Competition among Sellers in Securities Auctions

Alexander S. Gorbenko **Andrey Malenko**

Web Appendix

Proof of Lemma 1. See the proof of Lemma 2 in the web-appendix for DeMarzo, Kremer, and Skrzypacz (2005).

Proof of Lemma 2. Suppose for a moment that k is a continuous variable. If we prove that the lemma holds for any k, this will automatically imply that it holds for k taking values $0, 1, 2, ...$ First, let us prove that $V(k)$ is an increasing and concave function of k. Differentiating $V(k)$,

$$V'(k) = \int_{v_L}^{v_H} \left(\frac{1}{k} + \log F(v)\right) v d\left(F(v)^k\right) > 0,$$

because $\int_{v_L}^{v_H} \left(\frac{1}{k} + \log F(v)\right) d\left(F(v)^k\right) = 0$ as the first derivative of $\int_{v_L}^{v_H} d\left(F(v)^k\right) = 1$. Differentiating again,

$$V''(k) = \int_{v_L}^{v_H} \log F(v) \left(\frac{2}{k} + \log F(v)\right) v d\left(F(v)^k\right).$$

Note that $\int_{v_L}^{v_H} \log F(v) \left(\frac{2}{k} + \log F(v)\right) d\left(F(v)^k\right) = 0$ as the second derivative of $\int_{v_L}^{v_H} d\left(F(v)^k\right) = 1$. Also, $\log F(v) \left(\frac{2}{k} + \log F(v)\right)$ is positive for $v < F^{-1}\left(e^{-2/k}\right)$ and negative for $v > F^{-1}\left(e^{-2/k}\right)$. Hence, $V''(k) < 0$.

In a similar way, we can prove that $U^b(v, k, S)$ is decreasing and convex in k. Write

$U^b(v,k,S)$ as

$$\text{(B1)} \qquad U^b(v,k,S) = \int_{v_L}^{v} (v - ES(s(y,S),v))\, d\left(F(y)^{k-1}\right).$$

Differentiating,

$$\text{(B2)} \qquad U_k^b(v,k,S) = \int_{v_L}^{v} (v - ES(s(y,S),v)) \left(\frac{1}{k-1} + \log F(y)\right) d\left(F(y)^{k-1}\right).$$

Note that $\int_{v_L}^{v} \left(\frac{1}{k-1} + \log F(y)\right) d\left(F(y)^{k-1}\right) \leq 0$ for any v, because $\log F(y)$ is an increasing function of y and $\int_{v_L}^{v_H} \left(\frac{1}{k-1} + \log F(y)\right) d\left(F(y)^{k-1}\right) = 0$. Also, $s(y,S)$ is increasing in y by Lemma 1. Therefore, (B2) is negative. Differentiating (B1) again,

$$\text{(B3)} \quad U_{kk}^b(v,k,S) = \int_{v_L}^{v} (v - ES(s(y,S),v)) \log F(y) \left(\frac{2}{k-1} + \log F(y)\right) d\left(F(y)^{k-1}\right).$$

Note that $\log F(y)\left(\frac{2}{k-1} + \log F(y)\right)$ is positive for $y < F^{-1}\left(e^{-2/(k-1)}\right)$ and negative for $y > F^{-1}\left(e^{-2/(k-1)}\right)$. This and $\int_{v_L}^{v_H} \log F(y) \left(\frac{2}{k-1} + \log F(y)\right) d\left(F(y)^{k-1}\right) = 0$ imply that $\int_{v_L}^{v} \log F(y) \left(\frac{2}{k-1} + \log F(y)\right) d\left(F(y)^{k-1}\right) \geq 0$ for all v. Combining this with the fact that $s(y,S)$ is increasing in y, so $v - ES(s(y,S),v)$ is decreasing in y, yields $U_{kk}^b(v,k,S) > 0$.

Proof of Lemma 3. Equation (11), which determines $q\left(S,\tilde{S}\right)$, can be written in the following form:

(B4)

$$\sum_{k=1}^{n_b} \binom{n_b - 1}{k-1} \left[q^{k-1}(1-q)^{n_b-k} U^b(k,S) - \left(\frac{1-q}{n_s-1}\right)^{k-1} \left(1 - \frac{1-q}{n_s-1}\right)^{n_b-k} U^b\left(k,\tilde{S}\right)\right] = 0.$$

Denote the left-hand side of (B4) by $g\left(q,S,\tilde{S}\right)$. Note that

$$\text{(B5)}\quad g\left(0,S,\tilde{S}\right) = U^{b}\left(1,S\right)-\sum_{k=1}^{n_b}\binom{n_b-1}{k-1}\left(\frac{1}{n_s-1}\right)^{k-1}\left(1-\frac{1}{n_s-1}\right)^{n_b-k}U^{b}\left(k,\tilde{S}\right)>0;$$

$$\text{(B6)}\quad g\left(1,S,\tilde{S}\right) = U^{b}\left(n_b,S\right)-U^{b}\left(1,\tilde{S}\right)<0,$$

because when there is only one bidder, he captures all surplus from the auction above v_L. Differentiating (B4),

(B7)

$$\begin{aligned} g_q\left(q,S,\tilde{S}\right) &= \textstyle\sum_{k=1}^{n_b}\binom{n_b-1}{k-1}q^{k-1}\left(1-q\right)^{n_b-k}\left(\frac{k-1}{q}-\frac{n_b-k}{1-q}\right)U^{b}\left(k,S\right) \\ &+\tfrac{1}{n_s-1}\textstyle\sum_{k=1}^{n_b}\binom{n_b-1}{k-1}\left(\frac{1-q}{n_s-1}\right)^{k-1}\left(1-\frac{1-q}{n_s-1}\right)^{n_b-k}\left(\frac{k-1}{\frac{1-q}{n_s-1}}-\frac{n_b-k}{1-\frac{1-q}{n_s-1}}\right)U^{b}\left(k,\tilde{S}\right). \end{aligned}$$

Consider the first term of (B7). Since probabilities $\binom{n_b-1}{k-1}q^{k-1}\left(1-q\right)^{n_b-k}$ sum to one over k, taking the derivative of their sum means that for any q and n_b,

$$\sum_{k=1}^{n_b}\binom{n_b-1}{k-1}q^{k-1}\left(1-q\right)^{n_b-k}\left(\frac{k-1}{q}-\frac{n_b-k}{1-q}\right)=0.$$

Moreover, the terms of this sum are negative for $k<q\left(n_b-1\right)+1$ and positive, otherwise. Because it immediately follows from Lemma 2 that $U^{b}\left(k,S\right)$ is a decreasing function of k as an expectation of $U^{b}\left(v,k,S\right)$ with respect to v, the first term of (B7) is negative. By the same argument, the second term of (B7) is also negative. Hence, $g\left(q,S,\tilde{S}\right)$ is a monotonically decreasing function of q. Combining this with (B5) and (B6), we conclude that there exists a unique solution to (11). This proves part (a) of the lemma.

If S_1 is a steeper set of securities than S_2, $U^{b}\left(k,S_1\right)<U^{b}\left(k,S_2\right)$ for any $k>1$ with equality for $k=1$. This is because the total surplus is unaffected by the choice of security design, and the seller's revenues are higher when the security design is steeper. Thus, an increase in the steepness of the firm's own security design decreases $g\left(q,S,\tilde{S}\right)$. Similarly, an increase in the steepness of the other firms' security design increases $g\left(q,S,\tilde{S}\right)$. Because $g\left(q,S,\tilde{S}\right)$ is decreasing in q for any S and $\tilde{S}$, the point $q\left(S, \tilde{S}\right)$ at which g crosses zero moves in the same direction as $g\left(q, S, \tilde{S}\right)$ if S or $\tilde{S}$ is altered. This implies parts (b) and (c) of the lemma. To prove part (d), notice that $q = \frac{1}{n_s}$ is the solution of (B4) when $S = \tilde{S}$.

Proof of Proposition 5. Consider an ordered set of securities $S \neq S_{call}$ and r such that $v_r > v_L$. Because $S \neq S_{call}$ and $v_r > v_L$, we can reduce r by an infinitesimal amount and increase the steepness of S by an infinitesimal amount so that $q\left(S, r, S^*, r^*\right)$ defined by (19) does not change. The latter can be done, for example, by taking an ordered set of securities, in which for $s \in [s_L, s_H]$ each security with index s is a linear combination of a security in S with the same index and the corresponding security in S_{call}. From (20) and $V_r(k, r) < 0$ we conclude that $U^s\left(S, r, S^*, r^*\right)$ increases and the described deviation is profitable, so any reserve price r such that $v_r > v_L$ cannot be an equilibrium outcome. Finally, consider an ordered set of securities $S \neq S_{call}$ and r such that $v_r \leq v_L$. If S is not an equilibrium security design of the main model, then a pair (S, r) is not an equilibrium of the extended model, because the same deviation that is profitable in the main model is also profitable now. If S is an equilibrium security design in the main model, then a pair (S, r) is an equilibrium in the extended model. This is because any deviation to (S', r) for $S' \neq S$ is not profitable as it is not profitable in the main model, and any deviation to (S', r') for r' such that $v_{r'} > v_L$ is not profitable as it is worse than a deviation to (S'', r''), where $r'' = r' - \varepsilon$ for an infinitesimal ε and S'' is an ordered set of securities such that $q\left(S'', r'', S, r\right) = q\left(S', r', S, r\right)$.

Proof of Proposition 6. By analogy with (7) – (8), let $U^b\left(v_i, k, S, M\right)$ and $U^b\left(k, S, M\right)$ denote the interim and ex-ante bidder's expected surpluses from participating in an auction with k bidders, security design S, and procedure M. As in the main setting, consider a seller who chooses (S, M), when all other sellers choose $\left(\tilde{S}, \tilde{M}\right)$. For a potential bidder to select among auctions using mixed strategies, his expected payoff from choosing all sellers must be the same. Hence, the probability of choosing the deviator $q\left(S, M, \tilde{S}, \tilde{M}\right)$ solves

$$\text{(B8)} \qquad \begin{aligned} &\textstyle\sum_{k=1}^{n_b} \binom{n_b-1}{k-1} q^{k-1} (1-q)^{n_b-k} U^b(k, S, M) \\ &= \textstyle\sum_{k=1}^{n_b} \binom{n_b-1}{k-1} \left(\frac{1-q}{n_s-1}\right)^{k-1} \left(1 - \frac{1-q}{n_s-1}\right)^{n_b-k} U^b\left(k, \tilde{S}, \tilde{M}\right). \end{aligned}$$

Hence, the deviating seller's ex-ante expected surplus are equal to

$$\text{(B9)} \qquad \begin{aligned} U^s\left(S, M, \tilde{S}, \tilde{M}\right) &= \textstyle\sum_{k=0}^{n_b} \binom{n_b}{k} q^k (1-q)^{n_b-k} V(k) \\ &\quad -n_b q \textstyle\sum_{k=1}^{n_b} \binom{n_b-1}{k-1} \left(\frac{1-q}{n_s-1}\right)^{k-1} \left(1 - \frac{1-q}{n_s-1}\right)^{n_b-k} U^b\left(k, \tilde{S}, \tilde{M}\right). \end{aligned}$$

From (B9) we can see that the seller's expected surplus depends on her choice (S, M) only through the participation probability $q\left(S, M, \tilde{S}, \tilde{M}\right)$. Thus, we can rewrite $U^s\left(S, M, \tilde{S}, \tilde{M}\right)$ as $U^s\left(q, \tilde{S}, \tilde{M}\right)$ and reformulate the seller's problem in terms of choosing the probability q with which each bidder decides to participate in the auction. Mathematically, the seller's problem is

$$\text{(B10)} \qquad \max_{q \in \left[q_L\left(\tilde{S}, \tilde{M}\right), q_H\left(\tilde{S}, \tilde{M}\right)\right]} U^s\left(q, \tilde{S}, \tilde{M}\right),$$

where $q_L\left(\tilde{S}, \tilde{M}\right)$ and $q_H\left(\tilde{S}, \tilde{M}\right)$ are the lowest and highest participation probabilities the seller can achieve by altering the security design and auction format. For a given number of bidders, their expected surplus is the highest when the security design is cash and the lowest when the security design is call options in both first-price and second-price auctions (DeMarzo, Kremer, and Skrzypacz (2005)). When the auction is in pure cash, the revenue equivalence theorem holds. Hence, $q_H\left(\tilde{S}, \tilde{M}\right) = q\left(S_{cash}, SP\right) = q\left(S_{cash}, FP\right)$. When the auction is in call options, the first-price auction yields higher expected revenues for the seller than the second-price auctions.[1] Hence, $q_L\left(\tilde{S}, \tilde{M}\right) = q\left(S_{call}, FP\right)$. Therefore, (S^*, M^*) is the equilibrium if

[1] See Proposition 2 in DeMarzo, Kremer, and Skrzypacz (2005).

and only if

$$\text{(B11)} \qquad \frac{1}{n_s} = \arg \max_{q \in [q(S_{call}, FP, S^*, M^*), q(S_{cash}, SP, S^*, M^*)]} U^s\left(q, S^*, M^*\right)$$

Suppose that security design S^* is an equilibrium security design in the main setting of the model. This implies

$$\text{(B12)} \qquad \frac{1}{n_s} = \arg \max_{q \in [q(S_{call}, FP, S^*, SP), q(S_{cash}, SP, S^*, SP)]} U^s\left(q, S^*, SP\right)$$

Compared to (B11) with $M^* = SP$, (B12) has the same upper bound on q and a smaller lower bound on q. Also, as shown in the proof of Proposition 2, $U^s\left(q, S^*, SP\right)$ is a concave function of q. Therefore, if $S^* \neq S_{call}$, then the fact that S^* satisfies (B12) implies that (S^*, SP) satisfies (B11). Therefore, (S^*, SP) is the equilibrium pair of security design and auction format.