E C O N O M I C S B U L L E T I N

# Efficiency in the cake-eating problem with quasi-geometric discounting

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## *Abstract*

This paper shows that any equilibrium allocation in the cake-eating problem with quasi-geometric discounting is not Pareto efficient. However, efficiency can be established by introducing a planner who controls the initial endowment and makes transfers over time. It is shown than any Pareto efficient allocation can be supported by a perfect equilibrium with transfers.

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## 1. Introduction

Economic theories of intertemporal choice generally assume that individuals discount the future exponentially. However, experimental and field evidence suggests that many individuals have preferences that reverse as the date of decision making nears (Lowenstein and Thaler 1989; Ainslie 1992; Angeletos, Laibson, Repetto, Tobacman, and Weinberg 2001; DellaVigna and Malmendier 2003; Fang and Silverman 2004). This type of behavior has been modeled through quasi-geometric discount functions that generate time inconsistency. A solution to the decision problem is usually required to take the form of a subgame perfect equilibrium of a game where the players are the consumer and his future selves.

The purpose of this note is twofold. First, it studies efficiency in the cake eating problem with quasi-geometric discounting without commitment. Second, it investigates the possibility of obtaining efficient outcomes by introducing a planner who is able to distribute the initial endowment through a series of transfers over time.

The paper is organized as follows. We describe the model in section 2. In section 3 we introduce the concept of efficiency used throughout the paper. Sections 4 and 5, respectively, state formal theorems summarizing our inefficiency result and characterizing the set of efficient allocations that can be obtained in an equilibrium with transfers. Section 6 concludes. Section 7 provides proofs of the theorems.

## 2. The Model

Consider the following economy. Time is discrete and indexed by $t = 1, 2, \ldots, T$. There is one agent who lives for $T \geq 3$ periods. An initial endowment of one unit of a consumption good, $x$, is to be consumed over time.

In period $t$, preferences over consumption streams $x = (x_1, \ldots, x_T) \in \mathbb{R}_+^T$ are representable by the utility function

$$U_t(x) = u(x_t) + \beta \sum_{\tau=t+1}^{T} \delta^{\tau-t} u(x_\tau)$$

where $(\beta, \delta) \in (0, 1] \times (0, 1]$, and the instantaneous utility function, $u : \mathbb{R}_+ \to \mathbb{R}$, satisfies $u'(x) > 0$ and $u''(x) < 0$ over $(0, \infty)$.

The type of preferences represented by this model incorporates the so-called quasi-geometric discounting[1]. The parameter $\delta$ is called the *standard discount factor* and it represents the long-run, time consistent discounting; the parameter $\beta$ represents a preference for immediate gratification and is known as the *present-biased factor*. For $\beta = 1$ these preferences reduce to exponential discounting. For $\beta < 1$, the $(\beta, \delta)$ formulation implies discount rates that decline as the discounted event is moved further away in time implying time inconsistency.[2]

In the present analysis, we assume that the agent is sophisticated in the sense that she is fully aware of her time inconsistency problem. Similar to Strotz (1956), Peleg and Yaari (1973), Goldman (1980), Laibson (1997), and O'Donoghue and Rabin (2001) we model this problem by thinking of the agent as consisting of $T$ autonomous selves. For the ensuing games played between selves we consider subgame

[1] This type of preferences was originally proposed by Phelps and Pollack (1968).

[2] See Frederick et al. (2002) for a review of the $(\beta, \delta)$ formulation.

perfect equilibrium as our solution concept. Let $h_t = (x_1, \ldots, x_{t-1})$ be a feasible consumption history at time $t$. Let $S_t$ represent the set of feasible strategies for self $t$. Let $S = S_1 \times \ldots \times S_T$ be the joint strategy space of all selves. A consumption strategy for self $t$ is a function

$$s_t : S_t \longrightarrow \left[0, 1 - \sum_{i=1}^{t-1} x_i\right] \tag{1}$$

The allocation functions $c^t : S \to \mathbb{R}^t_+$ are given by

$$c^t(s_1, \ldots, s_t) = (c^{t-1}(s_1, \ldots, s_{t-1}, s_t(c^{t-1}(s_1, \ldots, s_{t-1}))) \tag{2}$$

with $c^1 = s_1$. An equilibrium allocation is a consumption vector $x^* \in \mathbb{R}^T_+$ supported by some perfect equilibrium $s^*$ of the intrapersonal game. Formally,

**Definition 1** *An equilibrium allocation is a consumption vector $x^* \in \mathbb{R}^T_+$ that satisfies $x^* = c^T(s^*)$ for some subgame perfect equilibrium $s^* \in S$ of the intrapersonal game.*

## 3. Welfare and time inconsistency

Time inconsistency implies that preferences today conflict with preferences tomorrow. Therefore, it is difficult to evaluate the welfare of a time inconsistent individual since her different temporal incarnations disagree about what is better for them. The difficulty with defining a welfare criterion for time inconsistent individuals is well illustrated by Goldman (1979): "The question of Pareto efficiency is especially vexing when the discussion of endogenous preferences is explicit. If we were to identify players...by both calendar time and the history of actions prior to the times of their decisions, then Pareto comparisons under alternative histories become impossible since the set of players changes". He remarks the importance of imposing some structure to indicate when two possible players are comparable.

We avoid the problem identified by Goldman (1979) by evaluating each self on her own terms. This approach, which has been applied by Phelps and Pollak (1968) and Goldman himself (1979), emphasizes the application of a Pareto criterion to evaluate equilibrium allocations

**Definition 2** *An equilibrium allocation, $x^*$, is Pareto efficient if there is no other feasible allocation that makes one self better off without making some other temporal selves worse off.*

Other authors have proposed alternative welfare criteria. O'Donoghue and Rabin (1999) advocate maximizing welfare from a "long-run perspective". It involves the existence of a "...(fictitious) period 0 where the person has no decision to make and weights all future periods equally." This approach incorporates the fact that most models of present-biased preferences try to capture situations in which people pursue immediate gratification. Other approaches privilege a subset of players in the intrapersonal game. For instance, welfare may be evaluated with respect to current self's perspective. This "dictatorship of the present" approach has been applied by Cropper and Laibson (1998), and Cropper and Koszegi (2001).[3]

## 4. Inefficiency of the equilibrium allocation

One implication of assuming that the individual is time consistent ($\beta = 1$) is that the optimal consumption path from self 1's perspective can be implemented in equilibrium: his future incarnations will consume and save the amounts he wants them to. On the other hand, if the individual is time inconsistent ($\beta < 1$), the equilibrium allocation may not be efficient because the strategic interaction of the temporal selves can originate a coordination failure with a suboptimal outcome as a result.

This conjecture has been explored by Goldman (1979), who shows that no interior equilibrium allocation other than one which is best for self 1 is Pareto efficient if some conditions are satisfied. He does not assume separability of the intertemporal utility function neither quasi-geometric discounting as we do here, however. Under separability, an efficient equiibrium allocation arises when $\beta = 1$. This equilibrium allocation coincides with the most preferred allocation from self 1's perspective. For $\beta < 1$, preferences are time inconsistent and the equilibrium allocation is always inefficient. Theorem 1 states this result:

**Theorem 1** *For $\beta < 1$, any equilibrium allocation is inefficient.*

What is driving this result? Given the time inconsistency problem, future selves may allocate resources between consumption and savings differently than the preferences of self 1 would dictate. Therefore, the first agent's decision to "overconsume" is his optimal response to this behavior: self 1 is willing to sacrifice present consumption if he could prescribe when in the future his extra savings will be consumed. Since past consumption does not affect current satisfaction, the Pareto improvement follows. More generally, self $t$ may be willing to sacrifice current consumption in order to increase the consumption of some future self if he were certain that future selves will not distort that decision. Consequently, the Pareto improvement implies the existence of an inflection point at which current consumption stops being reduced vis-a-vis the initial equilibrium.[4]

To illustrate the inefficiency result, consider an agent who lives for three periods, the minimum number of periods to have time inconsistency, and has preferences represented by a log utility function $u(x_t) = \ln x_t$. For the sake of simplicity, we assume that $\delta = 1$ and $\beta = 1/2$. It is not difficult to show that the equilibrium allocation is given by $c_1^* = 3/6$, $c_2^* = 2/6$ and $c_3^* = 1/6$. Notice that $1/18 = \arg\max \ln(3/6 - \tau) + 1/2 \ln(2/6) + 1/2 \ln(1/6 + \tau)$, so self 1 is willing to

[3]For an analysis of welfare criteria for time inconsistent individuals see Bhattacharya et al. (2004).

[4]I am very thankful to an anonymous referee for pointing me out the intuition behind the inefficiency result.

transfer $\tau = 1/18$ to self 3 to increase his own welfare. Clearly, this is a Pareto improvement.

## 5. Establishing efficiency through transfers

Assume the existence of a first-best scenario where a planner has full information and is able to distribute the endowment of the consumption good through a transfer scheme $\tau \in \{R_+^T : \sum_{t=1}^T \tau_t = 1\}$. We formally model this setting as a two-stage game where the players are the planner and the $T$ different selves. In stage 1, the planner announces how the endowment will be allocated through the transfer scheme. In stage 2, the different selves play an intrapersonal game. A strategy for self $t$ is a function

$$s_t : S_t \longrightarrow \left[0, \sum_{i=1}^{t} \tau_i - \sum_{i=1}^{t-1} x_i\right] \tag{3}$$

We define equilibria arising in this game as *perfect equilibria with transfers*.

The questions to be addressed are:

- Is it possible to observe efficient allocations in a perfect equilibrium with transfers?
- If so, what subset of efficient allocations can be obtained in equilibrium?

The answer to the first question is positive. Moreover, it can be shown that the planner can achieve any Pareto efficient allocation, $x^*$, by appropriately distributing the endowment over time

**Theorem 2** *Any efficient allocation $x^* \in \mathbb{R}_+$ can be supported by a perfect equilibrium with transfers.*

We prove this theorem by applying the following line of logic. First, notice that, on the equilibrium path, the agent has no incentive to transfer resources to the future *even if he actually could choose the point in time at which these resources will be consumed* if transfers coincide with an efficient allocation: i.e $\tau = x^*$ for some Pareto efficient allocation $x^* \in R_+^T$. From here we obtain the result that the chosen efficient allocation arises in equilibrium by doling out the consumption good appropriately. Therefore, the planner provides, through the transfer scheme, a mechanism that makes the individual commit to follow up an optimal consumption path.

## 6. Concluding remarks

The cake-eating problem offers a simple setting to analyze the consequences of time inconsistent behavior for efficiency. We have shown that by imposing constraints on future resource flows, efficiency could be established. In fact, most commitmet devices impose some inflexibility in the way resources will be consumed in the future. An illiquid asset a la Laibson (1997) is a good example of a commitment device that is used by current selves to limit consumer behavior.

There are some interesting extensions for the cake-eating problem with quasi-geometric discounting. For instance, one could introduce imperfect information in the sense that the planner cannot observe the degree of present-biased preference. We could also incorporate income shocks into to model. In that setting, the planner would have to play the roles of insurance provider and commitment enforcer simultaneously.

## 7. Proofs

**Proof of Theorem 1**. First, notice that if there exist periods $j$ and $t$, $j > t$, such that $u'(x_t^*) < \beta\delta^{j-t}u'(x_j^*)$, then the equilibrium allocation is inefficient. Hence it suffices to show that an allocation $x \in \mathbb{R}_+^T$ satisfying

$$u'(x_{T-2}) \geq \max[\beta\delta u'(x_{T-1}), \beta\delta^2 u'(x_T)] \tag{4}$$

cannot be an equilibrium allocation.

Let $x^* \in \mathbb{R}_+^T$ be an equilibrium allocation. Assume, towards a contradiction, that $x^*$ satisfies (4). Define the set $\Phi = \{\gamma \in \mathbb{R}^3 \mid \gamma_{T-2} = 1, \gamma_{T-1} \leq 0, \gamma_T \leq 0, \gamma_{T-1} + \gamma_T = -1\}$, and the function $\varphi(\tau) = u(x_{T-2}^* + \tau\gamma_{T-2}) + \beta\sum_{t=1}^{2}\delta^t u(x_t^* + \tau\gamma_t)$. Taking the second derivative of the function $\varphi(\cdot)$ we have

$$\varphi''(\tau) = u''(x_{T-2}^* + \tau) + \beta\gamma_{T-1}^2 u''(x_{T-1}^* + \gamma_{T-1}\tau) + \beta\gamma_T^2 u''(x_T^* + \gamma_T\tau)$$

hence, $\varphi(\tau)$ is strictly concave for all $\gamma \in \Phi$. Evaluating the first derivative of $\varphi(\tau)$ at $\tau = 0$, we have

$$\begin{aligned} \varphi'(0) &= u'(x_{T-2}^*) + \gamma_{T-1}\beta\delta u'(x_{T-1}^*) + \gamma_T\beta\delta^2 u'(x_T^*) \\ &> u'(x_{T-2}^*) - \max[\beta\delta u'(x_{T-1}^*), \beta\delta^2 u'(x_T^*)] \\ &\geq 0 \end{aligned}$$

Where the last inequality follows from the initial hypothesis. This shows that the optimum is strictly positive on $\Phi$: i.e. $\tau(\gamma) = \arg\max_{\tau\in\mathbb{R}_+}\varphi(\tau) > 0$, for all $\gamma \in \Phi$. Since $\Phi$ is a compact set and $\tau(\gamma)$ is a continuous function on $\Phi$ by the Maximum theorem, $\tau(\gamma)$ attains its minimum on $\Phi$ by Weierstrass theorem. Let $\upsilon = \min_{\gamma\in\Phi}\tau(\gamma)$, and take any $\bar{\tau} \in (0, \upsilon)$. To see that this is an optimal deviation, it suffices to show that $\gamma_{T-1} \in (-1, 0)$, where $\gamma_{T-1} = \frac{\Delta s_{T-1}^*}{\bar{\tau}}$ is the relative change in consumption at $T-1$ given the increase in consumption by $\bar{\tau}$ at $T-2$.

In period T-1, there is no dynamic inconsistency, so the optimal strategy is obtained by solving

$$s_{T-1}^* = \arg\max u(x_{T-1}) + \beta\delta u(x_T)$$

subject to the constraint $x_{T-1} + x_T = 1 - \sum_{t=1}^{T-2} x_t - \tau$. From the first order conditions we have that $\frac{ds_{T-1}^*}{d\tau} \in (-1, 0)$. By the Mean Value Theorem, there exists $\eta \in (0, \bar{\tau})$, such that $\frac{\Delta s_{T-1}^*}{\bar{\tau}} = s'_{T-1}|_{\tau=\eta}$. The result follows. □

**Proof of Theorem 2**. Let $x^*$ be a Pareto efficient allocation and set $\tau = x^*$. Let $s^*$ be a perfect equilibrium of the intrapersonal game. In order to prove the result, it suffices to show that $x_t \geq \tau_t$ for all feasible histories $h_t$. We prove this by induction. For period $T$, all resources left are consumed, hence $s_T \geq \tau_t$ for all $s_T \in S_T$. Next, take any history $h_t$ and assume that $x_i \geq \tau_i$ for all $i > t$. Assume, towards a contradiction, that $x_t < y_t$ for some history $h_t$. This implies that $u'(x_t) > u'(\tau_t) \geq \beta\delta^j u'(\tau_{t+j}) \geq \beta\delta^j u'(x_{t+j})$ for all $j > t$ since an efficient allocation satisfies $u'(x_t^*) \geq \beta\delta^j u'(x_{t+j}^*)$ for all $t, j \geq 1$. Clearly, self $t$ can improve his welfare by increasing his own consumption. Hence, by the one deviation property, this cannot be a perfect equilibrium. □

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