Portfolio Selection with Endogenous Estimation Risk

Diego Nocetti
The University of Memphis

Abstract

I explore how investors allocate mental effort to learn about the mean return of a number of assets and I analyze how this allocation changes the portfolio selection problem. I show that the endogeneity of estimation risk alters the comparative statics of portfolio choice and provides an explanation to Huberman's (2001) empirical findings that "Familiarity Breeds Investment".

Citation: Nocetti, Diego, (2006) "Portfolio Selection with Endogenous Estimation Risk." *Economics Bulletin*, Vol. 7, No. 6 pp. 1-9
Submitted: December 7, 2005. **Accepted:** September 26, 2006.
URL: http://economicsbulletin.vanderbilt.edu/2006/volume7/EB-05G10010A.pdf

1. Introduction

Standard models of portfolio selection under parameter uncertainty (Zellner and Chetty, 1965 and Klein and Bawa, 1976 are pioneers on the subject) are typically based on the assumption that investors learn about the true data generating process of asset returns using all available information. This assumption requires investors to have up to date databases of extremely large size. I would argue, however, that many investors do not use databases as econometricians, but make decisions based on the information *currently available "in their minds"*. In line with this argument, Nocetti (2005) presents a model where individuals exert mental effort to estimate the parameters of an economic model, by retrieving observations from a stock of memories. I take this hypothesis seriously to explore an economy where investors divide their attention to estimate the mean return of a number of assets.

An important difference of the divided attention model with respect to the standard treatment of parameter uncertainty is that individuals do not use all available information, but rely on their memory to infer the parameter estimates. Beyond providing a more realistic flavor to the inference problem, the advantage of such treatment is twofold. First, since the sample size is possibly small due to scarce cognitive resources, parameter uncertainty remains significant even if the data available is large and there are no structural shifts. Second, the endogeneity of estimation risk allows quantifying the magnitude and disentangling the determinants of the deviations from the canonical portfolio selection analysis in which investors know the true data generating process.

In the next section I review the standard Bayesian approach to portfolio selection with (exogenous) estimation risk. Section 3 considers how investors allocate mental effort to learn about mean excess returns and how, in the presence of scarce cognitive resources, they select the optimal portfolio shares. Section 4 establishes the main implications of the model. I first show how the optimal division of attention changes with the parameters of the economy. Second, I demonstrate that, like the case with exogenous estimation risk, the optimal portfolio allocation is observationally equivalent to the case with perfect knowledge of the economy, but with a higher degree of risk aversion. However, the comparative statics are strikingly different. In particular, the endogeneity of parameter uncertainty implies that: i) the effect of risk on the equity share is augmented by the existence of inattention; ii) an increase in the risk of one asset changes the holdings in all other assets in the portfolio, even if they are uncorrelated; and iii) investors optimally allocate a larger fraction of their portfolio in more familiar assets. This last result provides an explanation to Huberman's (2001) findings that "Familiarity Breeds Investment". In the present context, however, the bias towards more familiar assets is perfectly rational. Section 5 concludes with a discussion of ongoing work.

2. Bayesian Approach to Portfolio Selection

In this section I briefly describe the portfolio selection problem with estimation risk. I consider the simplest case where excess returns are i.i.d. and they follow a multivariate normal distribution with mean vector μ and known covariance matrix Σ. The representative investor does not known μ and has to estimate it using past data. As epitomized by Klein and Bawa (1976), the optimal portfolio with estimation risk is obtained by maximizing expected utility under the predictive distribution,

$$\begin{aligned} w &= \arg\max_w \int_{R_{t+1}} U(w) p\left(R_{t+1} \middle| \Theta_t\right) dR_{t+1} \\ &= \arg\max_w \int_{R_{t+1}} \int_{\mu} U(w) p\left(R_{t+1}, \mu \middle| \Theta_t\right) d\mu dR_{t+1} \end{aligned}, \tag{1}$$

where $U(w)$ is the utility function, $p\left(R_{P,t+1} \middle| \Theta_t\right)$ is the predictive density and $p\left(R_{t+1}, \mu \middle| \Theta_t\right) = p\left(R_{t+1} \middle| \mu, \Theta_t\right) p\left(\mu \middle| \Theta_t\right)$ is the posterior density of μ. Therefore, the Bayesian solution maximizes expected utility over the distribution of the parameters. As I shall demonstrate, the allocation of attention affects portfolio shares directly by changing the predictive density of excess returns.

It is simple to verify that, for an investor with CARA preferences [i.e. $U = -\exp(-\gamma W)$] the optimal portfolio is

$$w = \frac{1}{\gamma}\left[Var_t^*\left(R_{t+1}\right)\right]^{-1} E_t^*\left(R_{t+1}\right) \tag{2}$$

where E_t^* and Var_t^* denote the subjective expectation and variance at time t.

Without much lack of generality I assume that returns are uncorrelated. Then, in a rational equilibrium without estimation risk the optimal share in asset *i* is given by

$$w_i = \frac{\mu_i}{\gamma \sigma_i^2}, \tag{3}$$

where σ_i^2 is the variance of the ith asset excess returns.

Now consider the case with estimation risk and an exogenous number of observations n_i for asset *i*. Under a diffuse prior of excess returns, the predictive p.d.f. is $R_{t+1} \middle| \Sigma \sim N\left(\hat{\mu}_{t+1}, \Lambda\right)$, where $\hat{\mu}_{t+1}$ is the sample mean vector and Λ is the covariance matrix with diagonal elements $\sigma_i^2\left(1+\frac{1}{n_i}\right)$ and non-diagonal elements equal to zero.

It is therefore straightforward that, *given* a limited number of available observations, the Bayesian investor selects a portfolio with less risk. However, when the sample size is identical for all assets, as is usually assumed, the composition of the efficient frontier portfolios does not change. Therefore, the solution is observationally equivalent to a higher degree of risk aversion. That is, an investor that takes into account estimation risk is indistinguishable from an investor with risk aversion equal to $\gamma\left[1+(1/n)\right]$. I will therefore denote $\theta \equiv \gamma\left[1+(1/n)\right]$ as *effective* risk aversion.

This treatment of estimation risk assumes that the representative investor acts as an econometrician who uses all available information. In order to justify non-trivial adjustments one has to assume fairly small sample sizes. It is difficult to see, however, why an econometrician would not use the fairly long time series usually available in financial markets.

3. Portfolio Selection with Cognitive Constraints

The main assumptions regarding the cognition procedure are the same as those in Nocetti (2005), which I reproduce here:

a) Attention (mental effort) is a scarce resource (input).

b) The input is divisible (i.e. processing is parallel as opposed to serial) among activities which might differ in their demands.

c) The effort exerted to a given activity determines a particular output. The "production" of such output is achieved with a cognition technology.

d) The allocation of the input is done in an optimal way.

I further assume that the representative investor is endowed with a stock of memories of the entire history of excess returns. However, she relies on the *retrieval* of a subset of those memories to learn about μ_i. In particular, the representative investor exerts mental effort, e, to learn about the process of excess returns by retrieving a sample of size n from memory. A higher level of effort leads to a larger number of observations and, according to (1), higher expected utility.

Suppose that the cognition technology is Cobb-Douglas and that the output of the cognition procedure is the retrieval of n_i (random) past observations of excess returns,

$$\Phi_i e_i^{\alpha} = n_i \tag{4}$$

where Φ_i is a familiarity parameter and $\alpha \leq 1$. Equation (4) asserts that individuals are relatively more productive, in terms of effort exerted, in retrieving information about more familiar assets.

The attention capacity (k) constraint is

$$e_1 + e_2 + \dots\dots + e_m = k \ . \tag{5}$$

Then, the investor selects the optimal portfolio shares and the optimal division of attention subject to the *cognition/memory possibilities frontier*

$$\left(\frac{n_1}{\Phi_1}\right)^{\frac{1}{\alpha}} + \left(\frac{n_2}{\Phi_2}\right)^{\frac{1}{\alpha}} + \dots\dots + \left(\frac{n_m}{\Phi_m}\right)^{\frac{1}{\alpha}} = k \ . \tag{6}$$

The endogeneity of the sample size presents a new challenge to solve the portfolio problem. The reason is that the optimal attention level (i.e. the acquisition/retrieval of information that maximizes expected utility) must obviously be established prior to determining the conditional expectation of excess returns, and the optimal portfolio choice. I therefore assume that the representative investor uses the following procedure [see Muendler (2003) for a similar characterization]. First, given the prior estimate of

R_{t+1}, say $E_0^*\left(R_{t+1}\right)\ \forall i$ (recall the assumption of a diffuse prior), he decides the optimal level of attention and the (*ex-ante* optimal) portfolio shares jointly; Second, given the optimal n_i he finds the conditional estimate, $\hat{\mu}_{t+1}$, and the optimal portfolio shares[1].

The first order conditions are

$$E_0^*\left(R_{t+1}\right)-\gamma\sigma_i^2\left(1+\frac{1}{n_i}\right)w_i=0 \tag{7}$$

for the (*ex-ante*) portfolio share of asset *i*, and

$$\frac{\gamma w_i^2\sigma_i^2}{2n_i^2}-\lambda\frac{1}{\alpha}\left(\frac{n_i}{\Phi_i}\right)^{\frac{1-\alpha}{\alpha}}\frac{1}{\Phi_i}=0 \tag{8}$$

for the cognition problem of this asset. λ is the Lagrange multiplier and represents the change in the satisfaction received in equilibrium given a small change in the attention capacity constraint.

Equation (8) states that the marginal benefit of retrieving additional memories, a decrease in the variance of the predictive density of excess returns, is equalized to its marginal cost. Since it holds for all assets we have

$$\frac{\left(w_i^2\sigma_i^2/n_i^2\right)}{\left(w_j^2\sigma_j^2/n_j^2\right)}=\frac{\left(n_i/\Phi_i\right)^{\frac{1-\alpha}{\alpha}}1/\Phi_i}{\left(n_j/\Phi_j\right)^{\frac{1-\alpha}{\alpha}}1/\Phi_j}\qquad\forall j, \tag{9}$$

which is the usual optimality condition that the marginal rate of substitution equals the marginal rate of transformation among all possible actions.

Solving equations (7) and (8) simultaneously and using the optimal level of attention to obtain expected returns leads to

Proposition 1. *The optimal sample size and the optimal degree of attention to asset i is implicitly defined by*

$$\sigma_j^2\Phi_i^{1/\alpha}\left(1+\frac{c}{n_j}\right)^2 n_j^{\frac{1+\alpha}{\alpha}}-\sigma_i^2\Phi_j^{1/\alpha}\left(1+\frac{c}{n_i}\right)^2 n_i^{\frac{1+\alpha}{\alpha}}=0\ \forall j \tag{10}$$

and the (ex-post optimal) equity share in asset i is

$$w_i=\frac{\hat{\mu}_{t+1}}{\sigma_i^2\theta_i}, \tag{11}$$

[1] An alternative to this assumption would be to follow the literature on econometric learning whereby the optimization and forecasting problems are separated. Such separation, however, has no theoretical or empirical basis. Forecast errors affect the allocation of wealth only insofar they affect expected utility. See the discussion of corollary 1.

where $\theta_i \equiv \gamma\left[1+\frac{1}{n_i\left\{\sigma_i^2,\Phi_{i,},\sigma_j^2,\Phi_j,k\right\}}\right]$ *is effective risk aversion.*

4. Analysis

In the following subsections I use (10) and (11) to analyze the comparative statics of attention and to compare and contrast the divided attention framework with two benchmark models: the omniscient (infinite capacity) case with no estimation risk and the model with exogenous estimation risk, which I denote the "standard Bayesian" case.

4.1. Comparative Statics of Attention

Inspection of optimal attentiveness in (10) establishes:

Corollary 1. O*ptimal attention (and the number of retrieved observations) to asset i*:

a. Decreases (increases) with the dividends' variance of asset i (j)

b. Increases (decreases) with the productivity of retrieval of observations of asset i (j)

c. Increases with attention capacity

d. Is independent of the coefficient of risk aversion and the prior estimate of mean excess returns

The fact that attention to asset *i* falls with the variance of excess returns of the asset underscores the importance of considering the portfolio selection and cognition procedure jointly. In a framework where the statistical decision is separated from the economic decision an increase in the variance of the variable under consideration would increase the optimal allocation of mental effort. Then, since portfolio shares are negatively related to the variance of excess returns we would obtain that stocks with a lower share in the portfolio receive more attention. This is simply counterfactual. However, in the present framework, given an increase in the variance of the excess returns of asset *i* the investor reduces the holdings of this asset and invests a higher effort on other assets due to their relative increase in risk.

An increase in productivity produces a biased expansion of the cognition possibilities set and makes it optimal to increase (decrease) the effort exerted to the now relatively more (less) attention-intensive asset. In addition, because an increase in n_i reduces the relative variance of returns, the representative investor holds a larger share of this asset, creating a feedback effect.

Finally, as processing capacity increases effective risk aversion decreases, while the marginal contribution of γ and the prior estimates is the same for all assets.

Like the case with exogenous but finite samples, the *levels* of the shares on the risky assets are smaller due to estimation risk (note, however that this holds even for infinite available observations) and they are indistinguishable in the data from a higher degree of risk aversion. However, as I show next, the comparative statics are strikingly different.

4.2. Portfolio Shares and Risk

How does the divided attention framework, and in particular the endogenous characteristic of estimation risk, alter the comparative statics of the portfolio shares with respect to the volatility of excess returns? In the standard Bayesian case with exogenous *n* we have

$$\left(\frac{\partial w_i}{\partial \sigma_i^2}\right)^{SB} = -\frac{\hat{\mu}_{t+1}}{\theta \sigma_i^4} \tag{12}$$

where the superscript denotes "Standard Bayesian". By endogenizing effective risk aversion, the divided attention model disrupts the simple effect of volatility on the equity share. In particular, we obtain

$$\frac{\partial w_i}{\partial \sigma_i^2} = \left(\frac{\partial w_i}{\partial \sigma_i^2}\right)^{SB} + \frac{\hat{\mu}_{t+1}\left(1/n_i\right)\left(\partial n_i/\partial \sigma_i^2\right)\left(\sigma_i^2/n_i\right)}{\gamma \sigma_i^4\left[1+\dfrac{1}{n_i\left\{\sigma_i^2,\Phi_{i,}\sigma_j^2,\Phi_j,k,\mu_0\right\}}c\right]^2}. \tag{13}$$

From corollary 1 we know that $\left(\partial n_i/\partial \sigma_i^2\right) < 0$, implying that the elasticity of attention with respect to risk, $\left(\partial n_j/\partial \sigma_j^2\right)\left(\sigma_j^2/n_j\right)$, is negative. Therefore, compared to the standard Bayesian framework (and the infinite capacity case), *the effect of risk on the equity share is augmented by the existence of inattention.* Intuitively, an increase in risk reduces the holdings on this asset, which in turn feedback to decrease attention and reduce even more the shares.

The comparative statics give us another interesting implication. Because the assets are uncorrelated, in the two benchmark cases the share of one asset is completely unrelated to the process of the other assets. With attention limitations, however, the variance of asset *j* is involved in the determination of the share of asset *i*. In particular, we have $\partial w_i/\partial \sigma_j^2 > 0$. Intuitively, an increase in the variance of one asset makes the investor reduce the share of this asset, pay less attention to it and more to the other asset. This implies that, *a change in the volatility of one asset changes the holdings in all other assets in the portfolio, even though they are uncorrelated.*

4.3. Portfolio Diversification and Familiarity

A vast literature has provided evidence for the lack of international (e.g. French and Poterba, 1991) and intra-national (e.g. Coval and Moskowitz, 1999) portfolio diversification. Models of information asymmetries (e.g. Gehring, 1993; Brennen and Cao, 1997) and familiarity biases (e.g. Hubberman, 2001) have been the most successful empirically in explaining this lack of diversification. The present model can provide a foundation to those findings. As argued before, it seems reasonable that individuals are relatively more productive retrieving familiar information. This leads to

Proposition 2. *If individuals are more productive in the retrieval of observations that are from companies more familiar to them, the holdings of those familiar equities will be larger.*

The result follows directly from corollary 1 which states that attention is higher for those assets with higher productivity of recall (i.e. effective risk aversion is smaller). Thus, information asymmetries and the resulting lack of diversification arise endogenously in a model with attention constraints and memory deficits. This is not the case in the standard Bayesian setup.

Recent studies (e.g. Massa and Simonov, 2004; Kumar, 2004) distinguish familiarity effects between pure behavioral biases and a “rational” or information-based bias whereby more information in more familiar stocks leads the investor to *optimally* allocate a larger fraction of the portfolio in those stocks. Within the present model, however, such distinction is inappropriate because both effects are interlinked. The ease with which information is retrieved (a behavioral bias) introduces an information asymmetry which makes it optimal to invest a larger fraction of the portfolio in more familiar stocks.

5. Conclusion

This paper shows that introducing bounds to the attention/cognition process of an otherwise rational individual can produce strikingly different results to the standard cases of infinite attention capacity and exogenous estimation risk. However, the model has limitations that suggest at least two areas to pursue extensions.

First, I treat memory retrieval as a purely random process. Casual introspection and a large literature in psychology reveal that this is far from reality. Bringing memory biases can be done very simply. For example, it is well known that moods tend to cue memories that match in valence (e.g. positive mood, positive memories). This *mood-congruency* effect might have important implications for portfolio allocation and asset prices. For example, during periods in which investors are optimistic (pessimistic) they will tend to increase (decrease) their holdings on the risky assets, driving prices away from their fundamentals.

Second, I have not allowed for information sharing. One would expect that information sharing would drive the shares of the risky assets closer to the “standard Bayesian” case. Allowing for memory biases, however, might lead to completely different predictions. For example, in the context of mood-congruence effects, people might transmit a relatively larger amount of information that is consistent with their current mood. In such case, information sharing might exacerbate the effects memory biases.

References

Brennen, M. J. and Cao, H. (1997) “International Portfolio Investment Flows” *The Journal of Finance*, Vol. 52, No5, pp1851-1879.

Coval, J. D. and Moskowitz, T. (1999) “Home Bias at Home: Local Equity Preference in Domestic Portfolio” *The Journal of Finance*, Vol. 56, No 6, pp 2045-2073

Gehring, T. (1993) “An Information Based Explanation of the Domestic Bias in International Equity Investment” *Scandinavian Journal of Economics*, Vol. 95, No 1, pp 97-109

Huberman, G. (2001) “Familiarity Breeds Investment” *Review of Financial Studies*, Vol. 14, No 3, pp 659-680.

Klein, R.W., and Bawa, V.S. (1976) “The Effect of Estimation Risk on Optimal Portfolio Choice” *Journal of Financial Economics*, Vol 3, pp215-231.

Kumar, Alok (2004) “Is the Local Bias of Individual Investors Induced by Familiarity or Information-symmetry?” University of Notre Dame - Department of Finance

Massa, M. and Simonov, A. (2004) “Hedging, Familiarity and Portfolio Choice” CEPR discussion paper 4789

Muendler, M. (2003) “Rational Information Choice in Financial Market Equilibrium” Mimeo University of California, San Diego

Nocetti, D. (2005) "A Model of Mental Effort and Endogenous Estimation Risk." *Economics Bulletin*, Vol. 4, No. 14 pp. 1−10

Zellner, A. and Chetty, K. (1965) “Prediction and Decision Problems in Regression Models from the Bayesian Point of view” *Journal of the American Statistical Association* Vol. 60, pp 608-616