E C O N O M I C S B U L L E T I N

Linear, efficient and symmetric values for TU-games

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Abstract

In this paper, we study values for TU-games which satisfy three classical properties: Linearity, efficiency and symmetry. We give the general analytical form of these values and their relation with the Shapley value and the Egalitarian value.

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1 Introduction

The main result on values for TU-games is due to Shapley (1953) who defines a value which satisfies four properties: linearity, efficiency, symmetry and the dummy axiom. The aim of this paper is to study and characterize values which satisfy the first three classic properties. We obtain almost the same results as Hernandez-Lamoneda, Juarez and Sanchez-Sanchez (2008) which, independently, work in the same topic. We will focus our analysis on the discriminate properties of values to obtain new axiomatization of some classic values.

2 Notations and preliminaries

Let N be a finite set of $n-$players, we denote by:

- $P(N)$ the set of subsets of N and 2^N the set of non empty subsets of N.
- $\mathbb{R}$ the set of real numbers and $\Pi(N)$ the set of automorphisms of N.
- An n-person TU-game on N is a pair (N, v) where v is a mapping from $P(N)$ to $\mathbb{R}$ such that $v(\emptyset) = 0$

Γ will denote the $2^n - 1$ dimension linear space of all $n-$person TU-game on N.

- A value φ is a mapping from Γ to $\mathbb{R}^n$. $\varphi\ (v) = (\varphi_i(v))_{i \in N}$ is a vector of distribution of the payoffs obtained in v.

Let us give some properties of values ;

- φ is **linear** if : $\forall v, w \in \Gamma$, $\forall \alpha, \beta \in \mathbb{R}, \varphi(\alpha v + \beta w) = \alpha\varphi(v) + \beta\varphi(w)$
- φ is **efficient** if: $\forall v \in \Gamma, \sum_{i \in N} \varphi_i(v) = v(N)$
- φ is **symmetric** if: $\forall v \in \Gamma, \forall \pi \in \Pi(N), \forall i \in N, \varphi_{\pi(i)}(v) = \varphi_i(\pi v)$

where $\pi v(T) = v(\pi(T))\ \forall T \in 2^N$

- φ is **monotonic** if : $\forall v, w \in \Gamma, \forall i \in N, \begin{cases} v(S) \geq w(S) & \text{if } i \in S \\ v(S) = w(S) & \text{if } i \notin S \end{cases} \Longrightarrow \varphi_i(v) \geq \varphi_i(w)$
- φ is **covariant** if : $\forall v \in \Gamma, \forall \alpha \in \mathbb{R}, \forall \theta \in \mathbb{R}^n, \varphi(\alpha v + \theta) = \alpha\varphi(v) + \theta$

where $\alpha v + \theta$ is defined by : $(\alpha v + \theta)(S) = \alpha v(S) + \sum_{i \in S} \theta_i$

- φ **preserves additive games** if : $\forall \theta \in \mathbb{R}^n, \varphi(\theta(.)) = \theta$ where $\theta(.)$ is defined by : $\theta(S) = \sum\limits_{l \in S} \theta_l$

- φ is **non negative** if : $\forall v \in \Gamma, [\forall S \subseteq N, v(S) \geq 0] \Rightarrow \varphi_i(v) \geq 0 \ \forall i \in N$

- φ is **marginalist** if : $\forall v, w \in \Gamma, \forall i \in N,$

$[\forall S \subseteq N, v(S) - v(S - i) = w(S) - w(S - i)] \Longrightarrow \varphi_i(v) = \varphi_i(w)$

Let Ω be the set of $n+1-$vectors of real numbers $A = (A(k))_{k=0,..,n}$ such that $A(0)$ is a fixed real number and $A(n) = 1$.

For each element A of Ω, we define the value Ψ^A by :

$$\forall v \in \Gamma, \forall i \in N, \Psi_i^A(v) = \tfrac{v(N)}{n} + \sum_{k=1}^{n-1} \left[\tfrac{(n-k)!(k-1)!}{n!} A(k) \sum_{\substack{i \in S \\ |S|=k}} v(S) - \tfrac{(n-k-1)!(k)!}{n!} A(k) \sum_{\substack{i \notin S \\ |S|=k}} v(S) \right]$$

Let us give two others expressions of Ψ^A in the following lemma :

Lemma 1 : $\forall v \in \Gamma, \forall i \in N,$

1) $\Psi_i^A(v) = \sum\limits_{k=1}^{n} \left[\tfrac{(n-k)!(k-1)!}{n!} A(k) \sum\limits_{\substack{i \in S \\ |S|=k}} v(S) - \tfrac{(n-k-1)!(k)!}{n!} A(k) \sum\limits_{\substack{i \notin S \\ |S|=k}} v(S) \right]$

2) $\Psi_i^A(v) = \sum\limits_{k=1}^{n} \left[\sum\limits_{\substack{i \in S \\ |S|=k}} \tfrac{(n-k)!(k-1)!}{n!} \left[A(k)v(S) - A(k-1)v(S - \{i\}) \right] \right]$

We can therefore deduce the following relation between Ψ^A and the Shapley value denoted $Shap$.

Theorem 1 : $\forall v \in \Gamma, \forall i \in N, \forall A \in \Omega, \Psi^A(v) = Shap(v^A)$
where $v^A(S) = A(k)v(S)$ for each S such that $|S| = k$

Let us denoted by Φ the set of values Ψ^A for all elements A on Ω.

3 Geometric properties of Φ

It is easy to prove the following results :

Proposition 1 : Φ is a convex set and furthermore:$\forall A_1, A_2 \in \Omega, \forall \alpha \in [0, 1]$,
$A = \alpha A_1 + (1-\alpha)A_2 \Longrightarrow \Psi^A(v) = \alpha \Psi^{A_1}(v) + (1-\alpha)\Psi^{A_2}(v) \ \ \forall v \in \Gamma$

Proposition 2 : Φ is a $n-1$ dimension affine set. Furthermore $A = (A(k))_{k=1,..,n-1}$ is a vector of coordinates of $\Psi - \Psi^0$ in the basis $\left(\mu^k\right)_{k=1,..,n-1}$ where Ψ^0 is the egalitarian value and μ^k is defined by : $\forall v \in \Gamma, \forall i \in N \ \ \mu_i^k(v) = \tfrac{(n-k)!(k-1)!}{n!} \sum\limits_{\substack{i \in S \\ |S|=k}} v(S) - \tfrac{(n-k-1)!(k)!}{n!} \sum\limits_{\substack{i \notin S \\ |S|=k}} v(S)$

4 Characterization of Φ and discriminate properties

The main result of this section is the following (see Hernandez-Lamoneda, Juarez and Sanchez-Sanchez (2008) for the proof) :

Theorem 2 :A value φ is linear, efficient and symmetric if and only if φ is an element of Φ

To see the extend of our family of values, let us give some classical values which are elements of Φ and their corresponding elements of Ω.

Value	$A(1)$	$A(2)$	...	$A(k)$	...	$A(n-1)$	$A(n)$	**Authors**
Shapley	1	1		1		1	1	Shapley (1953)
Egalitarian	0	0		0		0	1	Brink R. Van den (2007)
Solidarity	$\frac{1}{2}$	$\frac{1}{3}$		$\frac{1}{k+1}$		$\frac{1}{n}$	1	Nowak and Radzik (1994)
Consensus	$\frac{n}{2}$	$\frac{1}{2}$		$\frac{1}{2}$		$\frac{1}{2}$	1	Yuan, Born and Ruys (2007)
C.I.S or Equal surplus	$n-1$	0		0		0	1	Driessen and Funaki (1991)

Let us give some results which are directly deduce from Theorem 1 :

Corollary 1 : A value φ is linear, efficient, symmetric and monotonic if and only if $\varphi = \Psi^A$ where A is positive (i.e. $A(k) \geq 0\ \forall k = 1,..,n$)

Corollary 2 : φ is linear, efficient, symmetric and preserves additive games if and only if $\varphi = \Psi^A$ with $\sum_{k=1}^{n-1} A(k) = n-1$

One of the subset of linear, efficient, symmetric values which preserve additive games is the set of $\alpha-$consensus value whose vector A is defined by : $A(k) = \begin{cases} 0 & \text{if } k=0 \\ \alpha + (1-\alpha)(n-1) & \text{if } k=1 \\ \alpha & \text{if } 2 \leq k \leq n-1 \\ 1 & \text{if } k=n \end{cases}$

The discrimination on values of Φ can be done in two different and equivalent ways :

- a particular vector $A = (A(k))_{k=0,..,n}$ of Ω is given and therefore we obtain a specific value Ψ^A.

- a property, called discriminate property, is added to linearity, efficiency and symmetry to obtain an unique element of Φ.

Let us introduce the non negativity and the marginalism as discriminate properties through the following results :

Corollary 3 : φ is linear, efficient, symmetric and non negative if and only if φ is the Egalitarian value.

Corollary 4 : φ is linear, efficient, symmetric and marginalist if and only if φ is the Shapley value.

Let us introduce another discriminate property used in the literature :

Let (N, v) a TU-game and $i \in N$,

- i is v-dummy player if : $\forall S \subseteq N - \{i\}, v(S) = v(S + i)$

Let φ be a value, $A = (A(k))_{k=0,..,n}$ a vector in Ω, and α a real number :

- φ satisfies the dummy property if : $\forall v \in \Gamma, \forall i \in N, [i$ is v-dummy player $\Rightarrow \varphi_i(v) = 0]$
- φ satisfies the $\alpha - A -$dummy property if : $\forall v \in \Gamma, \forall i \in N, \left[i\right.$ is v-dummy player $\left.\Rightarrow \varphi_i(v) = \alpha \Psi_i^A(v)\right]$

We obtain the following result :

Theorem 3 : φ is linear, efficient, symmetric and satisfies the $\alpha - A-$dummy property if and only if $\varphi = (1 - \alpha) Shap + \alpha \Psi^A$

We can therefore deduce some well-known results such as :

- the characterization of the Shapley value (Shapley, 1953) when $\alpha = 0$

- the theorem of Yang(1997) when Ψ^A is the egalitarian value;

- the characterization of the $\alpha-$consensus value (when Ψ^A is the equal surplus value) due to Yuan, Born and Ruys (2007)

Another discriminate property used in this paper is obtained by replacing the dummy player by the zero player as follows :

Let (N, v) a TU-game and $i \in N$,

- i is v-zero player if : $\forall S \subseteq N, i \in S \Rightarrow v(S) = 0$

Let φ be a value, $A = (A(k))_{k=0,..,n}$ a vector in Ω, α a real number :

- φ satisfies the zero player property if : $\forall v \in \Gamma, \forall i \in N, [i$ is v-zero player $\Rightarrow \varphi_i(v) = 0]$
- φ satisfies the $\alpha - A -$zero player property if : $\forall v \in \Gamma, \forall i \in N, \left[i\right.$ is v-zero player $\left.\Rightarrow \varphi_i(v) = \alpha \Psi_i^A(v)\right]$

We obtain the following result :

Theorem 4 : φ is linear, efficient, symmetric and satisfies the $\alpha - A-$zero player property if and only if $\varphi = (1 - \alpha)\Psi^0 + \alpha \Psi^A$ where Ψ^0 is the egalitarian value.

We can deduce the characterization of egalitarian value when $\alpha = 0$, result due to Van den Brink (2007).

The last discriminate property in this paper can be seen as the dual of the previous one and is defined by :

Let (N, v) a TU-game and $i \in N$,

- i is v-winning player if : $\forall S \subseteq N, v(S - \{i\}) = 0$

One can extend this property as the previous ones and obtain new values.

5 Conclusion

The paper has outlined the properties of the class of values which are linear, efficient and symmetric. In particular, it has been proved that the class is an affine linear space for which a "canonical" basis has been identified. On the one hand, this has been helpful in bringing out some common analytic expression of all the values in the class. We have obtained that every value in the class can be put in the random order form, similar to the well-known expression of the Shapley value. On the other hand, we have explained conditions for a given value which is presumed as member of the class to satisfy some basic properties as well as recasting in simplify and extensive terms some well-known results.

References

[1] Brink, R. Van Den (2007), Null or Nullifying players: The difference between Shapley value and the equal division solutions. Journal of Economic Theory , 136: 767-775.

[2] Driessen TSH, Funaki Y.(1991) Coincidence of and Collinearity between game theoretic solutions. OR Spektrum 13: 15-30.

[3] Hermandez-Lamoneda, Luis, Ruben Juarez, and Francisco Sanchez-Sanchez (2008) " solution Without dummy axiom for TU cooperative games" Economics Bulletin, Vol.3, No.1 pp 1-9

[4] Nowak A.S. and Radzik T.(1994) A solidarity Value for n-person Transferable Utility games. Int Jour of Game Theory , 23: 43-48.

[5] Shapley L.S (1953), A value for n-person games. In: Kuhn H, Tucker AW (eds) Contributions to the theory of games, vol. II. Princeton University Press, Princeton,NJ.

[6] Yang C.H (1997), A Family of Values for n-person Cooperative Transferable Utility Games. MS. Thesis. University at Buffalo.

[7] Young. H.P (1985). Monotonic Solutions of Cooperative games. Int. J. Game Theory 14: 65-72.

[8] Yuan Ju, P. Born and P. Ruys (2007),The Concensus Value: a new solution concept for cooperative games. Soc. Choice Welfare , 28 685-703.

6 Appendix

Lemma 1 :

1) is obvious

$$2)\ \Psi_i^A(v) = \sum_{k=1}^{n}\left[\tfrac{(n-k)!(k-1)!}{n!}A(k)\sum_{\substack{i\in S\\|S|=k}} v(S) - \tfrac{(n-k-1)!(k)!}{n!}A(k)\sum_{\substack{i\notin S\\|S|=k}} v(S)\right]$$

$$= \sum_{k=1}^{n}\tfrac{(n-k)!(k-1)!}{n!}A(k)\sum_{\substack{i\in S\\|S|=k}} v(S) - \sum_{k=1}^{n-1}\tfrac{(n-k-1)!(k)!}{n!}A(k)\sum_{\substack{i\in S\\|S|=k+1}} v(S-i)$$

$$= \sum_{k=1}^{n}\tfrac{(n-k)!(k-1)!}{n!}A(k)\sum_{\substack{i\in S\\|S|=k}} v(S) - \sum_{p=2}^{n}\tfrac{(n-p)!(p-1)!}{n!}A(p-1)\sum_{\substack{i\in S\\|S|=p}} v(S-i)$$

$$= \sum_{k=1}^{n}\sum_{\substack{i\in S\\|S|=k}}\tfrac{(n-k)!(k-1)!}{n!}\left[A(k)v(S) - A(k-1)v(S-i)\right] \qquad \blacksquare$$

Corollary 1 : Suppose $A(k) \geq 0\ \ \forall k = 1, 2, ...n,$ for any $i\in N$, and for any $v, w \in \Gamma$,

$$\left.\begin{array}{c} v(S) \geq w(S)\ \forall S \ni i \\ v(S) = w(S)\ if\ i\notin S \end{array}\right\} \Rightarrow \text{for all } k = 1, 2, ...n \quad \left.\begin{array}{c} A(k)v(S) \geq A(k)w(S)\ \forall S \ni i, |S| = k \\ -A(k-1)v(S) = -A(k-1)w(S)\ if\ i\notin S, |S| = k \end{array}\right\} \Rightarrow$$

$$\sum_{\substack{i\in S\\|S|=k}}\tfrac{(n-k)!(k-1)!}{n!}\left[A(k)v(S) - A(k-1)v(S-i)\right] \geq \sum_{\substack{i\in S\\|S|=k}}\tfrac{(n-k)!(k-1)!}{n!}\left[A(k)w(S) - A(k-1)w(S-i)\right]$$

$$\Rightarrow \sum_{k=1}^{n}\left[\sum_{\substack{i\in S\\|S|=k}}\tfrac{(n-k)!(k-1)!}{n!}\left[A(k)v(S) - A(k-1)v(S-i)\right]\right] \geq \sum_{k=1}^{n}\left[\sum_{\substack{i\in S\\|S|=k}}\tfrac{(n-k)!(k-1)!}{n!}\left[A(k)w(S) - A(k-1)w(S-i)\right]\right.$$

$\Rightarrow \Psi_i^A(v) \geq \Psi_i^A(w).$

Conversely, suppose Ψ^A is monotonic,

for any $i \in N$, for all $k = 1, 2, ..., n$, there exists at least one coalition $S_i \subseteq N$ such that :

$S_i \ni i$ and $|S_i| = k$.Consider the games $v(S) = \begin{cases} 1\ if\ S = S_i \\ 0\ if\ S \neq S_i \end{cases}$ and $w(S) = \begin{cases} \frac{1}{2}\ if\ S = S_i \\ 0\ if\ S \neq S_i \end{cases}$

It is obvious that $\left.\begin{array}{c} v(S) \geq w(S)\ \forall S \ni i \\ v(S) = w(S)\ if\ i \notin S \end{array}\right\}$. As Ψ^A is monotonic,this implies

$\Psi_i^A(v) \geq \Psi_i^A(w) \Leftrightarrow \frac{(n-k)!(k-1)!}{n!}A(k) \geq \frac{(n-k)!(k-1)!}{2n!}A(k)$

$\Leftrightarrow A(k) \geq \frac{A(k)}{2} \Rightarrow A(k) \geq 0. \quad \blacksquare$

Corollary 2 : .φ linear, efficient, symmetric and preserves additive games

$\Leftrightarrow \varphi = \Psi^A$ by Theorem 2 and Ψ^A preserves additive games

Ψ^A preserves additive games $\Leftrightarrow \forall\theta \in \mathbb{R}^n, \varphi(\theta(.)) = \theta \Leftrightarrow \Psi_i^A(\theta(.)) = \theta_i\ \ \forall\vec{i}\in \mathbf{N}$ and $\forall\boldsymbol{\theta} \in\mathbb{R}^n$

$$\Leftrightarrow \sum_{k=1}^{n}\left[\sum_{\substack{i\in S\\|S|=k}}\tfrac{(n-k)!(k-1)!}{n!}\left[A(k)\theta(S) - A(k-1)\theta(S-i)\right]\right] = \theta_i\ \ \forall i \in N \text{ and } \forall\theta \in \mathbb{R}^n$$

$$\Leftrightarrow \sum_{k=1}^{n}\left[\sum_{\substack{i\in S\\|S|=k}}\frac{(n-k)!(k-1)!}{n!}\left[A(k)\left(\theta(S-i)+\theta_i\right)-A(k-1)\theta(S-i)\right]\right]=\theta_i \quad \forall i\in N,\quad \forall\theta\in\mathbb{R}^n$$

$$\Leftrightarrow \left\{\begin{array}{l} \sum_{k=1}^{n}\left[\sum_{\substack{i\in S\\|S|=k}}\frac{(n-k)!(k-1)!}{n!}\left[A(k)-A(k-1)\right]\sum_{j\in S-i}\theta_j\right]=0 \\ \qquad\qquad \sum_{k=1}^{n}\left[\sum_{\substack{i\in S\\|S|=k}}\frac{(n-k)!(k-1)!}{n!}A(k)\right]=1 \end{array}\right. \quad \forall i\in N,\quad \forall\theta\in\mathbb{R}^n$$

$$\Leftrightarrow \sum_{k=1}^{n}\frac{(n-k)!(k-1)!}{n!}\left[A(k)-A(k-1)\right]\frac{(n-2)!}{(k-2)!(n-k)!}\theta_j=0 \;\text{ and }\; \sum_{k=1}^{n}A(k)=n \qquad \forall\boldsymbol{\theta}_j\in\mathbb{R}$$

$$\Leftrightarrow \sum_{k=1}^{n}(k-1)\left[A(k)-A(k-1)\right]\theta_j=0 \;\text{ and }\; \sum_{k=1}^{n-1}A(k)=n-1 \qquad \forall\boldsymbol{\theta}_j\in\mathbb{R}$$

$$\Leftrightarrow \sum_{k=1}^{n}(k-1)\left[A(k)-A(k-1)\right]=0 \;\text{ and }\; \sum_{k=1}^{n-1}A(k)=n-1$$

$$\Leftrightarrow \sum_{k=1}^{n-1}A(k)=n-1 \qquad \blacksquare$$

Corollary 3 : Suppose φ is linear, efficient, symmetric and non negative, $\forall v\in\Gamma, \forall i\in N$,

$$\varphi_i(v)=\Psi_i^A(v)=\frac{v(N)}{n}+\sum_{k=1}^{n-1}\left[\frac{(n-k)!(k-1)!}{n!}A(k)\sum_{\substack{i\in S\\|S|=k}}v(S)-\frac{(n-k-1)!(k)!}{n!}A(k)\sum_{\substack{i\notin S\\|S|=k}}v(S)\right]$$

Suppose $\varphi\neq\Psi^0$**,** then there exist at least one $k_0\in\{1,2,...,n-1\}$ such that $A(k_0)\neq 0$.

If $A(k_0)>0$, for $i_0\in N$, consider the game $v_0(S)=\begin{cases} 1 \; if\; i\notin S \text{ and } |\mathbf{S}|=k_0 \\ 0 \qquad\qquad \text{elsewhere}\end{cases}$

It is straightfoward that $\varphi_{i_0}(v_0)<0$

If $A(k_0)<0$, $for\; i_0\in N$, consider the game $w_0(S)=\begin{cases} 1 \; if\; i\in S \text{ and } |\mathbf{S}|=k_0 \\ 0 \qquad\qquad \text{elsewhere}\end{cases}$

It is straightfoward that $\varphi_{i_0}(w_0)<0$ $\blacksquare$

Corollary 4 : Suppose $\varphi=\Psi^A$ is linear, efficient, symmetric and marginalist, consider any games $\forall v,w\in\Gamma$ such that, $\forall i\in N, [\forall S\subseteq N, v(S)-v(S-i)=w(S)-w(S-i)]$

As φ is marginalist, $\varphi_i(v)=\varphi_i(w) \Leftrightarrow \sum_{k=1}^{n}\left[\sum_{\substack{i\in S\\|S|=k}}\frac{(n-k)!(k-1)!}{n!}\left[A(k)v(S)-A(k-1)v(S-i)\right]\right]=$

$$\sum_{k=1}^{n}\left[\sum_{\substack{i\in S\\|S|=k}}\frac{(n-k)!(k-1)!}{n!}\left[A(k)w(S)-A(k-1)w(S-i)\right]\right]$$

$$\Leftrightarrow \sum_{k=1}^{n}\left[\sum_{\substack{i\in S\\|S|=k}} \tfrac{(n-k)!(k-1)!}{n!}\left[A(k)(v(S)-w(S))+A(k-1)(w(S-i)-v(S-i))\right]\right]=0$$

$$\Leftrightarrow \sum_{k=1}^{n}\left[\sum_{\substack{i\in S\\|S|=k}} \tfrac{(n-k)!(k-1)!}{n!}\left[\left(A(k)-A(k-1)\right)\left(v(S-i)-w(S-i)\right)\right]\right]=0$$

Since this condition must be satisfied for any games $v,w\in\Gamma$ such that,

$\forall i\in N, [\forall S\subseteq N, v(S)-v(S-i)=w(S)-w(S-i)]$

$\Rightarrow A(k)-A(k-1)=0$ for all $k=2,3,...,n$ and $A(n)=1$

$\Rightarrow A(k)=1$ for all $k=1,2,...,n$

$\Rightarrow \varphi = Shap.$ ■

Theorem 3 : φ is linear, efficient, symmetric and satisfies the $\alpha-A-$dummy property if and only if , for any $v\in\Gamma$ and for any $v-$dummy player $i\in N, \varphi_i(v)=\Psi_i^{A'}(v)=\alpha\Psi_i^{A}(v)$

$$\Leftrightarrow \varphi_i(v)=\sum_{\substack{i\in S\\|S|=k}} \tfrac{(n-k)!(k-1)!}{n!}\left[A'(k)v(S-i)-A'(k-1)v(S-i)\right]$$

$$=\alpha\sum_{\substack{i\in S\\|S|=k}} \tfrac{(n-k)!(k-1)!}{n!}\left[A(k)v(S-i)-A(k-1)v(S-i)\right]$$

$$\Leftrightarrow \varphi_i(v)=\sum_{\substack{i\in S\\|S|=k}} \tfrac{(n-k)!(k-1)!}{n!}\left[A'(k)-A'(k-1)\right]v(S-i)$$

$$=\alpha\sum_{\substack{i\in S\\|S|=k}} \tfrac{(n-k)!(k-1)!}{n!}\left[A(k)-A(k-1)\right]v(S-i)$$

$\Leftrightarrow A'(k)-A'(k-1)=\alpha\left(A(k)-A(k-1)\right)$ for all $k=2,3,...,n$ and $A(n)=A'(n)=1$

$\Leftrightarrow A'(k)-A'(k-1)=\alpha\left(A(k)-A(k-1)\right)$ for all $k=2,3,...,n-1$

and $A'(n-1)=(1-\alpha)+\alpha A(n-1)$

$\Leftrightarrow A'(k)=(1-\alpha)+\alpha A(k)$ for all $k=1,2,...,n$

$\Leftrightarrow \varphi=\Psi^{A'}=(1-\alpha)Shap+\alpha\Psi^{A}$ ■

Theorem 4 : φ is linear, efficient, symmetric and satisfies the $\alpha-A-$zero player property if and only if for any $v\in\Gamma$ and for any $v-$zero player $i\in N, \varphi_i(v)=\Psi_i^{A'}(v)=\alpha\Psi_i^{A}(v)$

$$\Leftrightarrow \varphi_i(v)=\sum_{k=1}^{n}\left[\sum_{\substack{i\in S\\|S|=k}} \tfrac{(n-k)!(k-1)!}{n!}\left[A'(k)v(S)-A'(k-1)v(S-i)\right]\right]$$

$$=\alpha\sum_{k=1}^{n}\left[\sum_{\substack{i\in S\\|S|=k}} \tfrac{(n-k)!(k-1)!}{n!}\left[A(k)v(S)-A(k-1)v(S-i)\right]\right]$$

$$\Leftrightarrow \sum_{k=1}^{n}\left[\sum_{\substack{i\in S\\|S|=k}} \tfrac{(n-k)!(k-1)!}{n!}A'(k-1)v(S-i)\right]=\alpha\sum_{k=1}^{n}\left[\sum_{\substack{i\in S\\|S|=k}} \tfrac{(n-k)!(k-1)!}{n!}A(k-1)v(S-i)\right]$$

$\Leftrightarrow A'(k-1) = \alpha A(k-1)$ for all $k = 2, 3, ..., n$ and $A'(n) = A(n) = 1$

$\Leftrightarrow A'(k) = \alpha A(k)$ for all $k = 1, 2, ..., n-1$

$\Leftrightarrow \varphi = \Psi^{A'} = (1-\alpha)\Psi^0 + \alpha\Psi^A$ ■