

E C O N O M I C S   B U L L E T I N

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## Mixed Oligopoly and Productivity-Improving Mergers

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### *Abstract*

This paper investigates productivity improving merger activities between a public firm and a private firm in mixed oligopoly. We assume that the merged firm has two plants (formerly, firms). We show that both owners of a public firm and a private firm want to merge by coordinating their shareholding ratios in the merged firm, whenever the number of private firms is larger than a critical value, while the public firm does not want to merge without the effect of improving the productivity of the merged firm.

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We are grateful to the associate editor (Patrick Legros) and a anonymous referee for their helpful comments and suggestions. We appreciate the financial support from the Japanese Ministry of Education, Culture, Sports, Science and Technology under Waseda University 21st-COE GLOPE project. Needless to say we are responsible for any remaining errors.

**Citation:** Nakamura, Yasuhiko and Tomohiro Inoue, (2007) "Mixed Oligopoly and Productivity-Improving Mergers." *Economics Bulletin*, Vol. 12, No. 20 pp. 1-9

**Submitted:** May 8, 2007. **Accepted:** September 11, 2007.

**URL:** <http://economicsbulletin.vanderbilt.edu/2007/volume12/EB-07L20004A.pdf>

## 1 Introduction

The literature on horizontal mergers is roughly divided into two categories. The first deals with the profit effects of mergers. Salant et al. (1983) and Deneckere and Davidson (1985) examine whether mergers are beneficial with regard to the profits of the participants in a quantity and price setting game, respectively. The second category deals with the welfare effects of mergers. In particular, Farrell and Shapiro (1990) indicate that mergers may have welfare-improving effects by redistributing production from less efficient to more efficient firms.

Except for Bárcena-Ruiz and Garzón (2003), there exist few studies on the decision to merge by public and private firms in a mixed oligopoly. They explore the case in which a public and a private firm merge into a multiproduct firm and show that both firms want to merge when the shareholding ratio of the owner of the public firm takes an intermediate value and the substitutability of the goods produced by both the public and private firms is sufficiently low.

In contrast, they ignore the case where mergers improve production efficiency. Several reasons exist why mergers may lead to an improvement of productivity. One is the learning effect, in which a partner to the merger learns from the other partner's patents, management expertise, *etc.* Despite assuming that there are economies of scale,<sup>1</sup> Bárcena-Ruiz and Garzón (2003) disregard this improvement of productivity. However, if firms combine some form of "capital" between their facilities after a merger, it certainly results in improving productivity of the merged firm when economies of scale exist. There are no existing studies on *productivity-improving mergers* in the context of a mixed oligopoly. This study aims to fill this gap and have an impact on the subject. For this purpose, we investigate the productivity-improving merger as considered in McAfee and Williams (1992) under the assumption that firms have identical technologies represented by the quadratic cost function.

In our model, there exist one public firm and  $n$  identical private firms in a homogeneous goods market; this is in contrast to Bárcena-Ruiz and Garzón (2003), who explore a mixed duopoly in a differentiated goods market. We show that if a merger improves productivity, both a public and a private firm want to merge when the shareholding ratio of the owner of the public firm takes an intermediate value after the merger, even though there exist only a few private firms in the market. In addition, we find that if the number of private firms is sufficiently large, the owner of the public firm is always willing to merge whenever its shareholding ratio in the merged firm is lower than a critical value.

One example in the real world as our subject of research is presented by European automobile industry. In particular, the German public firm Volkswagen acquired the Spanish firm SEAT in 1986. Similarly, Renault, which was privatized in 1986, owned parts of equities in both Nissan Motor and Nissan Diesel in 1999.

This paper has four sections and an appendix. Section 2 sets up the model. We refer to McAfee and Williams (1992) for the cost function of the merged firm.<sup>2</sup> In Section 3, we explore the problem of a merger between a public firm and a single private firm. Our purpose here is to analyze whether the public and the private firm want to merge, when the merger has an effect

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<sup>1</sup>They assume that firms have identical technologies represented by the quadratic cost function.

<sup>2</sup>They assume that the cost of firm  $i$  ( $i = 1, \dots, n$ ) is equal to  $(q_i)^2/2k_i$ , where  $k_i$  is the firm's capital stock. In addition, we assume that the capital stock of each firm is normalized to 1, *i.e.*,  $k_0 = k_1 = \dots = k_n = 1$ .

of productivity improvement. Section 4 provides the conclusion. In Appendix, we investigate in detail the case of a merger without any improvement in productivity.

## 2 The model

We consider a mixed market in which  $(n + 1)$  firms produce a homogeneous good. One of the firms is a welfare-maximizing public firm (denoted by firm 0), and the others are symmetric profit-maximizing private firms (denoted by firm 1, firm 2,  $\dots$ , and firm  $n$ ). We assume the following linear inverse demand function:

$$P(Q) = a - Q \quad a > 0,$$

where  $Q$  is the total output of the good. Each firm produces the good using identical technology, and the cost function of firm  $i$  is given by

$$C_i(q_i) = (q_i)^2 \quad i = 0, 1, \dots, n,$$

where  $q_i$  ( $i = 0, 1, \dots, n$ ) is the output of each firm. The profit of firm  $i$  is expressed as

$$\pi_i = P(Q) - C_i(q_i) = (a - Q)q_i - (q_i)^2 \quad i = 0, 1, \dots, n. \quad (1)$$

Each private firm chooses its output level in order to maximize (1). On the other hand, the public firm chooses its output to maximize social welfare. Social welfare is represented by the sum of consumer surplus (denoted by  $CS$ ) and profits of all firms as follows:

$$W = CS + \sum_{i=0}^n \pi_i, \quad (2)$$

$$\text{where } CS = \int_0^Q P(z)dz - P(Q)Q = \frac{1}{2}Q^2.$$

We assume that the public firm and one of the private firms decide whether to merge and set up a multiplant firm whose ownership is shared by the owners of the public and private firms. For simplicity, we describe the owner of the public firm after the merger as *the public sector* and the owner of the private firm as *the private sector*. Since the private firms are symmetric, we assume that firm 1 can merge with the public firm without loss of generality. We consider that the merged firm (denoted by firm  $m$ ) has two plants, one of which is owned by the public firm and the other by the private firm before the merger. Thus, the merged firm can produce the good at lower cost than the other firms. The cost function of the merged firm is given by<sup>3</sup>

$$C_m(q_m) = \frac{1}{2}(q_m)^2,$$

where  $q_m$  is the output of the merged firm. The profit of the firm is expressed as

$$\pi_m = (a - Q)q_m - \frac{1}{2}(q_m)^2.$$

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<sup>3</sup>The merged firm may be regarded as a multiplant firm, operating the two former firms as “plants.” In this paper, we assume that a multiplant merged firm operates under a situation in which both plants perform most efficiently (see McAfee and Williams, 1992). We assume that the productivity of the public and private firms is symmetric, *i.e.*, the cost function of each firm is represented by the quadratic form of its own output. Therefore, a merged firm has technology that is twice as efficient as that of the two pre-merger firms.

Note that the total number of firms is reduced from  $(n + 1)$  to  $n$  by the merger.

The public and private sectors share the ownership of the merged firm. Let  $\alpha \in [0, 1]$  denote the shareholding ratio of the public sector and let the merged firm choose its output  $q_m$  to maximize the weighted average of social welfare and its own profit as in Matsumura (1998). This objective function is given by

$$V = \alpha W + (1 - \alpha)\pi_m. \quad (3)$$

Since the total number of the firms is reduced by the merger, social welfare is as follows:

$$W = CS + \sum_{k=2}^n \pi_k + \pi_m.$$

The profit of the merged firm is distributed according to the shareholding ratio. Thus, we assume that the private sector receives profit at the rate of  $(1 - \alpha)$ .

**Assumption 1.** *The payoff of the private sector that partially owns the merged firm is  $(1 - \alpha)\pi_m$ .*

When social welfare improves and the profit received by the private sector increases as the result of the merger, the public and the private firm merge.

We consider a two-stage game: In the first stage, both the public and the private firm decide whether to merge. In the second stage, all firms choose their own output levels.

### 3 The decision by firms to merge

We consider the following two cases: First, the firms do not merge, resulting in the competition between one public and  $n$  private firms. We denote this case as  $N$  (No merger). Second, the firms merge; this case is denoted as  $M$  (Merger).

We first examine the second stage of the game in case  $N$ . As stated in the previous section, the public firm chooses  $q_0$  to maximize (2), while the private firm  $j$  chooses  $q_j$  to maximize (1) ( $j = 1, \dots, n$ ). Solving these maximization problems simultaneously, we obtain the Nash equilibrium in the second stage:

$$\begin{aligned} q_0^N &= \frac{3a}{9 + 2n}, & q_j^N &= \frac{2a}{9 + 2n}, & \pi_0^N &= \frac{9a^2}{(9 + 2n)^2}, & \pi_j^N &= \frac{8a^2}{(9 + 2n)^2}, \\ CS^N &= \frac{a^2(3 + 2n)^2}{2(9 + 2n)^2}, & W^N &= \frac{a^2(27 + 28n + 4n^2)}{2(9 + 2n)^2}, & & & & j = 1, \dots, n. \end{aligned}$$

The output of the public firm is larger than that of each private firm regardless of the number of private firms,  $n$ . Consumer surplus and social welfare are increasing functions of  $n$ , while the profit of each firm is a decreasing function of  $n$ .

When the public firm (firm 0) and the private firm (firm 1) merge, they set up a multiplant firm that chooses  $q_m$  to maximize (3). The other firms choose their output level to maximize (1). As a result, we obtain the Nash equilibrium in the second stage:

$$q_k^M = \frac{a(2 - \alpha)}{7 + 2n - \alpha(2 + n)}, \quad q_m^M = \frac{3a}{7 + 2n - \alpha(2 + n)},$$

$$\begin{aligned}\pi_k^M &= \frac{2a^2(2-\alpha)^2}{[7+2n-\alpha(2+n)]^2}, & \pi_m^M &= \frac{9a^2(3-2\alpha)}{[7+2n-\alpha(2+n)]^2}, \\ CS^M &= \frac{a^2[1+2n-\alpha(n-1)]^2}{2[7+2n-\alpha(2+n)]^2}, \\ W^M &= \frac{a^2(2-\alpha)(6+10n+2n^2+3\alpha-2n\alpha-n^2\alpha)}{2[7+2n-\alpha(2+n)]^2}, & k &= 2, \dots, n.\end{aligned}$$

The output of the merged firm is larger than that of each private firm irrespective of  $n$  and  $\alpha$ . In addition, consumer surplus and social welfare are increasing functions of  $n$ , while social welfare decreases as  $\alpha$  increases when the value of  $\alpha$  is sufficiently high.<sup>4</sup> The rise of  $\alpha$  widens the output gap between each private firm and the merged firm. Although the productivity-improving merger enhances social welfare within the bounds of low  $\alpha$ , the widening gap reduces social welfare because of the convexity of the cost function when  $\alpha$  is sufficiently high. In addition, when the market is a monopoly after the merger ( $n = 1$ ), social welfare is maximized at  $\alpha = 1$ .

Next, we analyze both the public and private firm's incentives to merge in the first stage of the game. In order to explain these incentives, we employ two effects: *share effect* and *competition effect*. The share effect, which is represented by  $\alpha$ , affects the firms in case  $M$  through the weight of social welfare in objective function of the merged firm (see Bárcena-Ruiz and Garzón, 2003). The competition effect, which is represented by  $n$ , affects them in both cases  $N$  and  $M$ . When parameter  $n$  increases, for a given of parameter  $\alpha$ , consumer surplus and social welfare increase, while profits of all the firms decrease.

First, we examine whether the public firm wishes to merge with the private firm. Since the public firm aims at maximizing social welfare, it has an incentive to merge if  $W^M > W^N$ . Let  $\alpha_0^*$  and  $\alpha_0^{**}$  denote the values of  $\alpha$  such that  $W^M = W^N$ :

$$\begin{aligned}\alpha_0^* &= \frac{378 + 122n + 4n^2 + 3(9 + 2n)\sqrt{27 - 2n + 2n^2}}{351 + 166n + 14n^2}, \\ \alpha_0^{**} &= \frac{378 + 122n + 4n^2 - 3(9 + 2n)\sqrt{27 - 2n + 2n^2}}{351 + 166n + 14n^2}.\end{aligned}$$

We obtain the following proposition using  $\alpha_0^*$  and  $\alpha_0^{**}$ .

**Proposition 1.**  $W^M > W^N$  if and only if  $\alpha_0^{**} < \alpha < \alpha_0^*$ .

**Proof.** Subtracting  $W^N$  from  $W^M$ , we obtain the following equation:

$$W^M - W^N = \frac{-a^2[(351 + 166n + 14n^2)\alpha^2 - (756 + 244n + 8n^2)\alpha + 351 + 76n - 4n^2]}{2(9 + 2n)^2[7 + 2n - \alpha(2 + n)]^2}.$$

The sign of RHS depends on that of its numerator. Since this numerator is a quadratic concave function of  $\alpha$  and is equal to zero when  $\alpha = \alpha_0^*$  or  $\alpha = \alpha_0^{**}$ ,  $W^M > W^N$  if and only if  $\alpha_0^{**} < \alpha < \alpha_0^*$ .  $\square$

This proposition shows that if the number of private firms is greater than or equal to 6 ( $n \geq 6$ ), the public firm does not want to merge at  $\alpha = 1$ , since  $\alpha_0^*|_{n=6} < 1$  and  $\alpha_0^*$  is a

<sup>4</sup>Since  $\frac{\partial W^M}{\partial \alpha} = \frac{3a^2[8+n-\alpha(7+2n)]}{[7+2n-\alpha(2+n)]^3}$ ,  $W^M$  decreases as  $\alpha$  rises when  $\alpha > \frac{8+n}{7+2n}$ . In addition,  $\frac{8+n}{7+2n}$  is a decreasing function of  $n$ .

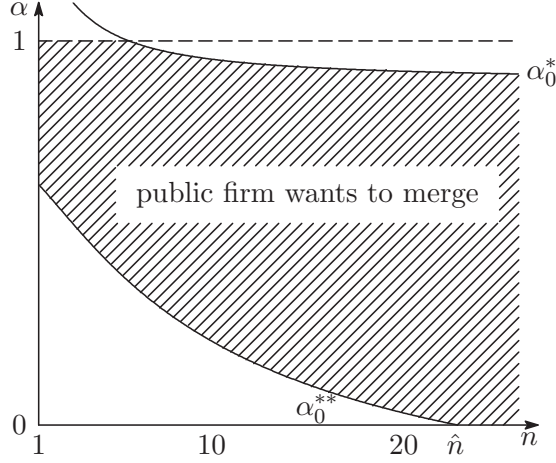


Figure 1: Illustration of Proposition 1

decreasing function of  $n$ . In addition, when the number is greater than or equal to 23 ( $n \geq 23$ ), the public firm wants to merge at  $\alpha = 0$ , because  $\alpha_0^{**}|_{n=23} < 0$  and  $\alpha_0^{**}$  is a decreasing function of  $n$ . In other words, even if the public sector does not have a share of the merged firm, the public firm has an incentive to merge in  $n \geq 23$ . Figure 1 illustrates this incentive in relation with parameters  $n$  and  $\alpha$ . The shaded area represents the range in which the public firm wants to merge. This range broadens as  $n$  increases until  $n = \hat{n}$ , but when  $n > \hat{n}$ , it narrows conversely.<sup>5</sup>

The increase in the number of private firms reduces the public firm's contribution to consumer surplus, but the output gap between the public and private firms remains. Since the gap decreases social welfare, the increase enhances the public firm's incentive to merge. Thus, the shaded area widens as  $n$  increases. This logic coincides with that of De Fraja and Delbono (1989), who show that the privatization of a public firm can improve social welfare.

However, Proposition 1 depends heavily on our assumption that the merger improves the productivity of the firm. If we do not assume this effect, the public firm will not wish to merge with the private firm regardless of the number of private firms (see Appendix).

Next, we consider whether the private firm (firm 1) decides to merge with the public firm. By Assumption 1, the private firm decides to merge if  $(1 - \alpha)\pi_m^M > \pi_1^N$ . Let  $\alpha_1^*$  and  $\alpha_1^{**}$  denote the values of  $\alpha$  such that  $(1 - \alpha)\pi_m^M = \pi_1^N$ :

$$\alpha_1^* = \frac{3197 + 1268n + 116n^2 + 3(9 + 2n)\sqrt{3289 + 1156n + 100n^2}}{2(1394 + 584n + 56n^2)},$$

$$\alpha_1^{**} = \frac{3197 + 1268n + 116n^2 - 3(9 + 2n)\sqrt{3289 + 1156n + 100n^2}}{2(1394 + 584n + 56n^2)}.$$

We obtain the following proposition using these equations.

**Proposition 2.**  $(1 - \alpha)\pi_m^M > \pi_1^N$  if and only if  $\alpha < \alpha_1^{**}$ .

**Proof.** Subtracting  $\pi_1^N$  from  $(1 - \alpha)\pi_m^M$ , we obtain the following equation:

$$(1 - \alpha)\pi_m^M - \pi_1^N = \frac{a^2[(1394 + 584n + 56n^2)\alpha^2 - (3197 + 1268n + 116n^2)\alpha + 1403 + 524n + 44n^2]}{2(9 + 2n)^2[7 + 2n - \alpha(2 + n)]^2}.$$

<sup>5</sup>The critical value is  $\hat{n} = (19 + 2\sqrt{178})/2 \approx 22.8417$ .

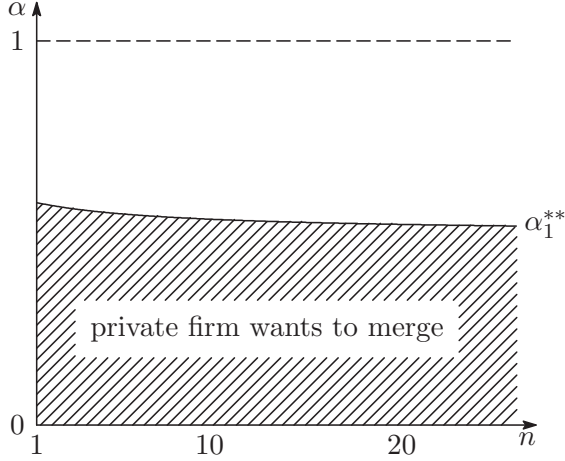


Figure 2: Illustration of Proposition 2

The sign of RHS depends on that of its numerator. Since this numerator is a quadratic convex function of  $\alpha$  and is equal to zero when  $\alpha = \alpha_1^*$  or  $\alpha = \alpha_1^{**}$ ,  $(1 - \alpha)\pi_m^M > \pi_1^N$  if  $\alpha > \alpha_1^*$  or  $\alpha < \alpha_1^{**}$ . However,  $\alpha_1^* > 1$  for all  $n$ , and thus the constraint of  $\alpha \in [0, 1]$  is violated. Therefore,  $(1 - \alpha)\pi_m^M > \pi_1^N$  if and only if  $\alpha < \alpha_1^{**}$ .  $\square$

Proposition 2 is illustrated in Figure 2. Since the increase in the number of private firms reduces the market price and the increment of profit by the merger, the private firm demands a higher profit distribution ratio to compensate the profit reduction. Therefore,  $\alpha_1^{**}$  is a decreasing function of  $n$  (in other words,  $(1 - \alpha_1^{**})$  is an increasing function of  $n$ ). Note that  $\lim_{n \rightarrow \infty} \alpha_1^{**} = 1/2$ ; thus, the private firm always decides to merge irrespective of  $n$  when the shareholding ratio of the private sector is more than  $1/2$ .

We present the following lemma in which we compare  $\alpha_1^{**}$  with  $\alpha_0^*$  and  $\alpha_0^{**}$  to determine whether the public and private firms merge.

**Lemma 1.**  $\alpha_0^* > \alpha_1^{**}$  for  $n \in [1, \infty)$  and  $\alpha_0^{**} > \alpha_1^{**}$  at  $n = 1$ , but there exists  $\tilde{n} \in (1, \infty)$  such that  $\alpha_1^{**} \geq \alpha_0^{**}$  for  $n \geq \tilde{n}$ .

**Proof.** See Appendix.

When the number of private firms is sufficiently small,  $\alpha_0^{**}$  is greater than  $\alpha_1^{**}$ . However, when the number exceeds the critical value  $\tilde{n}$ , this relation is reversed ( $\alpha_1^{**} \geq \alpha_0^{**}$ ). We obtain an approximate value of  $n$  such that  $\alpha_0^{**} = \alpha_1^{**}$  is 1.9907, *i.e.*, the firms do not merge in mixed “duopoly.” This coincides with the result of Bárcena-Ruiz and Garzón (2003).<sup>6</sup> By Propositions 1 and 2 and Lemma 1,<sup>7</sup> we obtain the following proposition:

**Proposition 3.** *The public firm 0 and the private firm 1 will merge when  $\alpha_0^{**} < \alpha < \alpha_1^{**}$ .*

<sup>6</sup>Bárcena-Ruiz and Garzón (2003) do not consider the productivity-improving merger. However, even if the merger improves the productivity of the merged firm, the firms do not merge in a mixed duopoly with a homogeneous good.

<sup>7</sup>Lemma 1 guarantees the existence of the range in which  $\alpha \in (\alpha_0^{**}, \alpha_1^{**})$ .

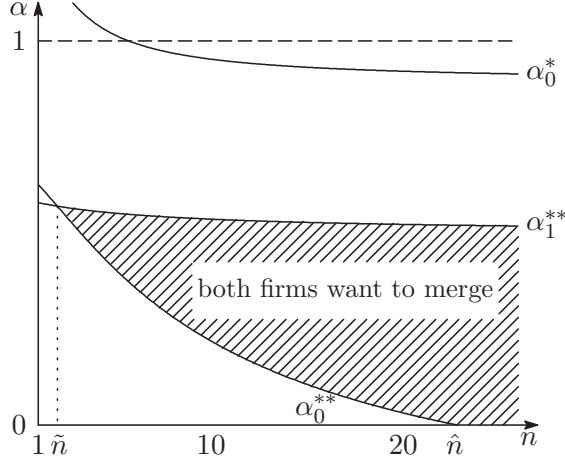


Figure 3: Illustration of Proposition 3

Figure 3 illustrates Proposition 3. If  $n \in (\tilde{n}, \hat{n})$ , the area in which both the public and private firm want to merge broadens as  $n$  increases.<sup>8</sup> In addition, when  $n$  is larger than  $\hat{n}$ , both firms want to merge even if the merged firm is owned only by the private sector (*viz.*,  $\alpha = 0$ ). This is because the welfare loss due to the excess production of the public firm is larger than the welfare improvement as a result of increasing consumer surplus as stated above.

The intuition behind Proposition 3 is as follows: The shareholding ratio is important for both the public and private sector of the merged firm because it directly affects the behavior of the firm. Thus, when parameter  $n$  is sufficiently small, the merger is not achieved around  $\alpha = 0$ , and around  $\alpha = 1$ , either. However, since an increase of  $n$  decreases  $q_0/Q$  and thus reduces the impact of the public firm on social welfare, the share effect decreases as  $n$  increases (interaction between the share effect and the competition effect). In addition to this, the existence of more efficient firm reduces total output cost. Therefore, the share effect for the public sector decreases with  $n$  and  $\alpha_0^{**}$  is severely reduced by increase of  $n$ . On the other hand, the share effect for the private sector is not affected by the competition effect so much because the payoff of the private sector,  $(1 - \alpha)\pi^M$ , is directly affected by the shareholding ratio. Thus, the share effect has a more crucial effect on the decision of the private sector than that of the public sector and  $\alpha_1^{**}$  is almost stable against an increase of  $n$ . Accordingly, the area in which both the public and private firms want to merge expands with the number of private firms.

#### 4 Conclusion

This paper investigated how a public and private firm's decision whether to merge depends on the shareholding ratio and the number of private firms. We showed that when the shareholding ratio of the public sector is  $\alpha \in (\alpha_0^{**}, \alpha_1^{**})$ , which is achieved in  $n > \tilde{n}$ , both firms decide to merge. Note that, in a mixed duopoly, the merger is not achieved.

Bárcena-Ruiz and Garzón (2003) demonstrate that the firms do not merge in a mixed duopoly with a homogeneous good. However, we proved that if mergers improve the efficiency of the firms and the number of private firms is sufficiently large, the result is not necessarily the same

<sup>8</sup>However, in  $n \geq \hat{n}$ ,  $\alpha_0^{**}$  is less than 0, and the area narrows as  $n$  increases by the constraint of  $\alpha \geq 0$ .



as theirs. In particular, the productivity-improving merger is critical to the result. If we do not assume this effect, the public firm does not choose to merge regardless of the number of private firms.

We assumed that each firm produces a homogeneous good using identical technology. We also briefly mention here the case of asymmetric technologies, in particular that of a public firm less efficient than a private firm. In this case, the public firm is even more willing to merge since it benefits from the higher productivity of the private firm. Thus, in the case that both firm's cost functions are asymmetric, a curve corresponding to  $\alpha_0^{**}$  in Figure 1 and 3 is more rapidly decreasing with  $n$  around 1. On the other hand, the private firm is reluctant to merge with the public firm, since the productivity improvement rate is lower than in the case where the firms have identical technologies. Thus, a curve corresponding to  $\alpha_1^{**}$  in Figure 2 and 3 shifts down below. Therefore, when a public firm is less efficient than a private firm, it is ambiguous whether the area in which both the public and private firms want to merge expands or not compared to the case where their cost functions are symmetric. However, the area still exists in  $(n, \alpha)$ -plane.

Our analysis contributes to the literature on mixed oligopoly by showing that a public firm may have an incentive to merge with a private firm in a homogeneous goods market. However, two interesting extensions of our model still remain. One is the situation in which the public firm merges with multiple private firms, and another is where there exist foreign shareholders of the private firms. Since it would appear that these situations would have an impact on the firms' decision to merge, the investigation of these situations is important for future studies of mergers in mixed markets.

## Appendix

### The public firm's decision without productivity improvement

We show that the public firm does not have an incentive to merge with the private firm in the case where the merger does not improve the productivity of the merged firm. In this case, social welfare before and after the merger is as follows:

$$W^N = \frac{a^2(27 + 28n + 4n^2)}{2(9 + 2n)^2},$$

$$W^M = \frac{a^2[36n + 9n^2 + (6 - 18n - 6n^2)\alpha - (3 - 2n - n^2)\alpha^2]}{2[9 + 3n - (2 + n)\alpha]^2}.$$

Subtracting  $W^N$  from  $W^M$ , we examine the public firm's incentive to merge with the private firm.

$$W^M - W^N = \frac{-a^2[(351 + 166n + 14n^2)\alpha^2 - (1458 + 576n + 36n^2)\alpha + 2187 + 810n + 54n^2]}{2(9 + 2n)^2[9 + 3n - (2 + n)\alpha]^2}.$$

The sign of RHS depends on that of its numerator. This numerator is a quadratic concave function of  $\alpha$  and the discriminant of this quadratic equation,  $D$ , is<sup>9</sup>

$$D = -432(9 + 2n)^2(27 + 14n + n^2) < 0.$$

Thus, for all  $n$ ,  $W^N$  is larger than  $W^M$  and the public firm does not want to merge.

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<sup>9</sup> $a^2$  is omitted for simplicity.

## Proof of Lemma 1

We divide this proof into three steps.

First, we prove that  $\alpha_0^* > \alpha_1^{**}$  for  $n \in [1, \infty)$ . Evaluating  $\alpha_0^*$  and  $\alpha_1^{**}$  at  $n = 1$ , then

$$\alpha_0^*|_{n=1} = \frac{56 + 11\sqrt{3}}{59} \approx 1.2721 > 0.5792 \approx \frac{509 - 11\sqrt{505}}{452} = \alpha_1^{**}|_{n=1}.$$

Using computer software, we obtain

$$\lim_{n \rightarrow \infty} \alpha_0^* = \frac{2 + 3\sqrt{2}}{7} \approx 0.8918 > \alpha_1^{**}|_{n=1}.$$

Since, in addition to this,  $\alpha_0^*$  and  $\alpha_1^{**}$  are decreasing functions of  $n$ , we obtain  $\alpha_0^* > \alpha_1^{**}$  for  $n \in [1, \infty)$ .

Second, we prove that  $\alpha_0^{**} > \alpha_1^{**}$  at  $n = 1$ . Evaluating  $\alpha_0^{**}$  at  $n = 1$ , we obtain

$$\alpha_0^{**}|_{n=1} = \frac{56 - 11\sqrt{3}}{59} \approx 0.6262.$$

Therefore,  $\alpha_0^{**}|_{n=1} > \alpha_1^{**}|_{n=1}$ .

Finally, we prove that there exists  $\tilde{n} \in (1, \infty)$  such that  $\alpha_1^{**} \geq \alpha_0^{**}$  for  $n \geq \tilde{n}$ . As mentioned in Section 3, both  $\alpha_0^{**}$  and  $\alpha_1^{**}$  are decreasing functions of  $n$ . In addition, we obtain the following limit relations:

$$\lim_{n \rightarrow \infty} \alpha_0^{**} = \frac{2 - 3\sqrt{2}}{7} \approx -0.3204 < 0.5 = \lim_{n \rightarrow \infty} \alpha_1^{**}.$$

Considering  $\alpha_0^{**}|_{n=1} > \alpha_1^{**}|_{n=1}$ , there exists  $\tilde{n} > 1$  such that  $\alpha_1^{**} \geq \alpha_0^{**}$  for  $n \geq \tilde{n}$ .  $\square$

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