19)$
$(15, 40)$	9	$(4.5, 4.5)$	$(6, 3)$	$(8.30, 0.69)$
$(15, 40)$	24	$(12, 12)$	$(16, 8)$	$(15, 9)$

3 Two properties. Strong composition down and consistency.

The next property has been widely studied in the claims problems literature. And it is particularly useful when some uncertainty over the estate exists. Let $e = (N, E', c)$ be the problem to solve, and let x be the awards vector selected by a rule S for that problem, $x = S(e)$. Imagine that when estimating the value of the estate, we were too optimistic, and the actual value E is smaller than expected, $E < E'$. Now, two claims vectors arise as legitimate demands. The first is the original claims vector c, and the second is the promised awards x. *Composition down* requires that, independently of which demands vector we consider, we end up with the same allocation.[5]

Composition down: For each $e \in \mathbb{C}$ and each $E' \in \mathbb{R}_+$ such that $C > E' > E$, then $S(e) = S(N, E, S(N, E', c))$.

Therefore, composition down stipulates as legitimate demands only two claims vectors, c and x. That is, in the formulation of the property it is implicitly assumed that, either all the agents unanimously demand the original claims c, or they unanimously demand the promised awards x. The claims vector represents not a collective right but a collection of individual rights. From this perspective, the next property recognizes as legitimate demands also the intermediate situations, where a consensus on c or x is not required. Now, let $T \subseteq N$ be a subset of the agents in N. Let $(x_T, c_{N \setminus T})$ be the vector of claims where the demands of agents in T is the promised awards x and agents not in T is the original claims c. In the spirit of composition down, *strong composition down* requires that we end up with the same allocation independently of which demands we consider, x, c or $(x_T, c_{N \setminus T})$ for any $T \subseteq N$.

Strong composition down: For each $e \in \mathbb{C}$, each $E' \in \mathbb{R}_+$ such that $C > E' > E$, and each $T \subseteq N$, then $S(e) = S(N, E, (S_T(N, E', c), c_{N \setminus T}))$.

Strong composition down looks very demanding. But, actually, many rules satisfy this property.

[5]This property was formulated by Moulin (2000).

Among them, the constrained and weighted constrained equal awards rules (the proportional rule, however, does not). Any fixed path rule also fulfils strong composition down.

Now let us consider a procedural property related to changes in the agent set. Suppose that, after solving a problem, some agents leave with their awards. The remaining agents re-valuate how to allocate the remaining estate among them. *Consistency* requires that each of these agents should receive the amount she received before the re-valuation.[6].

Consistency: For each $e = (N, E, c) \in \mathbb{C}$, each $N' \subset N$, and each $i \in N'$, $S_i(e) = S_i(N', \sum_{j \in N'} S_j(e), c_{N'})$.

Let us consider a rule S that, when $N = \{1, 2, 3\}$ coincides with the constrained equal awards, and when $N = \{1, 2\}$ coincides with the dictatorial rule that favors agent 1 (as the limit case of a weighted constrained equal awards rule). In comparing agent 1 with agent 2, this rule is quite fair for agent 2 when agent 3 is present, but extremely unfair when agent 3 is not. Consistency avoids this type of drawback. All the rules presented in the previous section, as they are defined there, are consistent.

Now we present our main result. It identifies the family of rules that satisfy strong composition down and consistency together. The proof is in the appendix.

Theorem 3.1. *The fixed path rules are the only rules satisfying strong composition down and consistency together.*

4 Some extensions

We explore in this section some other axioms, and we show how they are combined with strong composition down and consistency.

The first property stipulates that problems in which the claims and estate are small should be treated similarly to problems where the estate and claims are large. Although this criterion cannot be applied always, it is very desirable in many situations. *Homogeneity* requires that, if estate and claims are multiplied by the same positive amount, so the awards vector is.

Homogeneity. For each $e = (N, E, c) \in \mathbb{C}$ and each $\beta \in \mathbb{R}_+$, $S(N, \beta E, \beta c) = \beta S(N, E, c)$.

The weighted constrained equal awards rules are homogenous. Regarding the fixed path rules, some satisfy homogeneity and others do not.

The next property, *composition up*, represents the dual idea of composition down, and it is equally useful. Let $e = (N, E', c)$ be a problem to solve, and let x be the solution of the rule S for that problem, $x = S(e)$. Imagine that when estimating the value of the estate, we were too pessimistic, and the actual value E is greater than expected, $E > E'$. Now two possibilities are

[6]This property has been widely studied. Thomson (1998) is a survey.

open. One is to solve the problem with the revised estate, $S(N,E,c)$. The other is to assign the awards vector x first, and then to allocate the remaining estate $(E-E')$, after reducing the claims by the amounts just given, $c-x$. *Composition up* says that both ways result in the same awards.[7]

Composition up. For each $e \in \mathbb{C}$ and each $E' \in \mathbb{R}_+$ such that $E' < E$, then $S(e) = x + S(N, E-E', c-x)$, where $x = S(N, E', c)$.

The weighted constrained equal awards rules satisfy composition up. Regarding the fixed path rules, some satisfy the property while others do not.

By far, one of the most well-known results in claims problems literature is due to Moulin (2000). In its main result, that we reproduce below, the author characterizes the family of rules that satisfy composition down, composition up, homogeneity, and consistency together. Such a family is the so-called family $\mathcal{M}$ and it contains a wide variety of rules. Namely, the proportional, the weighted constrained equal awards, the weighted constrained equals losses, dictatorial rules,... and combinations of them.[8] Before defining the family $\mathcal{M}$, let us introduce the $\mathcal{D}$ rules.

Family $\mathcal{D}$. For $|N| = 2$. Each member of the family is defined as follows. Awards space is partitioned into cones; each non-degenerate cone is spanned by a homothetic family of piece-wise linear curves in two pieces, a segment containing the origin and contained in one of the boundary rays of the cone (the "first ray") and a half-line parallel to the other boundary ray (the "second ray"). (Cones can be degenerate, that is, can be rays.) For each claims vector, the path of awards of the rule is obtained by identifying the cone to which the claims vector belongs and the curve in the cone passing through it. The path is the restriction of the curve to the box having the origin and the claims vector as opposite vertices and whose sides are parallel to the axes.

Family $\mathcal{M}$. Each member of the family is defined as follows. The population of potential claimants is partitioned into priority classes; for each two-agent class, a rule in the family $\mathcal{D}$ is specified; to each class with three or more claimants, one of the following labels is attached: "proportional", or "constrained equal awards", or "constrained equal losses", and in each of the last two cases, a positive weight is specified for each claimant in the class. To solve each problem, we first identify the partition of the set of claimants actually present induced by the reference partition. The elements of this partition are handled in succession. For each class induced from a two-agent reference class, the rule in $\mathcal{D}$ specified for that class is applied; for each class induced from a three-or-more claimants reference class, the proportional, or weighted constrained equal awards, or weighted constrained equal losses rule is applied, according to the label attached to the class, with weights proportional to the weights that have been assigned to the agents who are present.

[7] This property was formulated by Young (1988).

[8] In order to avoid all the detailed technicalities needed to formally define the $\mathcal{M}$ rule (see Moulin (2000)) we reproduce the definition as in Thomson (2003).

Theorem 4.1 (Moulin (2000)). *A rule satisfies composition down, composition up, homogeneity, and consistency if and only if it is a $\mathcal{M}$-rule.*

The question that arises is the following. Which are the implications of substituting composition down for strong composition down in Theorem 4.1? The answer is our next result. It states that only the cea^{α} rules survive. That is, these are the only rules that satisfy strong composition down, composition up, homogeneity, and consistency together. Therefore, by introducing strong composition down, we separate the weighted constrained equal awards rules from all those characterized in Theorem 4.1. The proof is in the appendix.

Theorem 4.2. *The weighted constrained equal wards rules are the only rules satisfying*

(a) Strong composition down, composition up, and consistency together.
(b) Strong composition down, homogeneity, and consistency together.

Furthermore, as Theorem 4.2 suggests, (a) homogeneity can be obtained as a consequence of strong composition down, composition up, and consistency; and (b) composition up can be obtained as a consequence of strong composition down, homogeneity, and consistency.

Finally, we introduce a minimal requirement of fairness. *Equal treatment of equals* is very mild in this sense, and it simply requires equal agents to be treated equally. That is, agents with equals claims should receive equal awards.

Equal treatment of equals. For each $e \in \mathbb{C}$ and each $\{i, j\} \subseteq N$, if $c_i = c_j$ then $S_i(e) = S_j(e)$.

By adding this property to the axioms of Theorem 3.1, we obtain a new characterization of the constrained equal awards rule.

Corollary 4.1. *The constrained equal awards rule is the only rule satisfying equal treatment of equals, strong composition down, and consistency together.*

Most of the characterizations of the constrained equal awards rule provided in the existing literature use three types of properties, one of each type. First, those involving impartiality principles, as it is the case of equal treatment of equals. Second, stability with respect to changes in the estate, as it is the case of composition down, strong composition down, or composition up. Finally, the third type of properties, and the most controversial one, refers to very particular value judgements. As an illustration, Herrero and Villar (2001) contains two examples of these properties, *conditional full compensation* and *exemption*, where agents with small claims are deliberately protected. The last corollary avoids the latter type of principles, imposing only impartiality and stability.

5 Final remarks

In this work we have introduced the property of strong composition down as a revision of composition down. We have also described the fixed path rules for claims problems as a generalization of the weighted constrained equal awards family. Strong composition down and consistency characterize the fixed path family. Moreover, by adding homogeneity or composition up, we end up with the weighted constrained equal awards family. And, by adding equal treatment of equals, we end up with the constrained equal awards rule. Interestingly, strong composition down is enough to separate the cea^{α} rules in Moulin (2000)

For the present paper, we have implicitly considered rules from the point of view of gains. Nevertheless, it is quite common in the literature to make the dual analysis as well. Two rules are dual if one of them assigns awards in the same way the other one assigns losses. The dual of the constrained equal awards and weighted constrained equal awards rules are the so-called *constrained equal losses* and *weighted constrained equal losses rules*, respectively.[9] Similarly, we may define the dual of the fixed path family. The notion of duality applies to the properties as well. Two properties are dual if whenever a rule satisfies one of the properties, the dual of the rule satisfies the other property. Homogeneity and equal treatment of equals are self-dual (the dual property is itself), while the dual of composition down is composition up. Again, we may consider the dual property of strong composition down, which goes along the lines of composition up in the same way strong composition down does with respect to composition down. In view of Theorem 3.1, and using the characterization-by-duality result in Herrero and Villar (2001), the dual of strong composition down and consistency characterize the dual of the fixed path family.

In considering composition down and strong composition down together with their dual properties, more things can be said. Only the serial dictatorial rules satisfy strong composition down and strong composition up together.

[9] The reader is referred to Thomson (2003) for a formal description of both.

Appendix A. Proofs

This appendix is devoted to the proofs of Theorems 3.1 and 4.2, preceded by some definitions and technical results.

Resource monotonicity stipulates that no agent should be penalized as a consequence of an increase in the estate.

Resource monotonicity. For each $(N, E, c) \in \mathbb{C}$, if $E' > E$ then $S(N, E', c) \geq S(N, E, c)$.

Let us consider a problem and an awards vector for it with the following feature. For each two-agent subset of the agents involved, the rule chooses the restriction of that vector for the associated reduced problem to this agent subset. *Converse consistency* requires that the allocation should be the one selected by the rule for the original problem.[10]

Let $c.con(e; S) \equiv \{x \in \mathbb{R}^N_+ : \sum_{i\in N} x_i = E$ and for all $N' \subset N$ such that $|N'| = 2, x_{N'} = S(N', \sum_{i\in N'} x_i, c_S)\}$

Converse consistency. For each $e \in \mathbb{C}$, $c.con(e; S) \neq \phi$, and if $x \in c.con(e; S)$, then $x = S(e)$.

Lemma 5.1 (Elevator Lemma, Thomson (1998)). *If a rule S is consistent and coincides with a conversely consistent rule S' in the two-agent case, then it coincides with S' in general.*

Proposition 5.1 (Chun (1999)). *Resource monotonicity and consistency together imply converse consistency.*

Proof of Theorem 3.1.

It is not difficult to check that any fixed path rule satisfies *strong composition down* and *consistency*. It is also straightforward that *strong composition down* implies *composition down* and the latter *resource monotonicity*. Therefore, by Proposition 5.1, any fixed path rule is *converse consistent*. Let S be a rule fulfilling *strong composition down* and *consistency*. We define the N-path as the path of awards of S for claims vector Γ_N when the estate varies from 0 to $n\gamma$. Let p be the fixed path once we consider the N-paths for all $N \in \mathcal{N}$, which is well defined because S is *consistent*. We show that $S = B^p$. By the Elevator Lemma, it is enough to prove the result in the two-agent case. Let $N = \{i, j\}$ and $c \in \mathbb{R}^{\{i,j\}}$ such that $c_i \leq c_j$. Let $y \in p(\Gamma) = p^N$ be a vector such that $y_i = c_i$ and $y_j < c_j$. And let $z \in p(\Gamma)$ be a vector such that $z_j = c_j$ and $z_i > c_i$. We distinguish several cases.

Case 1. If $c \in p(\Gamma)$. Let $E \in [0, \sum_{i\in N} c_i]$. Note that, since $c \in p(\Gamma)$, $S(N, \sum_{i\in N} c_i, \Gamma) = c$ by definition of a path of awards. Then $S(N, E, c) = S(N, E, S(\sum_{i\in N} c_i, \Gamma))$. By *strong composition down*, $S(N, E, S(\sum_{i\in N} c_i, \Gamma)) = S(N, E, \Gamma) = B^p(N, E, \Gamma) = B^p(N, E, c)$. Therefore, $S(N, E, c) = B^p(N, E, c)$.

[10] This property was formulated by Chun (1999).

Case 2. If $c \notin p(\Gamma)$ and $E \leq \sum_{i \in N} y_i$. By Case 1, $S(N, \sum_{i \in N} y_i, z) = B^p(N, \sum_{i \in N} y_i, z) = y$. Then, notice that $c = (y_{-n}, z_n) = \left(S_{-n}(\sum_{i \in N}, z), z_n\right)$. By *strong composition down*, $S(N, E, z) = S\left(N, E, \left(S_{-n}(N, \sum_{i \in N} y_i, z), z_n\right)\right) = S(N, E, c)$. Since $z \in p(\Gamma)$, by Case 1, we know that $S(N, E, z) = B^p(N, E, z) = B^p(N, E, c)$. Therefore, $S(N, E, c) = B^p(N, E, c)$.

Case 3. If $c \notin p(\Gamma)$ and $E > \sum_{i \in N} y_i$. Note that, on the one hand, by *strong composition down*, $S(\sum_{i \in N} y_i, c) = S(\sum_{i \in N} y_i, S(E, c)) \leq S(E, c)$; and on the other hand, $S(\sum_{i \in N} y_i, c) = B^p(\sum_{i \in N} y_i, c) = y$. Then, $c \geq S(E, c) \geq y$. Therefore, $S(E, c) = B^p(E, c)$.

Therefore, S and B^p coincide in the two-agent case, and thus they do so in general.

[Insert Figure 3 about here.]

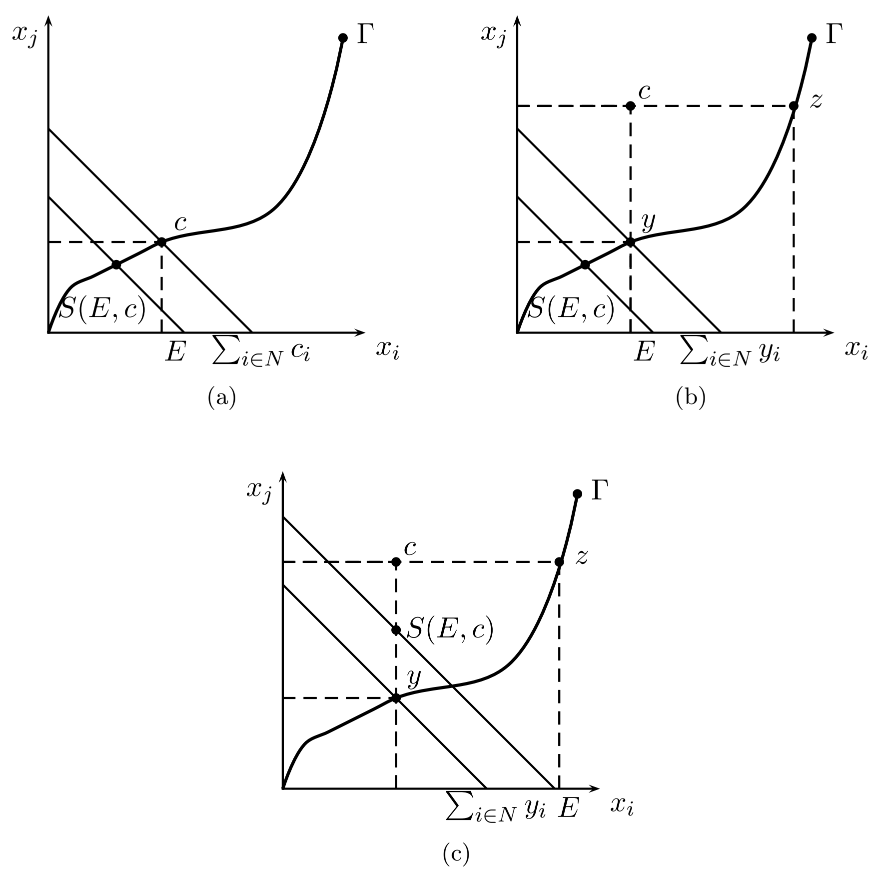

Figure 3: Illustration of the proof for the two-agent case. (a) Case 1. (b) Case 2. (c) Case 3.

Remark 5.1. The proportional rule satisfies consistency but violates strong composition down. A rule satisfying strong composition down but not consistency can be defined as follows.

$$S(e) = \begin{cases} cea^{(2,1)}(e) & \text{if } N = \{1,2\} \\ cea(e) & \text{otherwise} \end{cases}$$

The properties that characterize the fixed path family are, therefore, independent.

Proof of Theorem 4.2.

(a) The weighted constrained equal awards rules trivially fulfil the three properties. We only need to prove the converse for the two-agent case, $N = \{i, j\}$, since, in application of the Elevator Lemma, it will be extended to the general case. Let S be a rule satisfying *strong composition down*, *composition up*, and *consistency*. By Theorem 3.1, S is a fixed path rule; and then it is completely described by its paths. Let $\alpha \in \mathbb{R}^2$ be the point where the path $p(\Gamma)$ reaches the square determined the four vertices $(0,0)$, $(0,\gamma)$, $(\gamma,0)$ and $(\gamma,\gamma) = \Gamma$ (see Figure 4). Without loss of generality, we assume that $\alpha_j = \gamma$. We probe that if $z \in p(\Gamma)$ then $\alpha - z \in p(\Gamma)$, which implies that the path of awards from 0 to α is a straight line. For any $z, z' \in \mathbb{R}^2$, let $\overline{zz'}$ denotes the segment joining z and z'. We distinguish several cases.

Case 1. If $p(\Gamma)$ is always above the segment $\overline{0\alpha}$ (see Figure 4,(a)). Let $z \in p(\Gamma)$, and so $S(N, z_i + z_j, \alpha) = z$. There exists $E \in \mathbb{R}_{++}$ such that $S(N, E, \alpha - z) = (E - \alpha_j + z_j, \alpha_j - z_j)$ and $E - \alpha_j + z_j < \alpha_i - z_i$. By *composition up,*

$$\begin{aligned} S(N, E + (z_i + z_j), \alpha) &= S(N, z_i + z_j, \alpha) + S(N, E, \alpha - z) \\ &= (z_i, z_j) + (E - \alpha_j + z_j, \alpha_j - z_j) \\ &= (E + z_i + z_j - \alpha_j, \alpha_j) \\ &= (E + z_i + z_j - \gamma, \gamma) \end{aligned}$$

Since we imposed that $E - \gamma + z_j < \alpha_i - z_i$, then $E + z_i + z_j - \gamma < \alpha_i$. This is a contradiction with the definition of α.

Case 2. If $p(\Gamma)$ is sometimes above and sometimes below the segment $\overline{0\alpha}$ (see Figure 4, (b)). Without loss of generality let us suppose that it is above firstly (a similar argument can be applied if it is starts from below). Let $z \in p(\Gamma)$ be the point where $p(\Gamma)$ firstly crosses the segment $\overline{0\alpha}$. By *composition up* we have that

$$\begin{aligned} S(N, E + z_i + z_j, \alpha) &= S(N, z_i + z_j, \alpha) + S(N, E, \alpha - z) \\ &= z + S(N, E, \alpha - z) \end{aligned}$$

Note that $S(N, E, \alpha - z)$ is above $\overline{0\alpha}$, and so is $S(N, E + z_i + z_j, \alpha)$. This is contradicts the definition of z.

Case 3. If $p(\Gamma)$ is always below the segment $\overline{0\alpha}$ and there exists a point β as in Figure 4, (c). Using *composition up*, we get that for each $E \in [\beta_i + \beta_j, 2\gamma]$

$$\begin{aligned} S(N,E,\Gamma) &= S(N,\beta_i+\beta_j,\Gamma) + S(N,E-(\beta_i+\beta_j),\Gamma-\beta) \\ &= \beta + S(N,E-(\beta_i+\beta_j),\Gamma-\beta) \end{aligned}$$

Then, $S(N,E-(\beta_i+\beta_j),\Gamma-\beta) = S(N,E,\Gamma)-\beta$ for all $E \in [\beta_i+\beta_j, 2\gamma]$. That means that the path of awards of S for $\Gamma - \beta$ is as in Figure 4. But this contradicts what a fixed path rule is.

Case 4. Finally, we will show that the point β of the previous Case always exists (see Figure 4, (d)). Suppose that it does not. Let $z \in p(\Gamma)$, $z = S(N, z_i + z_j, \alpha)$. For all $E \in (z_i + z_j, \alpha_i + \alpha_j)$, by *composition up*, we have that

$$\begin{aligned} S(N,E,\alpha) &= S(N,z_i+z_j,\alpha) + S(N,E-(z_i+z_j),\alpha-z) \\ &= z + S(N,E-(z_i+z_j),\alpha-z) \end{aligned}$$

It is not difficult to check that $\alpha - z$ is above the segment $\overline{0\alpha}$. Then, there exits $E' \in (z_i + z_j, \alpha_i + \alpha_j)$ such that

$$S(N,E'-(z_i+z_j),\alpha-z) = (\alpha_i - z_i, E' - (z_i+z_j) - (\alpha_i+\alpha_j)).$$

Therefore,

$$\begin{aligned} S(N,E',\alpha) &= z + S(N,E-(z_i+z_j),\alpha-z) \\ &= (z_i,z_j) + (\alpha_i - z_i, E'-(z_i+z_j)-(\alpha_i-z_i)) \\ &= (\alpha_i, E'-\alpha_i) \end{aligned}$$

But we can then define β as $\beta = (\alpha_i, E' - \alpha_i)$.

[Insert Figure 4 about here.]

(b) This proof comes from Thomson (2006). He characterizes the set of homogeneous rules. By using their result it is not difficult to check that the only homogeneous rules within the fixed path family are the weighted constrained equal awards rules. Alternatively, one may use an argument similar to Case (a) to show that the fixed path should be like the weighted constrained equal awards rules.

Proof of Corollary 4.1.

By Theorem 3.1 is enough to show that the constrained equal awards rule is the unique fixed path rule satisfying *equal treatment of equals*. A rule satisfies *equal treatment of equals* if and only if the diagonal is its path of awards vector Γ, that is, if $(\lambda, \ldots, \lambda) \in p^N$ for all $\lambda \leq \gamma$. Therefore, $B^p = cea$.

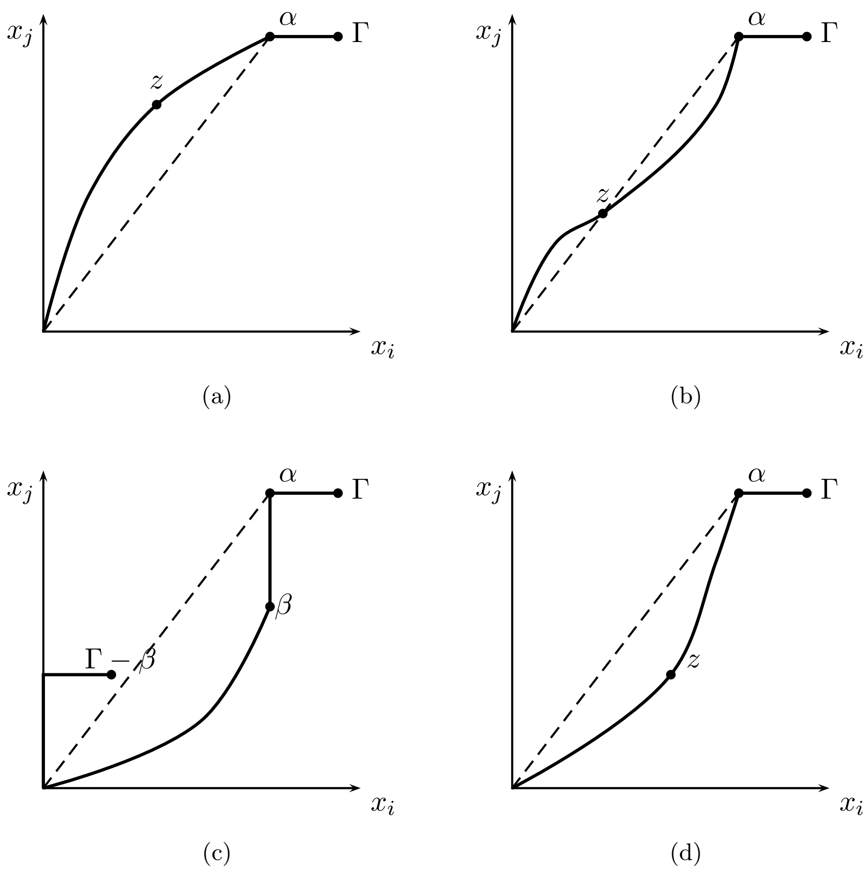

Figure 4: Illustration of the proof for the two-agent case. (a) Case 1. (b) Case 2. (c) Case 3. (d) Case 4.

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