E C O N O M I C S B U L L E T I N

Equilibrium efficiency in the Uzawa−Lucas model with sector−specific externalities

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Abstract

This note shows that the competitive equilibrium is efficient in the Uzawa−Lucas endogenous growth model with sector−specific externalities associated to human capital in the goods sector for a large class of goods production technologies.

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1. Introduction

Most developed countries heavily subsidize education. This is typically justified on the basis of the perceived positive external effects of human capital accumulation which create a wedge between the social and private returns to education. Therefore, one may inquire whether the presence of such external effects does provide an incontrovertible rationale for the government intervention from a theoretical viewpoint. In a recent paper, Gómez (2004) shows that the competitive equilibrium is optimal in the Uzawa-Lucas model (Uzawa, 1965, Lucas, 1988) when there are sector-specific externalities associated with human capital in the goods sector, so government intervention is not justified. However, he makes the rather restrictive assumption that output is produced with a Cobb-Douglas technology.

Recent empirical studies conclude that the assumption of a unitary elasticity of substitution between capital and labor in production is not supported by the data (e.g., Duffy and Papageorgiou, 2000, and Masanjala and Papageorgiou, 2004). This raises the question on whether the optimality result is satisfied for less restrictive goods production technologies as well. This note shows that Gómez's (2004) result also holds under the assumption that the goods production technology is such that the private marginal productivity of human capital devoted to the production of goods is proportional to its social marginal productivity. Although at first sight this assumption might seem to be rather restrictive, it is satisfied by production functions that are commonly used in both theoretical and applied work; in particular, by CES technology. The intuition for this result is simple: The time allocation decision of the private agent is the same as that of the central planner, since the return to and the cost of investing in human capital change in the same proportion.

The remainder of this note is organized as follows. Section 2 describes the decentralized economy. Section 3 describes the centrally planned economy, shows that the competitive equilibrium is optimal, and provides an intuitive explanation. Section 4 concludes.

2. The decentralized economy

2.1. Private agents

The economy is populated by a large number of identical infinitely lived representative agents who derive utility from the consumption of a private consumption good *C*. Each agent has a constant flow of one unit of time per period which can be allocated to work or learning. For simplicity, we assume that population is constant and normalized to one. The agent's preferences are described by the intertemporal utility function

$$\int_0^{\infty} \mathrm{e}^{-\rho t} \frac{C^{1-\sigma}-1}{1-\sigma} dt \qquad \rho > 0,\ \sigma > 0, \tag{1}$$

where ρ is the subjective discount rate, and $1/\sigma$ is the intertemporal elasticity of substitution. The agent maximizes (1) subject to the flow budget constraint

$$\dot{K} = rK + wuH + \pi - C, \tag{2}$$

and the constraint on human capital accumulation

$$\dot{H} = G(1-u)H. \tag{3}$$

Here, *K* denotes physical capital, *H*, human capital, *r*, the interest rate, *w*, the wage rate, *π*, firms profits, *u*, work time, 1–*u*, learning time, and *G* is a positive, C^2, concave and strictly increasing function, so that $G' > 0$ and $G'' \le 0$.

Let *J* be the current value Hamiltonian of the agent's optimization problem:

$$J = (C^{1-\sigma} - 1)\big/(1-\sigma) + \lambda(rK + wuH + \pi - C) + \theta G(1-u)H \ .$$

The first-order conditions for an interior solution are

$$C^{-\sigma} = \lambda \, , \tag{4a}$$

$$\lambda w H = \theta \, G'(1-u)H \, , \tag{4b}$$

$$\dot{\lambda} = (\rho - r)\lambda \, , \tag{4c}$$

$$\dot{\theta} = (\rho - G(1-u))\theta - \lambda w u \, , \tag{4d}$$

plus the transversality conditions

$$\lim_{t\to\infty} e^{-\rho t}\lambda K = 0 \, , \tag{4e}$$

$$\lim_{t\to\infty} e^{-\rho t}\theta H = 0 \, . \tag{4f}$$

2.2. Firms

Output, Y, is produced with a linearly homogeneous, C^2, concave production function

$$Y = F(K, uH, \overline{uH}) \, , \tag{5}$$

where $\overline{uH}$ is the average human capital in the goods sector, and expresses sector-specific externalities associated with human capital employed by the sector producing goods. This function exhibits positive, but diminishing, marginal productivity in all factors, $F_i > 0$, $F_{ii} > 0$, where F_i and F_{ii} (i=1,2,3) are, respectively, the first and second derivatives of F with respect to its ith argument.

We shall also assume that technology is such that the private marginal productivity of human capital in the goods sector, $F_2(K,uH,uH)$, is proportional to its social marginal productivity, $F_2(K,uH,uH) + F_3(K,uH,uH)$:

$$F_2(K,uH,uH) + F_3(K,uH,uH) = \alpha \, F_2(K,uH,uH) \qquad \alpha > 1. \tag{6}$$

Although this might seem to be a rather restrictive assumption, it is satisfied by production functions which are commonly used in both theoretical and applied work, as CES technology

$$F(K, uH, \overline{uH}) = A[\, aK^{\eta} + b(uH)^{\eta} + (1-a-b)(\overline{uH})^{\eta} \,]^{1/\eta} \, ,$$

technologies *á la* Jones and Manuelli (1990) as, for example,

$$F(K, uH, \overline{uH}) = AK + BK^{\eta}(uH)^{\beta}(\overline{uH})^{1-\eta-\beta} \, ,$$

and nested production functions as, for example,

$$F(K, uH, \overline{uH}) = A[\, aK^{\eta} + (1-a)(b(uH)^{\beta} + (1-b)(\overline{uH})^{\beta})^{\eta/\beta} \,]^{1/\eta} \, .$$

Profit maximization by competitive firms implies that capital and labor are used up to the point at which marginal product equates marginal cost:

$$r = F_1(K, uH, \overline{uH}) \, , \tag{7a}$$

$$w = F_2(K, uH, \overline{uH}) \, , \tag{7b}$$

where r is the rate of return on physical capital and w is the wage rate. Since the production function exhibits decreasing returns-to-scale at the private level, the competitive firm earns positive profits

$$\pi = F(K, uH, \overline{uH}) - F_1(K, uH, \overline{uH})\,K - F_2(K, uH, \overline{uH})\,uH \, . \tag{7c}$$

We assume that these profits are distributed back to households as dividends.

2.3. Equilibrium

Hereafter let $\gamma_v = \dot{v}/v$ denote the growth rate of the variable *v*. In what follows, the condition $\overline{uH} = uH$ will be imposed. In order to characterize the transitional dynamics, we transform the model by defining the new variables $z = K/(uH)$, the ratio of physical capital to human capital in the goods sector, and $s = C/K$, the ratio of consumption to physical capital. Furthermore, let $x = K/H$ be the ratio of physical to human capital, and $p = \theta/\lambda$, the relative price of human capital in units of goods.

In equilibrium, since *F* is linearly homogeneous it can be transformed as follows

$$Y = uH\,F(z,1,1) = uH\,f(z), \tag{8}$$

where *f* is C^2, positive, $f' > 0$, and $f'' < 0$. Hence, the marginal productivity of physical capital and the social marginal productivity of effective time, *v*, can be expressed as

$$r = F_1(K, uH, uH) = f'(z), \tag{9a}$$

$$v = F_2(K, uH, uH) + F_3(K, uH, uH) = f(z) - z f'(z). \tag{9b}$$

The assumption (6) entails that

$$w = F_2(K, uH, uH) = (f(z) - z f'(z))/\alpha. \tag{10}$$

From (2), using (7) and (8), we can obtain the overall resource constraint of the economy

$$\gamma_K = f(z)/z - s. \tag{11}$$

Using (3) and (11), the growth rate of $x = K/H$ is

$$\gamma_x = f(z)/z - s - G(1-u). \tag{12}$$

The growth rate of $p = \theta/\lambda$ can be obtained from (4b)-(4d) as

$$\gamma_p = f'(z) - G(1-u) - G'(1-u)u. \tag{13}$$

Log-differentiating (4a) with respect to time, and using (4c) and (9a), we derive the growth rate of consumption:

$$\gamma_C = (f'(z) - \rho)/\sigma. \tag{14}$$

Using (10), equation (4b) can be expressed as $f(z) - z f'(z) = \alpha\, p G'(1-u)$. Log-differentiating this expression with respect to time, we have $\xi\gamma_z = \gamma_p + \eta\gamma_u$, where η and ξ are defined by $\eta = -G''(1-u)u/G'(1-u)$ and $\xi = -z^2 f''(z)/(f(z) - z f'(z))$. Note that $\eta \geq 0$ and $\xi > 0$. Since $\gamma_z = \gamma_x - \gamma_u$, the growth rates of *z* and *u* can be expressed as

$$\gamma_z = (\gamma_p + \eta\gamma_x)/(\xi + \eta), \tag{15a}$$

$$\gamma_u = -(\gamma_p - \xi\gamma_x)/(\xi + \eta). \tag{15b}$$

Using (11) and (14), the growth rate of $s = C/K$ is

$$\gamma_s = f'(z)/\sigma - f(z)/z + s - \rho/\sigma. \tag{15c}$$

The system (15) characterizes the dynamics of the decentralized economy in terms of *z*, *s*, and *u*, recalling (12) and (13). Note that *x* and *p* must also be constant in the steady state. This will enable to simplify the computation of the steady state and the stability analysis.

The steady state can be computed as follows. Equating (12) to zero we obtain

$$s^* = (f(z^*) - z^* G(1-u^*))/z^*. \tag{16a}$$

Substituting *s** from (16a) into (15c) and equating the result to zero, we have

$$f'(z^*) = \rho + \sigma G(1-u^*), \tag{16b}$$

which, after substitution in (13), entails that the steady state level of u is determined by the equation

$$(1-\sigma)G(1-u^*)+G'(1-u^*)u^*-\rho=0\,. \tag{16c}$$

Let $Q(u)=(1-\sigma)G(1-u)+G'(1-u)u-\rho$. Since $Q'(u)>0$, equation (16c) has a unique solution $u^*\in(0,1)$ if and only if the condition $(1-\sigma)G(1)<\rho<(1-\sigma)G(1)+G'(0)$ is met. We shall assume henceforth that equation (16b) has a positive solution $z^*>0$. A sufficient condition for (16b) to have a positive solution z^* for any $u^*\in(0,1)$ is that the production function F satisfy the Inada conditions, which entail that $\lim_{z\to 0} f'(z)=\infty$ and $\lim_{z\to\infty} f'(z)=0$. Equation (16a) can be expressed as $s^*=(f'(z^*)-G(1-u^*))+(f(z^*)-z^*f'(z^*))/z^*$. But equation (13) entails that $f'(z^*)-G(1-u^*)>0$, and $(f(z^*)-z^*f'(z^*))/z^*>0$, so that $s^*>0$ if $u^*\in(0,1)$ and $z^*>0$. For the transversality conditions (7e) and (7f) to be satisfied it must be that $-\rho+\gamma_\lambda^*+\gamma_K^*<0$ and $-\rho+\gamma_\theta^*+\gamma_H^*<0$, respectively. These conditions can be readily shown to be equivalent to $-G'(1-u^*)u^*$, which is negative if $0<u^*<1$.

To analyze the stability of the steady state, we linearize the system (15) around the steady state (z^*,s^*,u^*). This yields

$$\begin{pmatrix}\dot z\\ \dot s\\ \dot u\end{pmatrix}=\begin{pmatrix}\dfrac{z^*f''(z^*)}{\xi^*} & -\dfrac{\eta^*z^*}{\xi^*+\eta^*} & 0\\ \dfrac{s^*f''(z^*)(\xi^*-\sigma)}{\sigma\xi^*} & s^* & 0\\ 0 & -\dfrac{\xi^*u^*}{\xi^*+\eta^*} & G'(1-u^*)u^*\end{pmatrix}\begin{pmatrix}z-z^*\\ s-s^*\\ u-u^*\end{pmatrix}=J\begin{pmatrix}z-z^*\\ s-s^*\\ u-u^*\end{pmatrix},$$

where ξ^* and η^* stand for the variables ξ and η evaluated at the steady state. The last diagonal element, $G'(1-u^*)u^*>0$, is a positive real eigenvalue of the matrix J. The other two eigenvalues of J are the eigenvalues of the 2×2 upper left submatrix of J,

$$\bar{J}=\begin{pmatrix}\dfrac{z^*f''(z^*)}{\xi^*} & -\dfrac{\eta^*z^*}{\xi^*+\eta^*}\\ \dfrac{s^*f''(z^*)(\xi^*-\sigma)}{\sigma\xi^*} & s^*\end{pmatrix}.$$

The determinant of $\bar{J}$ is negative,

$$\det=\frac{(\eta^*+\sigma)f''(z^*)s^*z^*}{(\eta^*+\xi^*)\sigma}<0\,.$$

Taking into account the definition of ξ^*, and using (16a) and (16b) to substitute for s^* and $f'(z^*)$, respectively, and $G(1-u^*)=(\rho-G'(1-u^*)u^*)/(1-\sigma)$ from (16c), the trace of $\bar{J}$ can be computed as $\mathrm{tr}=G'(1-u^*)u^*>0$, which is positive. Hence, $\bar{J}$ has one real positive eigenvalue and one real negative eigenvalue, and so, the coefficient matrix of the linearized system, J, has one negative real eigenvalue and two real positive eigenvalues, implying that the steady state is locally saddle-path stable.

3. The centrally planned economy

The central planner possesses complete information and chooses all quantities directly, taking all the relevant information into account. The planner maximizes (1) subject to (3) and

$$\dot{K} = F(K, uH, uH) - C \,. \tag{17}$$

Let J be the current value Hamiltonian of the planner's maximization problem, and let λ and θ be the multipliers for the constraints (17) and (3), respectively:

$$J = (C^{1-\sigma} - 1)\big/(1-\sigma) + \lambda\,(F(K, uH, uH) - C) + \theta G(1-u)H \,.$$

The first-order conditions for an interior solution are similar to that of the market economy (4), but (4b) and (4d) should be substituted, respectively, with

$$\lambda\,(F_2(K, uH, uH) + F_3(K, uH, uH))H = \theta G'(1-u)H \,, \tag{4b'}$$

$$\dot{\theta} = (\rho - G(1-u))\theta - \lambda\,(F_2(K, uH, uH) + F_3(K, uH, uH))u \,. \tag{4d'}$$

Using (4b'), (4c) and (4d'), the growth rate of $p = \theta/\lambda$ is again obtained as (13). Hence, proceeding in a similar manner to that in the case of the market economy, we find that the system (15) characterizes the dynamics of the centralized economy in terms of *z*, *s*, and *u*, recalling (12) and (13). Caballé and Santos (1993) show that there is a unique and saddle-path stable steady state if and only if $(1-\sigma)G(1) < \rho < (1-\sigma)G(1) + G'(0)$, under the assumption that the production function satisfies the Inada conditions. Although we have not required the fulfillment of the Inada conditions, we shall assume that the steady state of (15) is feasible, so the analysis of Caballé and Santos (1993) applies as well.

The system (15) describes both the dynamics of the market economy and the dynamics of the centrally planed economy. Hence, the decentralized economy replicates the first-best optimum attainable by a central planner, so we can state our main result.

PROPOSITION 1: *The competitive equilibrium is Pareto optimal in the Uzawa-Lucas model with sector-specific externalities associated with human capital in the goods sector when the production function is such that the private marginal productivity of human capital devoted to the production of goods is proportional to its social marginal productivity.*

4. Discussion

The result in Proposition 1 can be intuitively explained as in Gómez (2004). Lucas (1990) showed that a constant flat-rate tax on labor income is neutral in the market economy since the introduction of a wage tax with a constant rate reduces in the same proportion the returns and the cost of investment in human capital. A similar argument can be used to explain the optimality of the competitive equilibrium in presence of sector-specific externalities.

Comparison of (4b) and (4b') shows that, in equilibrium, the difference in perception between the individual agent in the market economy and the central planner is that the *private* return to effective work time, as seen by the individual agent, is the wage rate,

$$w = F_2(K, uH, uH) = (f(z) - z f'(z))/\alpha \,,$$

whereas its *social* return, as seen by the central planner, is

$$v = F_2(K, uH, uH) + F_3(K, uH, uH) = f(z) - z f'(z) = \alpha w \,. \tag{18}$$

The private return is lower than the social return as the representative agent does not take into account the effect of the sector-specific externality. In this model, human capital is produced using time and human capital as inputs and, therefore, the sole cost of investing in human capital is foregone earnings. Thus, although the agent does not realize that the private return to effective time (the wage rate) is lower than its social return, as the return to and the opportunity cost of human capital accumulation augment in the same proportion, the time allocation decision of the individual agent is the same as that of the central planner.

The former argument can be more formally stated by considering the time allocation margin of choice. As noted by Lucas (1990), the allocation of time between working in the output sector and learning new skills must be such that the value of a unit of time spent producing at each date is equal, on the margin, to the value of spending that unit of time accumulating new human capital that will enhance production in the future. In the market economy, this condition can be obtained as follows.

Denoting $p = \theta/\lambda$, equations (4c) and (4d) entail that

$$\dot{p} = p(r - G(1-u)) - wu \,.$$

Solving this differential equation yields

$$p(t) = p(T)e^{\int_t^T (G(1-u(\tau))-r(\tau))\,d\tau} + \int_t^T e^{\int_t^\kappa (G(1-u(\tau))-r(\tau))\,d\tau}\, w(\kappa)\,u(\kappa)\,d\kappa \,. \tag{19}$$

The fulfillment of the transversality condition (4e) entails that

$$-\rho + \gamma_\lambda^* + \gamma_K^* = -\rho + \gamma_\lambda^* + \gamma_H^* = G(1-u^*) - r^* < 0 \,.$$

Hence, taking the limit of $p(t)$ as T goes to infinity in (19), taking into account that $\lim_{T\to\infty} p(T) = p^* < \infty$, where p^* denotes the steady state value of p, we have

$$p(t) = \int_t^\infty e^{\int_t^\kappa (G(1-u(\tau))-r(\tau))\,d\tau}\, w(\kappa)\,u(\kappa)\,d\kappa \,. \tag{20}$$

Eq. (3) entails that $H(\kappa) = H(t)\, e^{\int_t^\kappa G(1-u(\tau))\,d\tau}$. Thus, multiplying both sides of (20) with $H(t)$, and using the former expression, we have

$$p(t)\,H(t) = \int_t^\infty e^{-\int_t^\kappa r(\tau)\,d\tau}\, w(\kappa)\,u(\kappa)H(\kappa)\,d\kappa \,. \tag{21}$$

Equation (4b) can be expressed as $wH = pG'(1-u)H$, which, using (21) yields

$$w(t)\,H(t) = G'(1-u(t)) \int_t^\infty e^{-\int_t^\kappa r(\tau)\,d\tau}\, w(\kappa)\,u(\kappa)H(\kappa)\,d\kappa \,. \tag{22}$$

In the centrally planned economy, the equivalent condition can be obtained in a similar manner. Equation (4b') can be expressed as

$$\lambda v = \theta G'(1-u) \,, \tag{23a}$$

where v is the social return to effective time defined in (18). Using (18), Eq. (4d') can be expressed as:

$$\dot{\theta} = (\rho - G(1-u))\theta - \lambda v u \,. \tag{23b}$$

Similarly to the case of the market economy, from (4c), (23b), (6) and (23a), we can obtain

$$v(t)\,H(t) = G'(1-u(t)) \int_t^\infty e^{-\int_t^\kappa f'(z(\tau))\,d\tau}\, v(\kappa)\,u(\kappa)H(\kappa)\,d\kappa \,. \tag{24}$$

The social return to physical capital and its private return coincide, i.e., $r = F_1(K, uH, uH) = f'(z)$. By assumption (6), the social return to effective labor is proportional to its private return, $v = \alpha w$. Thus, comparison of (22) and (24) clearly shows that the time allocation decision of the representative agent is the same as that of the central planner, since the return to and the cost of investing in human capital have changed in the same proportion.

5. Conclusions

We have shown that the competitive equilibrium is efficient in the Uzawa-Lucas endogenous growth model when the average human capital employed in the production of goods has an external effect in this sector. Thus, this type of externalities does not provoke a market failure and does not provide a rationale for government intervention. This result holds up under the assumption that the production technology is such that the private marginal productivity of human capital devoted to the production of goods is proportional to its social marginal productivity. This condition is satisfied by production functions commonly used in both theoretical and applied work as, for instance, the CES technology.

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