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A Variant of Uzawa's Theorem

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Abstract

Uzawa (1961) has shown that balanced growth requires technological progress to be strictly Harrod neutral (purely labor−augmenting). This paper offers a slightly more general variant of the theorem that does not require assumptions about savings behavior or factor pricing and is much easier to prove.

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Uzawa's (1961) theorem states, broadly speaking, that balanced growth requires technological progress to be Harrod neutral (purely labor-augmenting) along the equilibrium growth path. This is an extremely restrictive, and consequently extremely decisive, requirement, establishing that steady-state growth is a highly singular and therefore highly improbable case.[1] Yet textbooks mention the issue only in a cavalier manner, if at all.[2] This may be caused by the original proof being quite intricate. The purpose of this note is to provide a very short proof for a more general variant of the theorem. The theorem establishes that exponential growth implies Harrod neutrality. ("Exponential growth" refers to the case that all key variables grow exponentially; "balanced growth," requiring certain variables to grow in proportion, is covered as a special case.) In contrast to the classical statement by Uzawa (1961) and the more recent reformulation by Jones and Scrimegour (2004), the theorem does not involve assumptions about factor pricing (such as marginal productivity theory) or savings behavior.

Consider an economy with a neoclassical production function F. This function relates, at any point in time t, the quantity produced, denoted by Y_t, to labor input N_t and capital input K_t. The production function is assumed to exhibit, at any point in time, constant returns to scale. Due to technological progress, it shifts over time, and we write:

$$Y_t = F(N_t, K_t, t) \tag{1}$$

with

$$F(\lambda N, \lambda K, t) = \lambda F(N, K, t) \quad \text{for all} \quad (N, K, t, \lambda) \in \mathbb{R}_+^4. \tag{2}$$

Labor input N grows exponentially at rate n:

$$N_t = e^{nt} N_0. \tag{3}$$

Consumption at time t is denoted by C_t. Investment equals savings $(Y_t - C_t)$. The capital stock is augmented by savings and reduced by depreciation at the rate δ. Hence the capital stock changes over time according to

$$\dot{K}_t = Y_t - C_t - \delta K_t. \tag{4}$$

[1] As Aghion and Howitt (1998, 16 n.) remark, "there is no good reason that technological change takes that form." This singularity is *not* removed by theories about an induced bias in technological progress (Kennedy 1964, Samuelson 1965, von Weizsäcker, 1966, Drandakis and Phelps 1966, Acemoglu 2003). Theses theories require a "innovation possibility frontier" remaining invariant over decades if not centuries. This seems even less probable than assuming Harrod-neutrality right away. On the other hand, disposing of the assumption would lead to a model that could be fitted to *any* devolopment, just by postulating a suitable bias in technological change. The "new" growth theory favors, perhaps for that reason, the direct assumption. I recollect that many theorists (including myself) abandoned "old" growth theory around 1970 because they were not prepared to build their theories on such shaky foundations.

[2] Books like Abel and Bernanke (2005, 362-5), Agénor (2004, 440), Aghion and Howitt (1998, 16, 65), Barro (1997, 429), Blanchard (2006, 248), Blanchard and Fischer (1989, 3-4), Branson (1989, 638 f.), Burmeister and Dobell (1970, 78), Burda and Wyplosz (1997, 112-24), Froyen (2005, 78-85), Gärtner (2003, 238-41), Hacche (1979, 101), Mankiw (2003, 208-9), Romer (1996, 7), or Williamson (2005, 185-212) do not treat the problem in any intelligible way, while some older books like Barro and Sala-i-Martin (1995, 54-5) and Neumann (1994, 40) try to convey an idea about the issue.

Theorem (Variant of Uzawa's theorem of 1961). If the system (1)-(4) possesses a solution where Y_t, C_t, and K_t are all nonnegative and grow with constant growth rates rates y, c, and k, respectively, we can write

$$F(N_t, K_t, t) = G\left(N_t e^{(y-n)t}, K_t\right). \tag{5}$$

According to this theorem, exponential growth requires technological progress to be Harrod neutral (purely labor augmenting) along the growth path, with a rate of progress of $y - n$.

Proof. By assumption we have

$$\begin{aligned} Y_t &= Y_0 e^{yt} \\ C_t &= C_0 e^{ct} \\ K_t &= K_0 e^{kt}. \end{aligned} \tag{6}$$

From (4) and (6) we obtain

$$(k + \delta) K_t = Y_t - C_t \tag{7}$$

or

$$(k + \delta) K_0 = Y_0 e^{(y-k)t} - C_0 e^{(c-k)t} \tag{8}$$

for all t. Taking time derivatives yields

$$(y - k) Y_0 e^{(y-k)t} - (c - k) C_0 e^{(c-k)t} = 0$$

which implies

$$(y - k) Y_0 e^{(y-c)t} - (c - k) C_0 = 0$$

and therefore either $y = k$ and $c = k$, or $y = c$. If $y = c$, it follows that $(y - k)(Y_0 - C_0) = 0$. As $Y_0 = C_0$ would imply $K_0 = 0$ by (6) and (7) and this is ruled out by assumption, we must have $y = k$ in any case.

Define

$$G(N, K) := F(N, K, 0). \tag{9}$$

As $Y_0 = G(N_0, K_0)$, $Y_t = Y_0 e^{yt}$, $N_0 = N_t e^{-nt}$, $K_0 = K_t e^{-kt}$, and G is linear homogeneous, we can write

$$Y_t = G\left(N_t e^{(y-n)t}, K_t e^{(y-k)t}\right)$$

As $y = k$, this proves the theorem.

As noted in the proof, exponential growth requires production and consumption to grow at the common rate y. Hence the savings rate must be constant.

References

Abel, A. B. and Bernanke, B. S. (2005). *Macroeconomics.* Prentice Hall, Boston etc., 5 edition.

Acemoglu, D. (2003). Labor- and capital-augmenting technical change. *Journal of the European Economic Association*, 1(1):1–37.

Agénor, P.-R. (2004). *The Economics of Adjustment and Growth.* Harvard University Pres, Cambridge M. A., 2 edition.

Aghion, P. and Howitt, P. (1998). *Endogenous Growth Theory.* MIT Press, Cambridge M. A.

Barro, R. J. (1997). *Macroeconomics.* MIT Press, Cambridge M.A., 5 edition.

Barro, R. J. and Sala-i-Martin, X. (1995). *Economic Growth.* McGraw-Hill, New York etc.

Blanchard, O. (2006). *Macroeconomics.* Prentice Hall, Upper Saddle River, N.J., 4 edition.

Blanchard, O. J. and Fischer, S. (1989). *Lectures on Macroeconomics.* MIT Press, Cambridge, M.A. and London.

Branson, W. H. (1989). *Macroeconomic Theory and Policy.* Harper and Row, New York etc., 3 edition.

Burda, M. and Wyplosz, C. (1997). *Macroeconomics. A European Text.* Oxford University Press, Oxford, 2 edition.

Burmeister, E. and Dobell, A. R. (1970). *Mathematical Theories of Economic Growth.* Macmillan, London.

Drandakis, E. and Phelps, E. S. (1966). A model of induced invention, growth, and distribution. *Economic Journal*, 76(304):823–40.

Froyen, R. T. (2005). *Macroeconomics. Theories and Policies.* Prentice Hall, Upper Saddle River, N.J., 8 edition.

Gärtner, M. (2003). *Macroeconomics.* Prentice Hall, Harlow etc.

Hacche, G. (1979). *The Theory of Economic Growth. An Introduction.* Macmillan, London etc.

Jones, C. I. and Scrimegour, D. (2004). The steady-state growth theorem: A comment on uzawa (1961). Technical Report 10921, National Bureau of Economic Research, 1050 Massachusetts AvenueCambridge, MA 02138.

Kennedy, C. (1964). Induced bias in innovation and the theory of distribution. *Economic Journal*, pages 541–7.

Mankiw, N. G. (2003). *Macroeconomics.* Worth, New York, 5 edition.

Neumann, M. (1994). *Theoretische Volkswirtschaftslehre III.* Franz Vahlen, München, 2 edition.

Romer, D. (1996). *Advanced Macroeconomics.* McGraw-Hill, New York etc.

Samuelson, P. A. (1965). A theory of induced innovations along kennedy-weizsäcker lines. *Review of Economics and Statistics*, 47:343–56.

Uzawa, H. (1961). Neutral inventions and the stability of growth equilibrium. *Review of Economic Studies*, 28(2):117–24.

von Weizsäcker, C.-C. (1966). Tentative notes on a two sector model with induced technical progress. *Review of Economic Studies*, 33(3):245–51.

Williamson, S. D. (2005). *Macroeconomics.* Addison Wesley, Boston etc., 2 edition.