Online Appendix for

"Information and Prices with Capacity Constraints"

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February, 2010

Below I prove the two propositions from the paper, and offer extended analysis of the example introduced in Section 2. Equation numbers here refer to equations in the paper. All equations exclusive to the appendix are denoted with an "A."

A Proofs

A.1 Proof of Proposition 1

It was established in the text that for any $\lambda > 0$ and $V \in (0,1)$, an individual seller's optimal strategy will either be to set $p_H = 1$ or to set p_L given by (5), with $Q_L = -\ln(V)$. Therefore, there are only three candidate equilibrium: all sellers set some $p_L < \bar{p}(V^*)$ ($\alpha^* = 1$), all sellers set $p_H = 1$ ($\alpha^* = 0$), and sellers mix between the two strategies ($\alpha^* \in (0,1)$). It is straight-forward to establish that $\alpha^* = 0$ if and only if $\lambda = 0$.[1] Therefore, the only two possible cases are $\alpha^* = 1$ and $\alpha^* \in (0,1)$.

If all sellers set the same price, buyers randomize equally across them and $Q_L^* = b$. Thus, if $\alpha^* = 1$, $p_L^* = 1 - (be^{-b})/(1-e^{-b})$ and profits are $1-(1+b)e^{-b}$. This is optimal if and only if

$$\begin{aligned} 1-(1+b)e^{-b} &\geq 1-e^{-(1-\lambda)b} \\ \Leftrightarrow \lambda &\geq \ln(1+b)/b \equiv \hat{\lambda}. \end{aligned}$$

Therefore, $\lambda \geq \hat{\lambda} \Leftrightarrow \alpha^* = 1$.

Now consider a two-price equilibrium. When a fraction α of firms set price $p_L < \bar{p}$, informed buyers will randomize across only these firms, while uninformed buyers continue to randomize

[1] That $\lambda = 0 \Rightarrow \alpha^* = 0$ is immediate, as $\lambda = 0$ implies that demand is completely inelastic. To see that $\alpha^* = 0 \Rightarrow \lambda = 0$, suppose that $\alpha^* = V^* = 0$ and $\lambda > 0$. A seller could deviate to $p^d = 1 - \epsilon$ for any arbitrarily small $\epsilon > 0$, and all informed buyers would visit this seller (i.e. $Q^d = \infty$). Note that (a) such a deviation would be profitable, as $\pi^d = 1 - \epsilon > 1 - e^{-Q_H}$, and (b) the informed buyers' incentive constraint is not violated, as $\lim_{Q\to\infty} \eta(Q)(1-\epsilon) \geq V^* = 0$.

across all sellers. From the second equilibrium condition in definition 1, the expected number of informed buyers that will visit these sellers is determined by:

$$b = \int [q(p_L; V^*) + (1-\lambda)b] dF(p) \;\Rightarrow\; b = \alpha q(p_L; V^*) + (1-\lambda)b \;\Rightarrow\; q(p_L; V^*) = \frac{b\lambda}{\alpha}.$$

Therefore, the total expected queue length at sellers setting price p_L is a function of α :

$$Q_L(\alpha) = b\left[\frac{\lambda}{\alpha} + 1 - \lambda\right]. \tag{A-1}$$

Using (6), then, we can express the candidate equilibrium profits from setting p_L as a function of α:

$$\pi_L(\alpha) = 1 - [1 + Q_L(\alpha)]e^{-Q_L(\alpha)}.$$

Given the analysis above, $\pi_L(1) < \pi_H \;\Leftrightarrow\; \lambda < \hat{\lambda}$. Since π_L is clearly a continuous function on the domain $\alpha \in (0, 1]$, one can appeal to the intermediate value theorem by showing that (a) $\lim_{\alpha\to 0} \pi_L(\alpha) > \pi_H$, and (b) $\partial\pi_L/\partial\alpha < 0$ while $\partial\pi_H/\partial\alpha = 0$.

From equation (A-1), $\lim_{\alpha\to 0} Q_L(\alpha) = \infty$. By L'Hospital's rule, $\lim_{Q\to\infty}(1 + Q)e^{-Q} = 0$, so that $\lim_{\alpha\to 0} \pi_L(\alpha) = 1 > \pi_H$. Clearly $\partial\pi_H/\partial\alpha = 0$, so that it is left to show that $\partial\pi_L/\partial\alpha < 0$. Since

$$\frac{\partial\pi_L}{\partial\alpha} = -Q_L e^{-Q_L}\left(\frac{\lambda}{\alpha^2}\right),$$

π_L is strictly decreasing in α, and $\lambda < \hat{\lambda}$ implies that there exists a unique $\alpha^* \in (0, 1)$ such that $\pi_L(\alpha^*) = \pi_H$. That $\alpha^* \in (0, 1) \Rightarrow \lambda < \hat{\lambda}$ follows easily from the results above.

A.2 Proof of Proposition 2

Suppose $S - 1$ sellers post price p and a single seller deviates to price p_d. This seller's profits are

$$\tilde{\pi}(p_d; p) = \tilde{\mu}(\theta_d)p_d, \tag{A-2}$$

where θ_d satisfies

$$(1-p)\tilde{\eta}\left(\frac{1-\theta_d}{S-1}\right) = (1-p_d)\tilde{\eta}(\theta_d) \tag{A-3}$$

for $p_d \le \bar{p} \equiv 1 - \{(1-p)\tilde{\eta}[1/(S-1)]\}/\tilde{\eta}(0)$, and $\theta_d = 0$ for $p_d > \bar{p}$. The profit function $\tilde{\pi}$ is strictly concave on the domain $p_d \in [0, \bar{p}]$, so that the first order condition

$$\tilde{\mu}(\theta_d) + \frac{\partial\tilde{\mu}}{\partial\theta_d}\frac{\partial\theta_d}{\partial p_d}p_d = 0 \tag{A-4}$$

is both a necessary and sufficient condition to characterize the unique profit-maximizing price p_d, conditional on attracting informed buyers with strictly positive probability.

To establish the strict concavity of the profit function on this domain, note that

$$\frac{\partial^2 \tilde{\pi}}{\partial p_d^2} \propto N(1-\theta_d)^{N-1}\left[2\left(\frac{\partial \theta_d}{\partial p_d}\right) + p_d\left(\frac{\partial^2 \theta_d}{\partial p_d^2}\right)\right] - N(N-1)(1-\theta_d)^{N-2}\left(\frac{\partial \theta_d}{\partial p_d}\right)^2. \tag{A-5}$$

Since

$$\frac{\partial \theta_d}{\partial p_d} = \frac{\tilde{\eta}(\theta_d)}{\left(\frac{1-p}{S-1}\right)\tilde{\eta}'\left(\frac{1-\theta_D}{S-1}\right) + (1-p_d)\tilde{\eta}'(\theta_d)} < 0 \tag{A-6}$$

for $p \in [0, \bar{p}]$, clearly $\frac{\partial^2 \tilde{\pi}}{\partial p_d^2} < 0$ on this domain if θ_d is concave in p_d.[2] Manolis Galenianos & Philipp Kircher (2009) establish that this is true if (i) $\tilde{\mu}(\theta_d)$ is increasing and concave, (ii) $\tilde{\eta}(\theta_d)$ is decreasing and convex, and (iii) $\tilde{\eta}(\theta_d)^{-1}$ is convex.[3] The first of these properties is trivial to establish. To see that the second and third also hold, it is helpful to derive an alternative (equivalent) representation for $\tilde{\eta}(\theta_d)$. Following Melanie Cao & Shouyong Shi (2000), define

$$A(y) = \sum_{i=0}^{N-1}\sum_{k=0}^{U} y^{i+k} C_{N-1}^{i}(\theta)^i(1-\theta)^{N-1-i} C_U^k \left(\frac{1}{S}\right)^k \left(1-\frac{1}{S}\right)^{U-k} \frac{1}{i+k+1}.$$

Note that $A(0) = 0$, and that we want to compute $A(1)$. Since

$$\begin{aligned}\frac{\partial}{\partial y}[yA(y)] &= \sum_{i=0}^{N-1} C_{N-1}^{i}(\theta)^i(1-\theta)^{N-1-i}\sum_{k=0}^{U}\left(\frac{y}{S}\right)^k\left(1-\frac{1}{S}\right)^{U-k} \\ &= [y\theta + 1 - \theta]^{N-1}\left[\frac{y}{S} + 1 - \frac{1}{S}\right]^U,\end{aligned}$$

integrating yields

$$\tilde{\eta}(\theta) = \int_0^1 [y\theta + 1 - \theta]^{N-1}\left[\frac{y}{S} + 1 - \frac{1}{S}\right]^U dy. \tag{A-7}$$

If we denote the first and second derivatives of $\tilde{\eta}$ by $\tilde{\eta}'$ and $\tilde{\eta}''$, respectively, then

$$\tilde{\eta}'(\theta) = (N-1)\int_0^1 (y-1)[y\theta + 1 - \theta]^{N-2}\left[\frac{y}{S} + 1 - \frac{1}{S}\right]^U dy < 0 \tag{A-8}$$

$$\tilde{\eta}''(\theta) = (N-1)(N-2)\int_0^1 (y-1)^2[y\theta + 1 - \theta]^{N-3}\left[\frac{y}{S} + 1 - \frac{1}{S}\right]^U dy \geq 0, \tag{A-9}$$

so that condition (ii) is clearly satisfied. Condition (iii) is satisfied if and only if

$$2\left[\tilde{\eta}'(\theta)\right]^2 - \tilde{\eta}(\theta)\tilde{\eta}''(\theta) \geq 0 \tag{A-10}$$

[2] The inequality in (A-6) follows immediately from the fact that $\tilde{\eta}'(\theta) < 0$, which is established below.

[3] See Lemma 3 and its proof in Galenianos & Kircher (2009).

holds for $\theta \in (0,1)$, which is true.[4] Therefore, (A-5) uniquely determines the optimal price $p_d \in [0, \bar{p}]$ given p.

Given (A-7) and (A-8), it is easy to show that

$$\tilde{\eta}\left(\frac{1}{S}\right) = \frac{S\left[1-\left(1-\frac{1}{S}\right)^B\right]}{B} \tag{A-11}$$

$$\tilde{\eta}'\left(\frac{1}{S}\right) = -\frac{(N-1)S}{B-1}\left[\tilde{\eta}\left(\frac{1}{S}\right)-\left(1-\frac{1}{S}\right)^{B-1}\right]. \tag{A-12}$$

In order to characterize the symmetric strategy equilibrium, I impose the conditions $p_d = p$ and $\theta_d = 1/S$ to get

$$\frac{\partial \theta_d}{\partial p_d} = \frac{\tilde{\eta}\left(\frac{1}{S}\right)\left(1-\frac{1}{S}\right)}{\tilde{\eta}'\left(\frac{1}{S}\right)(1-p)}. \tag{A-13}$$

Substituting (A-13) into (A-5) and solving yields (16). Finally, one must check that a seller would not prefer to deviate to a price such that $\theta_s = 0$, but this is guaranteed precisely by the inequality in (15).

Having established that $\tilde{p}_L$ is the equilibrium price in this region of the parameter space, it is left to show that $\tilde{p}_L$ is increasing in N. However, since $\tilde{p}_L$ is of the form $\tilde{p}_L = \kappa_1/\{\kappa_1 + \kappa_2[N/(N-1)]\}$, where κ_1 and κ_2 are positive constants that depend only on B and S, it follows immediately that $\partial \tilde{p}_L/\partial N > 0$.[5]

[4] To prove this, note that the expression in (A-10) is weakly greater than

$$\left[\int_0^1 \left[\frac{y}{S}+1-\frac{1}{S}\right]^U dy\right] \times \left\{2(N-1)^2\left[\int_0^1 (y-1)\left[y\theta+1-\theta\right]^{N-2} dy\right]^2 - (N-1)(N-2)\left[\int_0^1 (y-1)^2\left[y\theta+1-\theta\right]^{N-3} dy\right]\left[\int_0^1 \left[y\theta+1-\theta\right]^{N-2} dy\right]\right\}.$$

This expression is decreasing in θ and converges to zero as $\theta \to 1$, so it is weakly positive for all $\theta \in [0,1]$.

[5] Clearly $\kappa_2 = [1-\tilde{\mu}(1/S)]\tilde{\eta}(1/S)[(B-1)/S] \geq 0$. To see that $\kappa_1 = \tilde{\mu}(1/S)[\tilde{\eta}(1/S)-(1-1/S)^{B-1}] \geq 0$, note that

$$\tilde{\eta}(1/S)-(1-1/S)^{B-1} \propto S-1-(S-1+B)\left(\frac{S-1}{S}\right)^B.$$

For any $\xi > 0$,

$$\left(1+\frac{1}{\xi}\right)^B \geq 1+\frac{B}{\xi} \quad \Rightarrow \quad \xi \geq (\xi+B)\left(\frac{\xi}{\xi+1}\right)^B.$$

Plugging in $\xi = S-1$ gives the desired result.

B Example: The Case of $N = 1$ and $U = 2$

Now suppose that only one buyer is informed, and that the other two buyers will select a seller at random. The optimal strategy of the informed buyer at the second stage is trivial: $\theta_1^*(p_1, p_2) = 0$ if $p_1 > p_2$, 1 if $p_1 < p_2$, and any value in $[0, 1]$ if $p_1 = p_2$.

Lemma 1. *A symmetric strategy equilibrium must be a distribution $F(p)$ that is (i) continuous, (ii) has connected support $\left[\frac{3}{4}, 1\right]$, and (iii) has the property that $F\left(\frac{3}{4}\right) = 0$.*

Proof: Note that a seller can always earn expected profits of at least 3/4 by setting a price equal to 1, so it can never be a best response to set a price $p < 3/4$ and we can restrict attention to prices in the domain $[3/4, 1]$. The proof proceeds as follows. That F has continuous, connected support over *some* interval $[p_{min}, p_{max}] \subseteq [3/4, 1]$ is stated without proof.[6] Given this, it will be established that (i) $p_{max} = 1$, (ii) $p_{min} = 3/4$, and (iii) $F(p_{min}) = 0$.

In a slight abuse of notation, denote the expected profits of seller s setting price p_s, given the other seller's strategy $F(\cdot)$, by

$$\pi_s(p_s; F) = p_s \left[1 - F(p_s) + F(p_s)\left(\frac{3}{4}\right)\right].$$

First, suppose that $p_{max} < 1$. Since $F(p_{max}) = 1$, $\pi_s(p_{max}; F) = \left(\frac{3}{4}\right) p_{max} < \left(\frac{3}{4}\right) 1 = \pi_s(1; F)$. Therefore, it must be that $p_{max} = 1$ and the expected equilibrium profits $\Pi_s^* = 3/4$. Second, suppose that $p_{min} > 3/4$, and let $\epsilon > 0$ be such that $p_{min} > 3/4 + \epsilon$. Then $F(3/4 + \epsilon) = 0$ and

$$\pi_s\left(\frac{3}{4} + \epsilon; F\right) = \frac{3}{4} + \epsilon > \frac{3}{4} = \Pi_s^*.$$

Therefore, it must be that $p_{min} = 3/4$. Lastly, suppose that $F(p_{min}) = \delta > 0$. Since $\theta_1^*(3/4, 3/4)$ can take on any value in the interval $[0, 1]$, it follows that

$$\min\left\{\pi_1\left(\frac{3}{4}; F\right), \pi_2\left(\frac{3}{4}; F\right)\right\} = \min\left\{\frac{3}{4}[\delta\theta_1^* + 1 - \delta], \frac{3}{4}[\delta(1 - \theta_1^*) + 1 - \delta]\right\} < \frac{3}{4},$$

a contradiction.■

Given this result, it is straight-forward to characterize the unique equilibrium strategy $F^*(p) = 4 - 3/p$ as the solution to the equality $\frac{3}{4} = \pi_s(p_s; F)$ for $p_s \in [3/4, 1]$. In this equilibrium, the average price is $\tilde{p}_L^C = \int_{3/4}^1 p dF^*(p) \approx .863$, and the average price paid by informed buyers $\mathbb{E}[\min\{p_1, p_2\}] = \int_{\frac{3}{4}}^1 [1 - F(p)]^2 dp + \frac{3}{4} \approx .822$.

[6] The argument is completely standard, following Burdett & Judd (1983).