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A simple framework for investigating the properties of policy games*

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Abstract

The paper extensively studies the static model of non-cooperative linear quadratic games in which a set of agents strategically chooses their instruments. We first derive the necessary and sufficient conditions for the existence of a Nash equilibrium as well as for multiple equilibria to arise. Furthermore, we study the general condition for policy neutrality and Pareto efficiency of the equilibrium by introducing a new concept of decisiveness.

JEL: C72, E52, E61

Keywords: Conflict of interest, Nash equilibrium existence, multiplicity, policy invariance, controllability, Pareto efficiency.

1 Introduction

The linear quadratic model is probably the most used setup in policy game applications. A linear model of the economy combined with well shaped linear-quadratic loss functions give intuitive but insightful linear first order conditions and provide this framework with the necessary simplicity to make it a privileged instrument for economic and policy analysis. Some key contributions to different fields of economics have made use of linear quadratic models. Some examples of pioneering studies in different and sometimes overlapping fields are Hamada (1976), and Canzoneri and Gray (1985) for international policy coordination; Barro and Gordon (1982) for monetary policy; Crawford and Sobel (1982) for signalling games; Anderson *et al.* (1998) for the analysis of public good provision; van der Ploeg and de Zeeuw (1992) for environmental policies; Alesina and

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Tabellini (1987) for public debt; Gylfason and Lindbeck (1986, 1994) for labor markets and unions.[1]

After some decades of profitable use and applications, however, a complete analysis and description of the main properties of equilibrium in this simple framework is still missing. Dasgupta and Maskin (1986), among others, stated sufficient conditions for equilibrium existence; but only recently, these conditions have been weakened by using the concept of controllability of an economic system introduced by Tinbergen (1952, 1956). Within the same simple framework also conditions for policy invariance have been stated.[2]

By analyzing the model extensively, we provide more general necessary and sufficient conditions for existence (and multiplicity) of Nash equilibrium. Furthermore, we study conditions for policy neutrality and the relationship between Nash equilibria and Pareto efficiency.

The paper is structured in three more sections. In section 2 we first describe the $2 \times 2 \times 2$ (2 agents, 2 instruments, 2 target variables) policy game and the conditions for equilibrium existence and uniqueness; we then investigate some specific issues (different targeting policies, policy symbiosis, partial effectiveness of instruments). In the third section, we provide general theorems for existence and multiplicity of equilibria, Pareto efficiency and policy neutrality, with reference to an $M_1 \times M_2 \times M_3$ model. A final section concludes.

2 The $2 \times 2 \times 2$ model

2.1 The basic framework

Our $2 \times 2 \times 2$ game is a linear quadratic policy game between two players with two instruments (one for each of them) constrained by a simple economy described by two variables. The game is static, or simultaneous, and non-cooperative.

Each player i sets an instrument, u_i, to minimize the following loss functions:

$$L_i = \frac{1}{2}\left[(x - \bar{x}_i)^2 + \beta_i (y - \bar{y}_i)^2\right] \qquad i = 1, 2 \tag{1}$$

defined over the deviations of two target variables (x, y) from some desired values $(\bar{x}_i, \bar{y}_i)$; β_i are given weight parameters.

From the above equation, we can define the marginal rate of substitution between the targets as:

$$MRS_{xy}^i = \frac{dy}{dx} = -\frac{\partial L_i / \partial (x - \bar{x}_i)}{\partial L_i / \partial (y - \bar{y}_i)} = -\frac{1}{\beta_i} \frac{x - \bar{x}_i}{y - \bar{y}_i} \qquad i = 1, 2 \tag{2}$$

Notice that, if $x \to \bar{x}_i$ for $y \neq \bar{y}_i$, $MRS_{xy}^i = 0$ and, if $y \to \bar{y}_i$ for $x \neq \bar{x}_i$, $MRS_{xy}^i \to \pm\infty$; while if both $(x, y)' \to (\bar{x}_i, \bar{y}_i)'$, MRS_{xy}^i is not defined as there

[1] A huge amount of studies, impossible to be mentioned, extended these studies in the last 30 years.

[2] See Acocella and Di Bartolomeo (2006), Acocella *et al.* (2006, 2007, 2009a, 2009b), Di Bartolomeo *et al.* (2008), Hughes Hallett *et al.* (2008, 2009).

is no change in the vector of targets that can compensate for a deviation from the first best outcome. This rate represents the slope of the indifference curve.[3]

In a similar manner, we can define elasticities of substitution among deviations:

$$\eta_{xy}^{i} = \frac{\partial L_i / \partial (x - \bar{x}_i)}{\partial L_i / \partial (y - \bar{y}_i)} \frac{y - \bar{y}_i}{x - \bar{x}_i} = \frac{1}{\beta_i} \qquad i = 1, 2 \tag{3}$$

The relationships between instruments and targets are summarized by a system of linear equations that describes the economy:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \end{bmatrix} \tag{4}$$

where a_{ji} are parameters, c_x and c_y are constants.

We can rewrite (4) in a compact-matrix form as:

$$z = Au + C \tag{5}$$

We assume that A is a full rank matrix. We assume that all the element a_{ij} are different from zero; we will introduce the case of lower rank and sparse (some zeros in A) matrices only later.

It is worth noticing that the full rank of A means that system (5) is controllable in the Tinbergen's terms, i.e., any given 2×1 vector $\bar{z}$ can be achieved (given $\bar{z}$, a unique vector $u^* = A^{-1} (\bar{z} - C)$ exists such that $z = \bar{z}$).

From system (4), we can define the marginal rates of transformation between the target variables with respect to the instrument of agent i. This marginal rate is:

$$MRT_{xy}^{i} = \frac{dy}{dx} = \frac{\partial y / \partial u_i}{\partial x / \partial u_i} = \frac{a_{2i}}{a_{1i}} \qquad i = 1, 2 \tag{6}$$

The marginal rate of transformation measures the impact on y of a marginal change in x induced by agent i; implicitly, it shows the way player i's instrument affects the target variables: if the signs of a_{2i} and a_{1i} are different, to increase a target variable one need to reduce the other, otherwise the two variables move together. Because of the linearity of the system (4), marginal rates of transformation are constant. Moreover, they are different between the two players because we have assumed A to be of full rank.

Formally, the Nash equilibrium is a vector u that minimizes (1) subject to (4) for both players. The first order conditions for the two agents can be described as follows:[4]

$$a_{1i} (x - \bar{x}_i) + \beta_i a_{2i} (y - \bar{y}_i) = 0 \qquad i = 1, 2 \tag{7}$$

[3] As usual, it measures the compensation in terms of the target variable y, for an infinitesimal variation of x, that makes agent i indifferent between accepting or not the change.

[4] Each player aims at equating the ratio of the outcome deviations from his target values to a constant, which is his marginal rate of transformation.

We refer to equations (7) as the quasi-reaction functions since they are the reaction functions defined in the space of the targets.[5] Optimization (7) clearly implies that both players equalize their marginal rate of substitution to their marginal rate of transformation.

We assume that the players are both active in the game to avoid trivial degenerative cases due to the non existence of the first order condition (7) of a player. This requires that $(a_{1i}, a_{2i}\beta_i)' \neq (0,0)'$ for $i = 1,2$. Note that if the βs are finite and different from zero, this is simply implied by the full rank assumption on the matrix A.

In a compact form the equation system (7) can be written as:

$$Zz - \bar{Z} = 0 \tag{8}$$

where $Z = \begin{bmatrix} a_{11} & \beta_1 a_{21} \\ a_{12} & \beta_2 a_{22} \end{bmatrix}$, $\bar{Z} = \begin{bmatrix} a_{11}\bar{x}_1 + \beta_1 a_{21}\bar{y}_1 \\ a_{12}\bar{x}_2 + \beta_2 a_{22}\bar{y}_2 \end{bmatrix}$.

Finally, we need to redefine the concept of policy neutrality. In fact, its classical definition (*we call it exogenous neutrality*), which implies that autonomous changes in an instrument have no influence on an outcome, has to be adapted to the realm of policy games, since in this setting instruments are endogenous variables.

In models without strategic interaction (the single agent case) the effect of the instruments set on the outcome is fully described by the instrument multipliers (the matrix A of equation (5)). By contrast, in models of strategic interaction a change in the instrument will also have an indirect effect through the induced reaction of the other players. Therefore, we must distinguish ex-ante effectiveness, which corresponds to *exogenous* neutrality and is fully described by matrix A, from ex-post effectiveness, interpreted as the capability of an instrument to affect a target variable in the equilibrium of the game. An instrument is ex-post effective (or *non neutral*) on some target variable if the equilibrium value of the target depends only on the will of the player gearing that instrument (as expressed by the parameters of his preference function) and not on that of other players. Ex-post ineffectiveness will thus be called *endogenous neutrality*, to distinguish it from the kind of neutrality that applies in a non-strategic context and in our context would be described by matrix A.

2.2 Existence, uniqueness, and multiple equilibria

We can state the following theorem.

Theorem 1 *A unique Nash equilibrium exists if and only if* $MRT^1_{xy}/MRT^2_{xy} \neq \eta^1_{xy}/\eta^2_{xy}$.

[5] In our example since all the outcomes can be achieved by an appropriate combination of instruments (recall the system is controllable in the Tinbergen's terms), if a Nash equilibrium exists in the space of quasi-reaction functions, there exists also a couple of instruments supporting it (i.e., a solution in the space of reaction function defined as usual in terms of control variables). We will investigate the problem further in the more general model presented in the next section.

Proof. Matrix Z is square and it is of full rank if and only if $\frac{a_{21}/a_{11}}{a_{22}/a_{12}} \neq \frac{\beta_2}{\beta_1}$, i.e. $\frac{MRT^1_{xy}}{MRT^2_{xy}} \neq \frac{\eta^1_{xy}}{\eta^2_{xy}}$. Then there is a unique vector of target variables that satisfies the optimality conditions for both players: $z^* = Z^{-1}\bar{Z}$. The unique Nash equilibrium follows from the full rank assumption on A. The corresponding Nash equilibrium is $u^* = A^{-1}\left(Z^{-1}\bar{Z} - C\right)$. ■

Less formally, conditions $MRT^1_{xy}/\eta^1_{xy} \neq MRT^2_{xy}/\eta^2_{xy}$ simply imply that the reaction functions of the two players are not parallel. Hence they can have an intersection only in one point, which is the Nash equilibrium.

This unique Nash equilibrium is generally not Pareto efficient. In fact, the equilibrium is obtained from $MRT^i_{xy} = MRS^i_{xy}$, $i = 1, 2$, (see (7)), but the marginal rates of transformation are constant and different between the two players (recall, in fact, that A is full rank); thus $MRS^1_{xy} \neq MRS^2_{xy}$, $i = 1, 2$. However, Pareto optimality requires $MRS^1_{xy} = MRS^2_{xy}$. Thus Nash non cooperative equilibria will not be, in general, efficient.

All the same, however, it can occur that an equilibrium exists where an agent reaches his desired bliss point. Specifically, this situation would occur, by chance, when the first best of an agent is on the quasi-reaction function of the other (and would then be the Nash equilibrium of the game).[6] In that case the equilibrium would be efficient. It is worth recalling that in this case the MRS is mathematically indeterminate and there is no need to define it since the agent reaches his first best and thus has not any incentive to trade off between its target variables.

Now consider the case:

$$\frac{MRT^1_{xy}}{\eta^1_{xy}} = \frac{MRT^2_{xy}}{\eta^2_{xy}} = \alpha \tag{9}$$

A unique equilibrium no longer exists (since $MRT^1_{xy}/MRT^2_{xy} \neq \eta^1_{xy}/\eta^2_{xy}$ does not hold) and multiple equilibria may emerge. Formally:

Theorem 2 *Multiple Nash equilibria exist if and only if* $(\bar{x}_1 - \bar{x}_2) = \alpha\,(\bar{y}_1 - \bar{y}_2)$.

Proof. If $(\bar{x}_1 - \bar{x}_2) = \alpha\,(\bar{y}_1 - \bar{y}_2)$, each desired allocation of an agent satisfies the first order condition of the other. The quasi-reaction functions of the two agents are thus the same and all points lying on them are potential Nash equilibria. By the full rank assumption of A, every z on the quasi-reaction function has a corresponding strategy vector u; thus infinite Nash equilibria arise. More formally, a candidate outcome for a Nash equilibrium must satisfy both quasi-reaction functions; if they have the same slope (i.e., $Rank\,[Z] = 1$), the only case of existence arises when the quasi-reaction functions are coincident. Rewriting the first order conditions as follows:

$$x - \bar{x}_i = \beta_i \frac{a_{2i}}{ai}\,(\bar{y}_i - y) \;\; i = 1, 2 \tag{10}$$

[6] Formally this requires that $a_{12}\bar{x}_1 + a_{22}\beta_2\bar{y}_1 = a_{12}\bar{x}_2 + a_{22}\beta_2\bar{y}_2$ *or* $a_{11}\bar{x}_2 + a_{21}\beta_1\bar{y}_2 = a_{11}\bar{x}_1 + a_{21}\beta_1\bar{y}_1$.

Subtracting the two (10) and using $\beta_1 \frac{a_{21}}{a_{11}} = \beta_2 \frac{a_{22}}{a_{12}} = -\alpha$ we obtain:

$$\bar{x}_1 - \bar{x}_2 = \alpha (\bar{y}_1 - \bar{y}_2) \tag{11}$$

as required by the condition of the theorem. ■

These equilibria are efficient if and only if the equilibrium outcome z^* happens to be the first best outcomes of one of the players, that is $z^* = (\bar{x}_1, \bar{y}_1)'$ or $z^* = (\bar{x}_2, \bar{y}_2)'$.

In summary, by the two theorems we have a complete taxonomy of existence and multiplicity in the canonical $2 \times 2 \times 2$ game (i.e., when A is of full rank and has no zeros, and the βs are strictly positive and finite). If $\frac{MRT^1_{xy}}{\eta^1_{xy}} \neq \frac{MRT^2_{xy}}{\eta^2_{xy}}$, a unique Nash equilibrium exists. If $\frac{MRT^1_{xy}}{\eta^1_{xy}} = \frac{MRT^2_{xy}}{\eta^2_{xy}} = \alpha$, either there are multiple equilibria (when $(\bar{x}_1 - \bar{x}_2) = \alpha (\bar{y}_1 - \bar{y}_2)$) or no equilibrium exists. In the next subsections we also consider some special cases departing from the canonical model.

2.3 Equilibrium properties of targeting policies

We can now investigate the properties of a Nash equilibrium with reference to a framework where for one or both players the elasticities of substitution among target deviations are infinite, i.e. $\beta_i \to 0$ or $\beta_i \to \infty$. This can be thought as equivalent to targeting player i's policies towards one objective only.

Formally, agent i is interested in one target only, if

1. $\beta_i \to 0$, the deviation of y from the preferred target $\bar{y}_i$ has no weight in his loss function and we can say that player i is only interested in variable x;

2. $\beta_i \to \infty$, the relative importance of the deviation of x from the preferred target shrinks to zero and the player will be interested only in the deviations of the target variable y.

We can consider three cases:

1. One player is interested in only one target variable.

2. Both players are interested in only one, but different, variable.

3. Both players are interested in only one and the same variable.

These cases are very common in the economic literature and exemplify some of the most important applications of the new theory of economic policy[7] in so far as the existence or other features of an equilibrium are concerned.

Consider the first case (one player is interested in both targets, and the other player is interested in only one). If the coefficients of the instruments are

[7] See Acocella *et al.* (2006).

different from zero for both agents, there will always be a Nash equilibrium. It will be Pareto efficient if the agents share the target value of the variable in common; it will be not efficient otherwise.

In the second case the agents are interested in different variables, i.e. $\beta_1 = 0$ and $\beta_2 = \infty$ or, conversely, $\beta_1 = \infty$ and $\beta_2 = 0$; they are thus not in conflict. Moreover, each agent controls his relevant subsystem and is able to get his first best allocation for any given strategy of the other player, as in the previous case. The first order conditions are compatible since there is no conflict and the unique Nash equilibrium is Pareto efficient (as it ensures the first best to each player).

Finally, if both agents are interested in the same variable,[8] $\frac{MRT^1_{xy}}{\eta^1_{xy}} = \frac{MRT^2_{xy}}{\eta^2_{xy}} = \alpha$ holds (with α either zero or infinite) and the condition for theorem 2 applies. Then no equilibrium exists, unless the players share the same desired value. In this case, $\bar{x}_1 - \bar{x}_2 = \alpha(\bar{y}_1 - \bar{y}_2) = 0$ or $\frac{\bar{x}_1 - \bar{x}_2}{\alpha} = \bar{y}_1 - \bar{y}_2 = 0$, and infinite equilibria exist.

2.4 Symbiosis and implicit coordination

Another specific case of interest is that of symbiosis, which arises when the two agents share the same desired values for all the target variables. In our context (of two players with two targets each) symbiosis occurs for $\bar{x}_1 = \bar{x}_2 = \bar{x}$ and $\bar{y}_1 = \bar{y}_2 = \bar{y}$. It implies that there is no "conflict" between the players.[9]

By assuming symbiosis, a trivial solution in terms of outcomes emerges, i.e., $z = \bar{z}$ where $\bar{z} = [\bar{x}, \bar{y}]'$, which is the same efficient solution that can be obtained if there were only one player (or a social planner) setting both instruments. Formally, the system of first order conditions is now:

$$Z(z - \bar{z}) = 0 \tag{12}$$

If Z is of full rank, only one Nash equilibrium exists (see theorem 1). The two players will choose their best policy and, in doing so, they will implicitly coordinate to achieve their first best. In fact, the first order conditions they must satisfy are the same as those of a single player endowed with the two instruments as linear independent instruments (i.e., A is of full rank) ensures a single valued correspondence between the vector of target variables and that of instruments.

By contrast, if Z is not of full rank, multiple equilibria arise (see theorem 2) as the first order conditions are linearly dependent, i.e. both the reaction functions and the quasi-reaction functions are the same for the two agents. A coordination failure à la Cooper and John (1988) emerges since equilibria are related to different and rankable outcomes.[10]

[8] I.e., $\beta_1 = \beta_2 = 0$ or $\beta_1 = \beta_2 = \infty$.

[9] For a more general case of symbiosis see Di Bartolomeo *et al.* (2008) and Acocella *et al.* (2009b).

[10] Acocella *et al.* (2009b) show how announcements can be used as an equilibrium selection device in this framework.

2.5 Partial effectiveness of instruments

Now we consider cases where agents can only partially influence the outcome of the economic system, i.e. are able to influence only some variables. As already said, this ex ante ineffectiveness of instruments can be viewed as an exogenous policy neutrality of an agent and is completely different from the endogenous policy neutrality. As exogenous neutrality arises when A contains at least one zero, marginal rates of transformation become infinite (or zero).[11] We also consider a further case of partial ineffectiveness which arises when the two agents affect the outcome in the same way, i.e. their instruments are not independent. In this case not all $z \in \mathbb{R}^2$ can be reached as the matrix A is not of full rank. In other words, the economic system is no longer globally controllable in the Tinbergen terms.[12]

Under partial effectiveness of instruments, four possible cases can arise.

1. Both agents are able to influence only one and the same variable,[13] and are obviously interested in that variable.[14] It is immediate to realize that each agent controls his relevant subsystem and, as a consequence, the Nash equilibrium exists if and only if both agents share the same target value for that variable. However, again, a problem of coordination of strategies arises. There are in fact infinite combinations of their instruments that support the achievement of the common desired targets.

2. Each agent influences only one, but a different, variable.[15] As we have assumed that players are both active, they should be interested at least in the variable for which their instrument is effective. Also in this case both agents control their relevant subsystem, but since the subsystems are dichotomous, there will be a Nash equilibrium for all possible target values of the agents. This equilibrium is Pareto efficient if and only if one agent is interested in only one variable or if, by chance, the first best of the agent is on the quasi-reaction function of the other one.

3. One agent influences both variables and the other only one. In this case, the latter is always able to control his relevant subsystem (as we have assumed that players are both active). Thus, no equilibrium exists if also the former agent controls the same target and does not share the same target values. In the case of symbiosis, an equilibrium always exists and is Pareto efficient. However, it will be unique only if the former player does not control the same target variable as the former.

[11] Obviously it does not make sense to consider the case in which one agent has no influence on any variable since it would violate the assumption that both agents are active in the game.

[12] The space of targets (quasi-reaction functions) is no longer isomorphic to that of the instruments (reaction functions).

[13] I.e. $a_{11} = a_{12} = 0$ (or, symmetrically, $a_{21} = a_{22} = 0$).

[14] Otherwise we would obtain a trivial non sense: first order condition would not exist if both were not interested in the only variable they can affect.

[15] Formally, $a_{12} = a_{21} = 0$ (symmetrically, $a_{11} = a_{22} = 0$).

4. The agents influence both variables, but with linearly dependent instruments. The whole system is not controllable in the Tinbergen terms. Thus even if there were a z^* for which the quasi-reaction functions intersect, there would be no guarantee that there is at least a vector of instruments u^* supporting it. Only if $Rank\,[A|z^*] = 1$, z^* could be achieved, but then multiple Nash equilibria would arise as, in this case, infinite combinations of instruments support z^*. (i.e., the reaction functions are coincident).

3 The general case: The $M_1 \times M_2 \times M_3$ model

3.1 The basic framework

In this section we extend the previous theorems for the existence of a Nash equilibrium to the general case of M_1 agents, M_2 instruments and M_3 variables.

The economy is described by the following linear system:

$$Ax = Bu = \sum_{i=1}^{M_3} B_i u_i \tag{13}$$

where each agent aims at minimizing a quadratic criterion defined on the deviation from a desired target vector ($\bar{x}_i$):[16]

$$L_i = (x - \bar{x}_i)' \, Q_i \, (x - \bar{x}_i) \tag{14}$$

where Q_i is a positive semi-definite diagonal matrix, which represents the weights that the agent places on deviations from the desired targets (we indicate the weight on the k-th element of vector x by $q_i\,(k)$). We do not require that Q_i is of full rank as one agent may not be interested in one or more target variables.

We assume that each agent is endowed with only one instrument, so there are $M_1 = M_2$ agents. This assumption is not restrictive at all; the case of an agent who is endowed with more than one instrument can be simply introduced by assuming more agents minimizing the same criterion (the case of symbiosis discussed before shows this equivalence*)*.[17]

We also assume that the total number of instruments is not greater than the total number of target variables (i.e., $M_2 \leq M_3$).[18]

[16] For the sake of brevity, without loss of generality, the criterion is assumed to be strictly quadratic. See Acocella and Di Bartolomeo (2004) for a discussion.

[17] Drawing from section 2.4, a situation with a player i having two targets, x and y, and two instruments, u_1 and u_2, is equivalent to the case in which there are two fictitious players, say i' and i'', that have one instrument each and the same target values.

[18] This assumption is introduced to rule out a trivial case that would lead to either infinite Nash equilibria or non existence. Informally, if $M_2 > M_3$, there are at least $M_2 - M_3$ linearly dependent instruments. Whenever a vector of outcomes x^* is consistent with a Nash equilibrium, there are infinite many combinations of instruments supporting it and, therefore, infinite Nash equilibria would arise. Multiple equilibria, however, would emerge only in the case of equal target values for the players; in general no equilibrium would exist. For a formal discussion of over-determined systems in this framework, see Acocella *et al.* (2008).

Finally, we assume that the instrument set by agent i is effective on his target variables; this formally requires $\left(A^{-1}B_i\right)^{'} Q_i \neq 0$, for all i. The assumption simply implies that each player is active in the game; otherwise his reaction function would not exist.

3.2 Existence, uniqueness and multiple equilibria

The system of quasi-reaction functions, i.e. the M_2 first order conditions in the outcome space, is the following:

$$Zx - \bar{Z} = 0 \tag{15}$$

where $Z = \begin{bmatrix} \left(A^{-1}B_1\right)^{'} Q_1 \\ \vdots \\ \left(A^{-1}B_i\right)^{'} Q_i \\ \vdots \\ \left(A^{-1}B_I\right)^{'} Q_I \end{bmatrix}$, $\bar{Z} = \begin{bmatrix} \left(A^{-1}B_1\right)^{'} Q_1\bar{x}_1 \\ \vdots \\ \left(A^{-1}B_i\right)^{'} Q_i\bar{x}_i \\ \vdots \\ \left(A^{-1}B_I\right)^{'} Q_I\bar{x}_I \end{bmatrix}$.

Lemma 3 *The system of quasi reaction functions (15) has a solution if and only if* $Rank\,[Z] = Rank\,\left[Z\,\middle|\bar{Z}\right]$.

Proof. The proof is trivial. ■

Notice that the existence of a target vector x^* that solves (15) is not necessarily a Nash equilibrium outcome: there might not be a vector of strategies that supports it.

Theorem 4 *A solution* $\bar{x}^*$ *of the system of quasi reaction function (15) is the outcome of a Nash equilibrium if and only if:* $Rank\left[\left(A^{-1}B\right)'\right] = Rank\left[\left(A^{-1}B\right)' |\bar{x}^*\right]$.

Proof. The proof is trivial. ■

The lemma assures that there exists a vector of values for the target variables that is on the quasi-reaction functions of all the agents, which implies that no agent has incentive to change his instrument and deviate. The theorem instead guarantees the feasibility of that outcome, i.e. there exists a strategy for the players that supports it. It follows that if u^* is a Nash equilibrium of the game, no agent has an incentive to deviate: so u^* should support a vector of targets for which the first order conditions are verified for all agents.

Corollary 5 *The Nash equilibrium is unique if and only if* $ZA^{-1}B$ *is full rank.*

Proof. The conditions to be satisfied for a Nash equilibrium are $Zx - \bar{Z} = 0$ and $x = A^{-1}Bu$; by substituting the latter equality into the former, we can say that uniqueness will arise if the matrix $ZA^{-1}B$ is invertible or, since it is square, of full rank. ■

3.3 Pareto efficiency

We can now state the necessary and sufficient conditions for a Nash equilibrium to satisfy Pareto efficiency.

Theorem 6 *An allocation associated with a Nash equilibrium is Pareto efficient if and only if it corresponds to the minimum of a lexicographic preference ordering over the loss functions of the agents.*

Proof. Define Π as the set of all possible orders of the agents (i.e., all the possible $M_1!$ permutations), $\pi \in \Pi$ is a given ordering of agents and $\pi(i)$ identifies the position of agent i. Then we define L a lexicographic function over π or a subset of it, e.g. $L\left(L_1, ..., L_{\pi(j)}, ..., L_{M_1}\right)$ or $L\left(L_1, ..., L_{\pi(j)}\right)$.

Now assume that there is a Nash equilibrium x' that is Pareto efficient but does not minimize a lexicographic function for any $\pi \in \Pi$. Therefore, for any π, there exists an agent i such that a) x' is not argmin of $L\left(L_1, L_2, ..., L_{\pi(i)}\right)$; b) x' is argmin of $L\left(L_1, L_2, ..., L_{\pi(i)-1}\right)$, i.e. i is the first agent of π for whom the lexicographic function is not minimized by x'. We define n is the number of players after $\pi(i)$.

We first take the case of $n = 0$, that is for all π such that agent i is the last agent of the permutation; it will hold that a) x' is not argmin of $L\left(L_1, ..., L_{M_1-1}, L_{\pi(i)}\right)$; b) x' is argmin of $L\left(L_1, L_2, ..., L_{M_1-1}\right)$ and therefore it is trivial that x' cannot be Pareto efficient.

We now consider the case of $n = 1$, that is for all π such that agent i is the second-last agent of the permutation; it will hold that a) x' is not argmin of $L\left(L_1, ..., L_{M_1-2}, L_{\pi(i)}\right)$; b) x' is argmin of $L\left(L_1, L_2, ..., L_{M_1-2}\right)$. There are two possible cases to consider. 1) If x' is argmin of $L\left(L_1, ..., L_{M_1-2}, L_{M_1}\right)$, we can define permutation π', by inverting M_1 with $\pi(i)$. for which $n = 0$ and therefore x' cannot be Pareto efficient. 2) If x' is not argmin of $L\left(L_1, ..., L_{M_1-2}, L_{M_1}\right)$, by the first order conditions of the Nash equilibrium we known that instruments are set such that $MRS = MRT$, since instruments are assumed to be independent MRS should be different among agents, therefore x' cannot be Pareto efficient as this requires that SMS are equal between agents.

Now we face the most general case. Let's assume that for any permutation π such that $n = k - 1$, x' is not Pareto efficient and consider a given ordering π' such that $n = k$. If among the k agents following i there is an agent j such that x' is argmin of $L\left(L_1, ..., L_{\pi(i)-1}, L_{\pi(j)}\right)$, we can define permutation π', by inverting $\pi(j)$ with $\pi(i)$. for which $n = k - 1$; therefore x' cannot be Pareto efficient. Otherwise, by the first order conditions of the Nash equilibrium we known that $MRS = MRT$ between at least a pair of agents, therefore x' cannot be Pareto efficient.

By an induction argument, an allocation associated with a Nash equilibrium is Pareto efficient if there exist an ordering for which it minimizes $L(L_1, ..., L_{\pi(j)}, ..., L_{M_1})$. The other side of the theorem (an allocation associated with a Nash equilibrium that corresponds to the minimum of a lexicographic preference ordering over the loss functions of the agents is Pareto efficient) is trivial. ■

This theorem points out that the Nash equilibrium is usually Pareto inefficient apart from few specific cases. Moreover, in these cases Pareto efficiency of the allocation is the minimum of an implicit lexicographic ordering over the losses of the agents.

3.4 Policy invariance and decisiveness

Given the definition of policy neutrality, we state ex-post effectiveness[19] of an instrument u_i relative to a target variable x_k, as the capability of agent i of influencing the outcome relative to the Nash equilibrium of the game by changing his policy: this is equivalent to requiring that either $\frac{\partial x_k}{\partial Q_i} \neq 0$ or $\frac{\partial x_k}{\partial \bar{x}_i} \neq 0$, i.e. the outcome of the Nash equilibrium is not independent of agent i's preferences.[20]

In order to derive the conditions for neutrality we consider a subgroup χ of agents. Given this subgroup, we can rearrange the system (13) as:

$$\begin{bmatrix} A_{x_\chi x_\chi} & A_{x_\chi x_{\bar{\chi}}} \\ A_{x_{\bar{\chi}} x_\chi} & A_{x_{\bar{\chi}} x_{\bar{\chi}}} \end{bmatrix} \begin{bmatrix} x_\chi \\ x_{\bar{\chi}} \end{bmatrix} = \begin{bmatrix} B_{x_\chi u_\chi} & B_{x_\chi u_{\bar{\chi}}} \\ B_{x_{\bar{\chi}} u_\chi} & B_{x_{\bar{\chi}} u_{\bar{\chi}}} \end{bmatrix} \begin{bmatrix} u_\chi \\ u_{\bar{\chi}} \end{bmatrix} \tag{16}$$

where u_χ is the vector of the instruments of the agents of group χ, while $u_{\bar{\chi}}$ is the vector of the instruments of the other agents; x_χ is the vector of all the target variables $x_\chi(k)$ for which there is at least an agent i of the group χ for which $\left[A^{-1}B_i\right]_k' q_i(k) \neq 0$. In other words, we select the vector of targets x_χ of group χ to include all and only the targets for which some agent in χ is both interested in and ex-ante effective.

From (16), we can extract the relationship between x_χ and u_χ, i.e.

$$A_{x_\chi x_\chi} x_\chi = B_{x_\chi u_\chi} u_\chi + C \tag{17}$$

where $C = B_{x_\chi u_{\bar{\chi}}} u_{\bar{\chi}} - A_{x_\chi x_{\bar{\chi}}} x_{\bar{\chi}}$.

Now we can introduce the following definitions.

Definition 7 *We define the χ-game as the non-cooperative simultaneous game among the agents of subgroup χ considering C as an arbitrary constant and where for all the target variables $x_\chi(k)$ it holds that $\left[A^{-1}B_i\right]_k' q_i(k) \neq 0$ for at least one agent $i \in \chi$.*

Notice that, since we assumed that $\left(A^{-1}B_i\right)' Q_i \neq 0$ (i.e., every agent is active in the game), and as a consequence of the way we selected the targets of the χ-game, all agents in χ will be active as well: in particular, it will hold that $\left(A^{-1}_{x_\chi x_\chi} B_{x_\chi u_\chi i}\right)' Q_i \neq 0$.

[19] As said, ex-ante effectiveness is related to the multipliers of the system (13). In the context of this system an instrument u_i is ex-ante effective with respect to target variable x_k if the multiplier $A^{-1}B_i \neq 0$;

[20] We can say that whenever an agent's instrument u_i is not ex-post effective for some target variable x_k in the Nash equilibrium, we will say that x_k is policy invariant with respect to instrument u_i or, equivalently, that u_i is policy neutral with respect to x_k.

Definition 8 *If an equilibrium of the* χ*-game exists and the outcomes associated to this equilibrium are independent of* C*, then the* χ *group is decisive with respect to the target variables* x_χ*.*

Theorem 9 *The* χ *group is decisive with respect to the target variables* x_χ *if and only if Lemma 3 holds for the* χ*-game and the matrix* $A^{-1}_{x_\chi x_\chi} B_{x_\chi u_\chi}$ *is full rank or left invertible.*

Proof. The Lemma 3 assures that an outcome from which agents have no incentive to deviate exists. Conditions on $A^{-1}_{x_\chi x_\chi} B_{x_\chi u_\chi}$ assure that, if such outcome exists, it is feasible and independent of C. Note that, as matrix $A^{-1}_{x_\chi x_\chi} B_{x_\chi u_\chi}$ is full rank or left invertible, the system of reaction functions is, in fact, either determinate or overdetermined. ■

Two corollaries follow.

Corollary 10 *When an equilibrium of the entire game exists, if the* χ *group is decisive with respect to the target variables* x_χ*, then the policies of agents that do not belong to that group are neutral with respect to* x_χ*.*

Corollary 11 *If there are more groups that are decisive with respect to a non empty intersection of target variables, then the equilibrium either does not exist or is multiple. The latter is the case if there is no implicit conflict among the two groups, i.e. the outcomes associated with the two* χ*-games are the same.*

The first corollary shows that the selection of the targets of the χ-game implies that the first order conditions of the agents of group χ for the χ-game are the same as for the original game. For the first corollary we assume that an equilibrium of the entire game exists: as agents in χ are decisive, whatever the outcome vector x_χ that satisfies lemma 3 for the χ-game is reached and is fully determined; since the first order conditions are invariant, any strategy change of the other agents (not belonging to χ) will be ineffective on the vector of targets x_χ and results in being policy neutral. The second corollary shows that, if two groups χ and χ' have overlapping targets ($x_\chi \bigcap x_{\chi'} \neq 0$), the equilibrium either does not exist or is multiple. In particular, since both groups are decisive, each of them has a set of outcomes that satisfies first order conditions (by Lemma 3) and can be achieved independently of the choices of the other agents. Therefore, if these sets of targets have a non-empty intersection, an equilibrium exists but it is not unique, since the system is overdetermined (it is at least determined for each group of agents); if, conversely, the intersection is empty, no equilibrium exists.

In a similar context, Acocella *et al.* (2009b) show that policy neutrality emerges when there is an implicit coalition[21] that controls its targets: if a coalition controls its subsystem, its members are able to individually set their instruments to achieve the first best outcome independently of the strategy set by non members; in some sense the players implicitly coordinate to get their first

[21] An implicit coalition is defined as a group of agents that have non contrasting interests..

best, thus offsetting the action of the other agents. Moreover, if there is a conflict between two implicit coalitions that control their targets, the equilibrium either does not exist or is multiple (non existence of a Nash equilibrium arises when there are two controlling implicit coalitions with conflicting interests).

Our corollaries extend the results in Acocella *et al.* (2009b). Implicit coalition controllability is a particular case of group decisiveness. It arises when the targets of interest to the members of the group coincide (symbiosis). Our result is more general, since controllability is not required for decisiveness. Decisiveness does not need symbiosis among the group members (then conflict is possible and controllability may not be achieved); it only requires that the conflict among them is completely interior to the group (i.e. independent of the policy of the non members).

4 Conclusions

We have provided an analysis of the linear-quadratic static policy game framework and shown that different scenarios may emerge depending on the number of variables of interest, the existence of a conflict,"symbiosis" or group decisiveness among the agents and the effectiveness of the instruments. Our results appear to be essential to fully understand policy games and for model building, as they state the conditions for the consistency of the optimal strategies of all the players (and thus the existence of the equilibrium of the game) as well as the effectiveness of policy instruments. In addition, they are relevant for institution building, as they can help us to show the conditions under which a decentralized equilibrium may fail to exist or to be Pareto efficient.

References

Acocella N. and G. Di Bartolomeo (2006), "Tinbergen and Theil meet Nash: Controllability in policy games," *Economics Letters*, 90: 213-218.

Acocella N., G. Di Bartolomeo and A. Hughes Hallett (2006), "Controllability in policy games: Policy neutrality and the theory of economic policy revisited," *Computational Economics*, 28: 91-112.

Acocella N., G. Di Bartolomeo and A. Hughes Hallett (2007), "Dynamic controllability with overlapping targets: Or why target independence may not be good for you," *Macroeconomic Dynamics*, 11: 202-213.

Acocella N., G. Di Bartolomeo and P.G. Piacquadio (2009a), "Conflict of interests, (implicit) coalitions and Nash policy games," *Economics Letters*, 105: 303-305.

Acocella N., G. Di Bartolomeo, A. Hughes Hallett and P.G. Piacquadio (2009b), "Announcement wars as equilibrium selection device," *CORE Working Paper* No 77, Catholic University of Louvain la Neuve.

Alesina A. and G. Tabellini (1987), “Rules and discretion with noncoordinated monetary and fiscal policies,” *Economic Inquiry*, 25: 619-630.

Anderson S.P, J.K. Goeree and C.A. Holt (1998), “A theoretical analysis of altruism and decision error in public goods games,” *Journal of Public Economics*, 70: 297-323.

Barro R.J. and D.B. Gordon (1983), “Rules, discretion and reputation in a model of monetary policy,” *Journal of Monetary Economics*, 12: 101-22.

Canzoneri M.B. and J.A. Gray (1985), “Monetary policy games and the consequences of non-cooperative behavior,” *International Economic Review*, 26: 547-564.

Cooper R. and A. John (1988), “Coordinating coordination failures in Keynesian models,” *Quarterly Journal of Economics*, 103: 441-463.

Crawford V. and J. Sobel (1982), “Strategic information transmission,” *Econometrica*, 50: 1431-1452.

Di Bartolomeo G., A. Hughes Hallett and N. Acocella (2008), “The existence of Nash equilibria in n-player LQ-games. Explaining the hegemony in international monetary and trade agreements,” *Department of Communication Working Paper* No 40, University of Teramo.

Gylfason G. and A. Lindbeck (1986), “Endogenous unions and governments,” *European Economic Review*, 30: 5-26.

Gylfason G. and A. Lindbeck (1994), “The interaction of monetary policy and wages,” *Public Choice*, 79: 33-46.

Hughes Hallett A., G. Di Bartolomeo and N. Acocella (2008), “When can central banks anchor expectations? Policy communication and controllability,” *CEPR Discussion Paper* No. 7078.

Hughes Hallett A., G. Di Bartolomeo and N. Acocella (2009), “Policy games, policy neutrality and Tinbergen controllability under rational expectations,” *Journal of Macroeconomics*, *forthcoming.*

Hamada K. (1976), “A strategic analysis of monetary interdependence,” *Journal of Political Economy*, 84: 677-700.

Tinbergen J. (1952), *On the theory of economic policy*, Amsterdam, North Holland.

Tinbergen J. (1956), *Economic policies. principles and design*, Amsterdam, North Holland.

van der Ploeg F. and A.J. de Zeeuw (1992), “International aspects of pollution control,” *Environmental & Resource Economics*, 2: 117-139.