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Bias−Corrected Bootstrap Inference for Regression Models with Autocorrelated Errors

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Abstract

A bootstrap bias−correction method is applied to statistical inference in the regression model with autocorrelated errors. It is found that this method substantially reduces small−sample size distortions relative to alternative methods proposed in the literature.

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1. Introduction

This paper is concerned with statistical inference for the coefficients of regression model when the error term is autocorrelated. Past studies have reported size distortion where the conventional t-test based on a standard feasible GLS (FGLS) technique over-rejects the true null hypothesis in small samples. This size distortion is particularly severe, when the data is trended and autocorrelation (AR) of error term is high (see Park and Mitchell, 1980; King and Giles, 1984; Veall, 1986; Kwok and Veall, 1988; Rayner 1991; and Rilstone, 1993). A brief review of this literature also appears in Li and Maddala (1996, Section 3.3). Past studies noted that the problem is caused mainly by downward bias in the estimation of the AR coefficient. In response to this, Kwok and Veall (1988) used bias-corrected estimators for the AR(1) coefficient based on the jackknife and bootstrap methods. Rayner (1991) conducted a bootstrap test procedure where the jackknife is used to estimate the AR(1) coefficient, while Rilstone (1993) considered iterated bootstrap confidence intervals. Although these authors reported some improvements, serious size distortions still remain especially when the error term is highly autocorrelated.

This paper proposes an improved bootstrap procedure when statistical inference is conducted for the regression model with AR(1) errors. It is distinct from the past studies on the following points. First, bias-correction is conducted in two stages of the bootstrap. That is, following Kilian (1998), pseudo-data sets of the bootstrap are generated using a bias-corrected estimator for the AR(1) coefficient, and then bias-correction is again given to the AR(1) coefficient estimate obtained from the pseudo-data sets. For this purpose, the bias-corrected estimators based on the bootstrap and jackknife methods are used. Secondly, the FGSL estimates for regression coefficients are re-calculated using the bias-corrected estimate for the AR(1) coefficient, again in two stages of the bootstrap. As a result, bias-correction is implemented to estimation of the regression coefficients as well as to the AR(1) coefficient. The third point is related to the way in which bootstrap inference is carried out. In view of the result obtained by van Giersbergen and Kiviet (2002), attention is paid to what they referred to as the test statistic approach as opposed to the confidence interval approach. With the former, resampling is conducted using the restricted parameter estimators and residuals under the null hypothesis. Other hybrid approaches, such as the one adopted by Rayner (1991), can show undesirable small sample properties, according to van Giersbergen and Kiviet (2002).

Monte Carlo simulations are conducted to compare size and coverage probabilities of alternative bias-corrected bootstrap inference based on the test statistic and confidence interval approaches. It is found that the bias-corrected bootstrap test based on the test statistic approach substantially improves size properties, regardless of whether bias-correction is conducted using the jackknife or bootstrap. It provides empirical size values fairly close to the nominal one for nearly all cases considered. This improvement is particularly strong when the data is trended and the error term is highly autocorrelated. The next section presents the methods of parameter estimation and bias-correction. Section 3 provides the details of the bias-corrected bootstrap inference. Section 4 presents Monte Carlo results, and Section 5 concludes the paper.

2. Parameter Estimation and Bias-Correction

The simple regression model with an AR(1) error term is considered, which can be written as

$$y_t = \beta_0 + \beta_1 x_t + u_t, \qquad t = 1, 2, 3,\dots n, \tag{1}$$

$$u_t = \rho u_{t-1} + e_t, \qquad |\rho| < 1, \tag{2}$$

where $e_t \sim$ iid $N(0,1)$ and x_t is non-random. For FGLS estimation of equation (1), the Cochrane-Orcutt iterative procedure is used with the first observations adjusted based on the Prais-Winsten transformation. The FGLS estimators for β_0, β_1, and ρ are denoted as $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\rho}$, while $\hat{e}_t$'s indicate the associated centred residuals from (2).

To correct for the downward bias associated with $\hat{\rho}$, the bias-corrected estimators based on the bootstrap and jackknife methods are used. The former can be calculated in three stages:

(i) Let $u_1^* = e^*/(1-\hat{\rho})^{0.5}$ where e^* is a random draw from $\{\hat{e}_t\}$. Generate $u_t^* = \hat{\rho}u_{t-1}^* + e_t^*$, where e_t^* is a random draw from $\{\hat{e}_t\}$ with replacement.

(ii) Generate pseudo-data as $y_t^* = \hat{\beta}_0 + \hat{\beta}_1 x_t + u_t^*$. Using $\{y_t^*, x_t\}_{t=1}^n$, calculate the bootstrap FGLS estimator $\hat{\rho}^*$ for ρ.

(iii) Repeat Stages (i) and (ii) B_1 times to obtain the bootstrap distribution $\{\hat{\rho}^*(i)\}_{i=1}^{B_1}$.

If $|\hat{\rho}| < 1$, the bootstrap bias-corrected estimator for ρ is calculated as

$$\hat{\rho}_B^c = \hat{\rho} - Bias(\hat{\rho}), \tag{3}$$

where $Bias(\hat{\rho}) = \overline{\rho}^* - \hat{\rho}$ and $\overline{\rho}^*$ is the sample mean of $\{\hat{\rho}^*(i)\}_{i=1}^{B_1}$. When $|\hat{\rho}_B^c| > 1$, $\hat{\rho}_B^c$ is set to 0.99 or –0.99 following the stationarity correction proposed by Kilian (1998). It has been found to be highly effective for the bias-corrected bootstrap of AR time series (see, for example, Berkowitz and Kilian, 2000). The impact of this adjustment is asymptotically negligible, because it effectively shrinks the bias estimate $Bias(\hat{\rho})$. No bias-correction is given to $\hat{\rho}$ when $|\hat{\rho}| \geq 1$, also following Kilian (1998).

As an alternative to bootstrap bias-correction, the half-sample jackknife of Quenoulle (1949) is used. It takes the following form:

$$\hat{\rho}_J^c = 2\hat{\rho} - (\hat{\rho}_1 + \hat{\rho}_2)/2, \tag{4}$$

where $\hat{\rho}_1$ and $\hat{\rho}_2$ are the estimators for ρ on each half of the sample. If $|\hat{\rho}_J^c| > 1$, Fisher's transformation $F(\rho) = 0.5 \log[(1+\rho)/(1-\rho)]$ is used so that $\hat{\rho}_J^c$ is appropriately bounded, following Kwok and Veall (1988). That is, $F(\hat{\rho}_J^c)$ can be jackknifed to obtain $\hat{F}(\hat{\rho}) = 2F(\hat{\rho}) - 0.5[F(\hat{\rho}_1) + F(\hat{\rho}_1)]$, and the bounded estimate $\hat{\rho}_J^c$ can be obtained by taking the inverse of $\hat{F}(\hat{\rho})$.

Let $\hat{\rho}^c \in \{\hat{\rho}_B^c, \hat{\rho}_J^c\}$. The regression coefficients are re-calculated using $\hat{\rho}^c$, in order to obtain the bias-corrected FGLS estimators $\hat{\beta}_0^c$ and $\hat{\beta}_1^c$ for β_0 and β_1. Let $se(\hat{\beta}_1^c)$ denote the standard error estimator for $\hat{\beta}_1^c$, and $\{\hat{e}_t^c\}$ the centred residuals calculated from (2) using $\hat{\rho}^c$. Based on these, we can obtain $T^c = (\hat{\beta}_1^c - \beta)/se(\hat{\beta}_1^c)$, which is a bias-corrected version of the conventional t-statistic $T = (\hat{\beta}_1 - \beta)/se(\hat{\beta}_1)$.

3. Bias-Corrected Bootstrap

The bias-corrected bootstrap is used to test for H_0: $\beta_1 = \beta$ against H_1: $\beta_1 \neq \beta$. First, the resampling scheme based on the confidence region approach (see van Giersbergen and Kiviet; 2002) is presented in three stages:

(i) Let $u_1^* = e^*/(1-\hat{\rho}^c)^{0.5}$ where e^* is a random draw from $\{\hat{e}_t^c\}$. Generate $u_t^* = \hat{\rho}^c u_{t-1}^* + e_t^*$, where e_t^* is a random draw from $\{\hat{e}_t^c\}$ with replacement.

(ii) Generate pseudo-data as $y_t^* = \hat{\beta}_0^c + \hat{\beta}_1^c x_t + u_t^*$. Using $\{y_t^*, x_t\}_{t=1}^n$, calculate the bootstrap FGLS estimators $\hat{\beta}_0^*, \hat{\beta}_1^*$, and $\hat{\rho}^*$ for β_0, β_1, and ρ. The bias-corrected version of $\hat{\rho}^*$ is obtained using either (3) or (4), and denoted as $\hat{\rho}^{*c}$. For bootstrap bias-correction, $Bias(\hat{\rho})$ is used as an approximation to $Bias(\hat{\rho}^*)$ following Kilian (1998). Using $\hat{\rho}^{*c}$, the bootstrap FGLS estimators for β_0 and β_1 are re-calculated and denoted as $\hat{\beta}_0^{*c}$ and $\hat{\beta}_1^{*c}$. Let se($\hat{\beta}_1^{*c}$) denote the bootstrap counterpart of $se(\hat{\beta}_1^c)$.

(iii) Repeat Stages (i) and (ii) B_2 times to obtain the bootstrap distribution $\{z_i = (\hat{\beta}_1^{*c}(i) - \hat{\beta}_1^c)/se(\hat{\beta}_1^{*c}, i)\}_{i=1}^{B_2}$.

From the above bootstrap distribution, 100(1-2α)% bias-corrected bootstrap confidence interval based on the percentile-t method (see Efron and Tibshirani, 1993) can be constructed as CI = [$\hat{\beta}_1^c - z_{1-\alpha} se(\hat{\beta}_1^c), \hat{\beta}_1^c - z_\alpha se(\hat{\beta}_1^c)$], where z_τ is the τ^{th} percentiles of $\{z_i\}_{i=1}^{B_2}$. The decision rule is to reject H_0 at the 2α level of significance if $\beta \notin CI$.

The resampling scheme based on the test statistic approach (van Giersbergen and Kiviet; 2002) is similar, but somewhat different from the confidence region approach. In addition to the unrestricted estimation and bias-correction given in Section 2, the restricted estimation under H_0 and the associated bias-correction should be conducted. Let $\hat{\beta}_0^r, \hat{\beta}_1^r = \beta$ and $\hat{\rho}^r$ denote the restricted FGLS estimators for β_0, β_1, and ρ under H_0. Their bias-corrected versions are denoted as $\hat{\beta}_0^{rc}, \hat{\beta}_1^{rc} = \beta$, and $\hat{\rho}^{rc}$, while $\{\hat{e}_t^{rc}\}$ is the restricted version of $\{\hat{e}_t^c\}$. Note that these bootstrap or jackknife bias-corrections are conducted following (3) or (4), but based on the restricted regression. In Stage (i), u^*'s are generated using $\hat{\rho}^{rc}$ and $\{\hat{e}_t^{rc}\}$ instead of $\hat{\rho}^c$ and $\{\hat{e}_t^c\}$. In Stage (ii), the pseudo data set is generated as $y_t^* = \hat{\beta}_0^{rc} + \beta x_t + u_t^*$. Again, in the case of bootstrap bias-correction, $Bias(\hat{\rho})$ is used as an approximation to $Bias(\hat{\rho}^*)$. In Stage (iii), the bootstrap distribution of interest is $\left\{m_i = (\hat{\beta}_1^{*c}(i) - \beta) / se(\hat{\beta}_1^{*c}, i)\right\}_{i=1}^{B_2}$. The decision rule is to reject H_0 at the 2α level of significance, if $T^c \notin [m_\alpha, m_{1-\alpha}]$ where m_τ is the τ^{th} percentiles of $\{m_i\}_{i=1}^{B_2}$.

4. Monte Carlo Results

Monte Carlo simulations are conducted to compare size properties associated with CI_B, CI_J, T^{cB}, T^{cJ} and T. CI_B and CI_J are CI's calculated using $\hat{\rho}_B^c$ and $\hat{\rho}_J^c$ respectively; while T^{cB} and T^{cJ} are the tests based on T^c using $\hat{\rho}_B^c$ and $\hat{\rho}_J^c$. The experimental design loosely follows that of Rayner (1991). The sample sizes n simulated are 20, 60 and 100, with $\rho \in \{0, 0.3, 0.6, 0.9, 0.95\}$ and $\beta_0 = \beta_1 = \beta = 1$. The number of Monte Carlo trials is set to 1000; and the numbers of bootstrap replications B_1 and B_2 are set to 500 and 2000 respectively. Note that x_t is fixed over Monte Carlo trials. The levels of significance 2α considered are 0.05 and 0.10. However, only the results associated with the former are reported, because those associated with the latter are found to be qualitatively similar.

The simulation results are presented in Table 1. We first consider two x_t designs simulated by Rayner (1991), which are time trend t and GNP data of Maddala and Rao (1973). As seen in previous studies, the conventional test T fails dramatically, showing grossly inflated size values as the value of ρ increases. This is evident for both x_t designs. CI_B and CI_J show much better size properties than T, but they suffer from serious size distortions when the value of ρ is high and the sample size is small. T^{cB} shows desirable size properties except when n = 20 and ρ is high. On the other hand, T^{cJ} performs well when n = 20 for nearly all ρ values. But its size values tend to be lower than 5% when the sample size is larger.

Two additional x_t designs, labelled DGP1 and DGP2, are simulated, whose details are given at the bottom of Table 1. The DGP1 is a stationary AR(1) time series with no linear trend, and the DGP2 is an AR(1) time series with linear trend. For DGP1, as might be expected, all CI's and T's show desirable size properties for all n and ρ, except for the conventional test T when n = 20. For DGP2, CI_B and CI_J show inflated size values when the value of ρ is high and the sample size is small. In contrast, both T^{cB} and T^{cJ} show highly desirable size properties. Note that T^{cB} tends to over-estimate 5% slightly, especially when the sample size is small and the value of ρ is high; while T^{cJ} tends to under-estimate 5% slightly. As expected, the conventional test T shows grossly inflated size values for the DGP2.

As a final note, the Hildreth-Lu grid search method was considered as an alternative method of FGLS estimation to the Cochrane-Orcutt. However, the size properties of alternative bias-corrected bootstrap tests are found to be insensitive to the choice of estimation method, and the details are not reported here.

5. Concluding Remarks

This paper finds that the bias-corrected bootstrap substantially improves size distortions of the statistical test in the regression model with autocorrelated errors. The bias-corrected bootstrap based on the test statistic approach is found to provide superior size properties to that based on the confidence region approach, especially when the sample size is small. With the test statistic approach, the bias-corrected bootstrap provides empirical size values fairly close to the nominal one even when the value of the AR coefficient is high. The extent of improvement is much higher than those reported by past studies such as Rayner (1991) and Rilstone (1993). Both the bootstrap and jackknife are found to be effective for bias-correction, but the results suggest that the bootstrap be preferred as a means of bias-correction when the sample size is more than moderately large.

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Table 1. Percentage of rejection of the null hypothesis (The level of significance $2\alpha = 0.05$)

x_t	ρ	n=20					n=60					n=100				
		CI_J	CI_B	T^{cB}	T^{cJ}	T	CI_J	CI_B	T^{cB}	T^{cJ}	T	CI_J	CI_B	T^{cB}	T^{cJ}	T
t	0	7.2	4.9	5.7	6.3	11.3	3.6	5.2	4.2	3.3	5.2	5.8	3.3	5.3	5.7	6.5
	0.3	7.5	6.1	6.0	4.7	14.7	3.9	5.2	5.0	3.4	6.3	5.5	3.6	5.4	5.4	7.2
	0.6	8.4	7.9	7.0	2.0	20.8	4.3	4.8	5.1	3.1	9.3	5.2	3.9	5.0	4.7	8.6
	0.9	11.7	14.1	7.1	4.2	38.2	6.9	6.7	5.6	2.1	25.9	6.5	5.9	6.1	3.0	17.2
	0.95	12.4	15.1	9.3	4.4	44.8	9.6	9.2	6.9	3.1	35.7	7.7	8.0	6.3	3.0	25.5
GNP	0	6.0	5.0	5.6	5.1	11.3	3.7	5.1	4.5	3.6	5.4	—	—	—	—	—
	0.3	6.2	5.6	5.5	4.4	15.1	3.9	4.7	5.3	3.5	7.0	—	—	—	—	—
	0.6	7.6	8.5	6.5	3.6	21.4	4.7	4.7	5.8	3.8	10.3	—	—	—	—	—
	0.9	9.1	12.5	8.6	4.4	33.9	5.8	6.1	6.2	2.8	23.2	—	—	—	—	—
	0.95	12.0	13.3	10.0	4.8	39.4	9.1	7.3	5.7	3.7	29.1	—	—	—	—	—
DGP1	0	5.8	5.7	5.6	6.4	10.5	4.7	5.4	5.2	4.4	6.9	6.0	4.9	4.8	5.5	6.3
	0.3	6.8	6.5	6.4	6.0	13.6	5.1	4.8	5.1	4.2	6.2	5.4	5.1	4.6	4.8	7.1
	0.6	5.6	6.9	5.4	4.8	17.5	4.5	5.7	5.1	4.3	6.6	5.3	4.9	4.7	5.5	6.6
	0.9	4.2	5.5	5.2	3.7	13.6	4.5	4.5	5.2	5.3	5.4	5.2	5.2	4.5	5.9	6.2
	0.95	4.7	4.6	5.2	3.2	13.2	4.6	4.4	5.2	4.9	5.2	5.4	5.0	4.6	5.9	6.0
DGP2	0	6.6	4.2	5.1	5.1	11.6	3.7	4.9	4.3	4.2	5.3	5.3	3.7	5.0	6.4	6.0
	0.3	7.2	5.4	6.1	3.6	15.4	3.9	4.8	4.4	3.6	6.4	5.3	3.6	5.0	5.6	7.2
	0.6	7.9	7.1	7.5	3.3	21.5	4.2	4.9	5.2	3.7	10.6	4.9	3.7	4.8	4.9	8.0
	0.9	12.7	12.4	7.9	5.6	31.3	6.5	6.1	6.7	4.5	26.1	5.3	5.4	5.3	3.1	16.7
	0.95	14.5	13.6	7.9	5.4	31.9	9.6	6.9	5.9	4.9	33.1	6.9	6.2	5.9	3.9	21.9

CI_J : bootstrap confidence interval, bias-correction based on the jackknife; CI_B : bootstrap confidence interval, bias-correction based on the bootstrap; T^{cB}: test based on the bias-corrected bootstrap, bootstrap bias-correction; T^{cJ}: test based on the bias-corrected bootstrap, bias-correction based on the jackknife; T: test based on FGLS estimation; DGP1: $x_t = 1 + 0.5\, x_{t-1} + v_t$, v_t ~iidN(0,1), DGP2: $x_t = 1 + 0.02t + 0.95\, x_{t-1} + v_t$, v_t ~iidN(0,1).

No results are reported for GNP when n = 100, because the sample size of the GNP series given in Maddala and Rao (1973) is less than 100.