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On the coincidence between the Shimomura's bargaining sets and the core

Josep M. Izquierdo, Carles Rafels

Adreça correspondència:

Departament de Matemàtica Econòmica, Financera i Actuarial
Facultat d'Economia i Empresa
Universitat de Barcelona
Av. Diagonal 690,
08034 BARCELONA
Telèfon + 34 93 4029090
Fax +34 93 4034893
e-mail: jizquierdoa@ub.edu, crafels@ub.edu

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Abstract

A necessary condition for the coincidence of the bargaining sets defined by Shimomura (1997) and the core of a cooperative game with transferable utility is provided. To this aim, a set of payoff vectors, called max-payoff vectors, are introduced. This necessary condition simply checks whether these vectors are core elements of the game.

Resum

En l'article es dona una condició necessària per a que els conjunts de negociacio definits per Shimomura (1997) i el nucli d'un joc cooperatiu amb utilitat transferible coincideixin. A tal efecte, s'introdueix el concepte de vectors de màxim pagament. La condició necessària consiteix a verificar que aquests vectors pertanyen al nucli del joc.

Keywords: Cooperative games ˙ Core ˙ Bargaining set ˙ Max-payoff vectors
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1 Introduction

An important issue in cooperative games with transferable utility is the problem of distributing the joint profit obtained by a set of agents. The core of the game consists of those distributions where each subgroup of agents is at least rewarded according its own capability to generate profit (that is, according its *worth*). Allocations outside the core of the game involve at least a subgroup of players that complain and would like to give up cooperation. From another perspective, the concept of bargaining set of a cooperative game refers to distributions where the complaint (or objection) of a subgroup of players is countered by the complaint (or counterobjection) of another coalition.

In the literature, several definitions of bargaining set have been introduced: the seminal one by Davis and Maschler (1967), the Mas-Colell bargaining set (1989), the Zhou bargaining set (1994), the reactive bargaining set (Granot, 2010) and the semireactive bargaining set (Sudhölter and Potters, 2001) are among the most relevant. For some classes of games with a non-empty core, some results on the coincidence between the core and the bargaining sets have been proved, mainly due to the rich structure of these games. This is the case of convex games (Shapley, 1971) with respect to the Davis-Maschler bargaining set (see Maschler et al., 1971) and with respect to the Mas-Colell bargaining set (see Dutta et al., 1989). It is also the case of average monotonic games (Izquierdo and Rafels, 2001) and of the assignment games (Shapley and Shubik, 1972) with respect to the Davis-Maschler and the Mas-Colell bargaining sets (see Solymosi, 1999 and 2008, respectively). Nevertheless, it is rather cumbersome to check whether, for an arbitrary game, these types of bargaining sets do or do not coincide with the core. Following this, Solymosi (1999) and Holzman (2001) give a necessary and sufficient condition so that the core of a game equals its Davis-Maschler bargaining set or its Mas-Colell bargaining set, respectively. The aim of this paper is to shed light on whether the Zhou bargaining set (the most unknown in terms of equivalences) coincides with the core, for an arbitrary game.

Besides the particular requirements on the counterobjecting coalitions, the Zhou bargaining set was originally defined not assuming that all players would finally join the grand coalition but allowing players to form subgroups (technically speaking, considering *coalition structures*). After Zhou's definition, Shimomura (1997) has introduced slightly modifications on the Zhou bargaining set that essentially imply three differences: firstly, that the grand coalition of all players forms; secondly, that players only will collaborate in objections and counterobjections if strictly higher payoffs are implemented; and, finally, that the final payoff should be *individually rational*. The first and third requirements make easier the comparison with other bargaining sets; the second one points out that individual incentives are important. The present paper analyzes this modification of the Zhou bargaining set and a similar one for the Mas-Colell bargaining set, also introduced by Shimomura, where the above three requirements are imposed.

We provide a necessary condition to check the coincidence between the core and the Shimomura's bargaining sets. This condition is based on the so-called *max-payoff* vectors. These vectors assign, following an a priori ordering, a minimum payoff to players preserving some core constraints. The coincidence between the Shimomura's bargaining sets and the core of a game implies that all max-payoff vectors are core elements of the game, that is, all core constraints must be satisfied.

In Section 2 some preliminary definitions and notations are given. In Section 3 we introduce and illustrate the computation of the max-payoff vectors and we state the main result of the paper. Section 4 concludes by analyzing the case of assignment games.

2 Notations and definitions

Let $N = \{1, 2, \ldots, n\}$ be a set of players. For all coalition $S \subseteq N$, $|S|$ denotes the number of players in S. A cooperative game with player set N is a function $v : 2^N \to \mathbb{R}$ assigning to each coalition $S \subseteq N$ a real number $v(S)$ such that $v(\varnothing) := 0$. The function v is called the characteristic function of the game and $v(S)$ is the worth of the coalition S. Let $\mathcal{G}^N$ be the class of cooperative games with transferable utility and player set N.

A game $v \in \mathcal{G}^N$ is monotonic if for all $S \subseteq T \subseteq N$, $v(S) \leq v(T)$. The monotonic cover of a game $v \in \mathcal{G}^N$ is the game $\hat{v} \in \mathcal{G}^N$ defined as $\hat{v}(S) := \max_{R \subseteq S}\{v(R)\}$. By definition, the monotonic cover is a monotonic game.

Let $\mathbb{R}^N$ stand for the real space of vectors $x = (x_i)_{i\in N}$ where x_i is interpreted as the payoff to player $i \in N$, x_S is the restriction of x to the members of S and $x(S)$ denotes $\sum_{i\in S} x_i$, with the convention $x(\varnothing) = 0$. The set of preimputations of a game $v \in \mathcal{G}^N$ is defined by $I^*(v) := \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$. The set of imputations of v is defined by $I(v) := \{x \in \mathbb{R}^N \mid x(N) = v(N)$ and $x_i \geq v(\{i\})$, for all $i \in N\}$ and its core is defined by $C(v) := \{x \in \mathbb{R}^N \mid x(N) = v(N)$ and $x(S) \geq v(S)$, for all $S \subseteq N\}$. A game with a non-empty core is called a balanced game. Let $\mathcal{B}^N \subseteq \mathcal{G}^N$ be the subclass of balanced games with player set N. Given a game v, a preimputation $x \in I^*(v)$ and a pair of players i and j, the maximum surplus of i against j at x is defined as

$$s^v_{ij}(x) = \max\{v(S) - x(S)|S \subseteq N,\, i \in S,\, j \notin S\}.$$

We say that player i outweighs player j at x if $s^v_{ij}(x) > s^v_{ji}(x)$. The prekernel of a game v (Davis and Maschler, 1965), $\mathcal{PK}(v)$, is always non-empty and consists of those preimputations x such that no player outweighs any other player at x. This is

$$\mathcal{PK}(v) = \{x \in I^*(v) \mid \text{ for all } i, j \in N, s^v_{ij}(x) = s^v_{ji}(x)\}.$$

As usual, the bargaining set is defined by means of an interaction of objections and counterobjections. Shimomura (1997) considers modifications of both the Mas-Colell bargaining set (1989) and Zhou bargaining set (1994). To define them, let $v \in \mathcal{G}^N$ and $x \in I(v)$. Following Shimomura (1997), an *objection* to x is a pair (S, y), $\varnothing \neq S \subseteq N$ and $y \in \mathbb{R}^S$ with $y(S) = v(S)$ such that $y_i > x_i$, for all $i \in S$. Shimomura also qualifies the original definitions of counterobjection: a *counterobjection* to an objection(S, y) *à la* Mas-Colell is now a pair (T, z), $\varnothing \neq T \subseteq N$, $z \in \mathbb{R}^T$ with $z(T) = v(T)$ such that $z_i > y_i$, for all $i \in T \cap S$, and $z_i > x_i$, for all $i \in T \setminus S$; on the other hand, a *counterobjection* to (S, y) *à la* Zhou is a pair (T, z), where $T \setminus S \neq \varnothing$, $S \setminus T \neq \varnothing$, $T \cap S \neq \varnothing$, and $z \in \mathbb{R}^T$ with $z(T) = v(T)$ such that $z_i > y_i$, for all $i \in T \cap S$, and $z_i > x_i$, for all $i \in T \setminus S$. Note that the bargaining process starts with an imputation x and involves strictly higher payoffs not only for all players involved in objections but also in counterobjections, and these are the main changes with respect to the original definitions.

Definition 1 *The Mas-Colell bargaining set (*à la *Shimomura) is defined as*

$$\mathcal{MB}_{Sh}(v) = \{x \in I(v) \mid \textit{each objection to } x \textit{ can be countered} \text{ à la } \textit{Mas-Colell}\}.$$

Definition 2 *The Zhou bargaining set (*à la *Shimomura) is defined as*

$$\mathcal{Z}_{Sh}(v) = \{x \in I(v) \mid \textit{each objection to } x \textit{ can be countered} \text{ à la } \textit{Zhou}\}.$$

If no confusion arises, we will refer to them simply as the Mas-Colell bargaining set and the Zhou bargaining set. By definition, these sets only consist of imputations (individually rational payoff vectors) and they always include the core. If the core is nonempty, obviously these bargaining sets are non-empty. Moreover, the following chain of inclusions holds: $C(v) \subseteq \mathcal{Z}_{Sh}(v) \subseteq \mathcal{MB}_{Sh}(v)$.

Shimomura also defines a subset of the Zhou bargaining set (the *steady bargaining set*, $\mathcal{SB}(v)$) by means of a domination binary relation between coalitions. He shows that the steady bargaining set can be rewritten as

$$\mathcal{SB}(v) := \left\{ x \in I(v) \;\middle|\; \begin{array}{l} \text{for all } S \subseteq N \text{ with } v(S) - x(S) > 0, \\ \text{there exists } T \subseteq N \,:\, S \cap T \neq \varnothing,\ S \setminus T \neq \varnothing, \\ T \setminus S \neq \varnothing \text{ and } v(T) - x(T) \geq v(S) - x(S) \end{array} \right\} \qquad (1)$$

and he also proves that $\mathcal{SB}(v) \subseteq \mathcal{Z}_{Sh}(v)$. This subsolution of the Zhou bargaining set will be useful to prove our results, since some imputations will be known to belong to the bargaining set by checking their inclusion in the steady bargaining set.

3 A necessary condition for the coincidence

We start by analyzing a game with four players. The characteristic function is defined by $v(\{i\}) = 0$, for all $i \in \{1, 2, 3, 4\}$, and $v(S) = |S|$, for all $S \subseteq N$, $|S| \geq 2$. It is easy to check that the core of the game consists of a unique point: $C(v) = \{(1, 1, 1, 1)\}$. Now, let us suppose players are ordered exogenously – for instance, take the ordering $\theta = (2, 3, 1, 4)$ – and let us assign a payoff to each player as follows. Give the first player, player 2, her individual worth:

$$x_2^\theta = v(\{2\}) = 0.$$

Then, the second player in the ordering, player 3, takes the maximum payoff of either staying alone or joining player 2, forming the coalition $\{2, 3\}$, and obtaining the worth of the coalition minus the payoff x_2^θ assigned to player 2 previously; hence,

$$x_3^\theta = \max\{v(\{3\}), v(\{2, 3\}) - x_2^\theta)\} = 2.$$

Player 1 comes in third place and also chooses among staying alone or joining a subset S (not necessarily all) of his predecessors and obtaining what is left from the worth of this coalition S after paying x_i^θ to each predecessor i in S. This is,

$$x_1^\theta = \max\{v(\{1\}), v(\{1, 2\}) - x_2^\theta, v(\{1, 3\}) - x_3^\theta, v(\{1, 2, 3\}) - x_2^\theta - x_3^\theta\} = 2.$$

Finally, the last player is just given what is left to reach efficiency, this is

$$x_4^\theta = v(N) - x_1^\theta - x_2^\theta - x_3^\theta = 0.$$

Thus, the payoff vector obtained is $x^\theta(v) = (2, 0, 2, 0)$. Notice $x^\theta(v)$ is not in the core of the game since $x_2^\theta + x_4^\theta = 0 < v(\{2, 4\})$. The claim we prove in the paper is that the Shimomura's bargaining sets of a game do not coincide with its core since at least one max-payoff vector relative to some ordering does not belong to the core. To this end we define several notions.

Let $v \in \mathcal{G}^N$ and let θ be an ordering of players in N, that is a bijection $\theta : \{1, \ldots, n\} \to N$ where $\theta(k) = i_k \in N$; we denote it by $\theta = (i_1, \ldots, i_n)$ and by Θ_N the set of all such orderings. Furthermore, given $\theta = (i_1, \ldots, i_n) \in \Theta_N$, let us define the set of predecessors of a player i_k by $P_{i_k}^\theta := \{i_1, \ldots, i_{k-1}\}$, for all $k \in \{2, \ldots, n\}$, while $P_{i_1}^\theta := \varnothing$. The set of followers of a player i_k is defined by $F_{i_k}^\theta := \{i_k, \ldots, i_n\}$, for all $k \in \{1, \ldots, n\}$.

We say that $x \in \mathbb{R}^N$ lexicographically precedes $y \in \mathbb{R}^N$ with respect to θ, $x \prec_\ell^\theta y$, if either $x_{i_1} < y_{i_1}$ or there exists $k \in \{2, \ldots, n\}$ such that $x_{i_k} < y_{i_k}$ and $x_{i_r} = y_{i_r}$, for all $r \in \{1, \ldots, k-1\}$. The *lexmin* solution over the core of a balanced game v relative to $\theta \in \Theta_N$ is defined as the (unique) payoff vector $\ell^\theta(v) \in C(v)$ that lexicographically precedes any other vector in the core of the game v, i.e. $\ell^\theta(v) \prec_\ell^\theta x$ for all $x \in C(v)$, $x \neq \ell^\theta(v)$.

A formula to compute the lexmin solution of an arbitrary game is not available for the general case. Nevertheless, we can define a recursive formula to obtain a payoff vector (we call it the *max-payoff vector*) such that, whenever it is in the core, it coincides with the lexmin solution.

The max-payoff vector $x^\theta(v) \in \mathbb{R}^N$ of v relative to θ is defined by

$$x^\theta_{i_k} := \max_{Q \subseteq P^\theta_{i_k}} \{v(\{i_k\} \cup Q) - x^\theta(Q)\}, \quad \text{for all } k \in \{1, \ldots, n-1\}, \text{ and}$$

$$x^\theta_{i_n} := v(N) - x^\theta(N \setminus \{i_n\}).$$

The relationship between the lexmin and the max-payoff vectors is summarized in the next proposition.

Proposition 1 *Let $v \in \mathcal{B}^N$ and $\theta \in \Theta_N$. Then*

$$x^\theta(v) \in C(v) \Leftrightarrow \ell^\theta(v) = x^\theta(v).$$

Proof We only need to prove the only if part. Suppose the vectors $x^\theta(v)$ and $\ell^\theta(v)$ are not equal, where $\theta = (i_1, \ldots, i_n)$. Comparing $x^\theta(v)$ and $\ell^\theta(v)$, and following the ordering θ, let player i_k be the first player with different payoffs. That is, $x^\theta_{i_1} = \ell^\theta_{i_1}, \ldots, x^\theta_{i_{k-1}} = \ell^\theta_{i_{k-1}}$ and $x^\theta_{i_k} \neq \ell^\theta_{i_k}$; notice $k \neq n$. If $x^\theta_{i_k} < \ell^\theta_{i_k}$ then $x^\theta(v) \prec^\theta_\ell \ell^\theta(v)$ but this contradicts the definition of the lexmin solution. If $x^\theta_{i_k} > \ell^\theta_{i_k}$, by definition of max-payoff vector $\ell^\theta_{i_k} < x^\theta_{i_k} = \max_{Q \subseteq P^\theta_{i_k}} \{v(\{i_k\} \cup Q) - x^\theta(Q)\} = v(\{i_k\} \cup Q^*) - x^\theta(Q^*) = v(\{i_k\} \cup Q^*) - \ell^\theta(Q^*)$ for some $Q^* \subseteq P^\theta_{i_k}$. Hence, $\ell^\theta(\{i_k\} \cup Q^*) < v(\{i_k\} \cup Q^*)$ which contradicts $\ell^\theta(v)$ to be a core element. $\square$

The coincidence between the lexmin and the max-payoff vector is in fact a necessary condition for the coincidence of the bargaining sets and the core.

Theorem 1 *For any arbitrary balanced game $v \in \mathcal{B}^N$ we have:*

1. *If $C(v) = \mathcal{Z}_{Sh}(v)$, then $x^\theta(v) \in C(v)$, for all $\theta \in \Theta_N$*

2. *If $C(v) = \mathcal{MB}_{Sh}(v)$, then $x^\theta(v) \in C(v)$, for all $\theta \in \Theta_N$.*

Proof Let us first prove item 1. For $|N| \leq 2$ the result is trivial. For $|N| \geq 3$, let us suppose we have $C(v) = \mathcal{Z}_{Sh}(v)$ and there exists an ordering $\theta = (i_1, i_2, \ldots, i_n) \in \Theta_N$ such that $x^\theta(v) \notin C(v)$ or equivalently, by Proposition 1, $\ell^\theta(v) \neq x^\theta(v)$. Therefore, there exists $k \in \{1, 2, \ldots, n-1\}$ such that $\ell^\theta_{i_k} \neq x^\theta_{i_k}$ and, for all $i_r \in P^\theta_{i_k}$, $\ell^\theta_{i_r}(v) = x^\theta_{i_r}(v)$. In fact, $\ell^\theta_{i_k} \neq x^\theta_{i_k}$ implies $\ell^\theta_{i_k} > x^\theta_{i_k}$, since $\ell^\theta(v)$ belongs to $C(v)$.

We first claim that $k \neq n-1$. To check this, let us suppose $k = n-1$ and so $\ell^\theta_{i_{n-1}} \neq x^\theta_{i_{n-1}}$ and $\ell^\theta_{i_r} = x^\theta_{i_r}$, for all $r \in \{1, 2, \ldots, n-2\}$. In fact, we have $\ell^\theta_{i_{n-1}} > x^\theta_{i_{n-1}}$. Hence, define $x \in \mathbb{R}^N$ as $x_{i_r} := \ell^\theta_{i_r}$, for all $r \in \{1, \ldots, n-2\}$, $x_{i_{n-1}} := \ell^\theta_{i_{n-1}} - \varepsilon$ and $x_{i_n} := \ell^\theta_{i_n} + \varepsilon$, where $0 < \varepsilon < \ell^\theta_{i_{n-1}} - x^\theta_{i_{n-1}}$. It is easy to check that $x \in C(v)$ and $x \prec^\theta_\ell \ell^\theta(v)$ which is a contradiction. Therefore, $k \leq n-2$.

Now, take $\varepsilon \in \mathbb{R}$ such that

$$0 < \varepsilon < \min \left\{ \ell^\theta_{i_k} - x^\theta_{i_k}, \min_{\substack{S \subseteq N : i_k \in S \\ \ell^\theta(S) - v(S) > 0}} \{\ell^\theta(S) - v(S)\} \right\}$$

and define the payoff vector $\alpha \in \mathbb{R}^N$ as $\alpha_{i_k} := \ell^\theta_{i_k} - \varepsilon$ and $\alpha_i := \ell^\theta_i$, if $i \in N \setminus \{i_k\}$. Let us remark that $\alpha(N) = v(N) - \varepsilon < v(N)$. Furthermore, let us define the excess game $(F^\theta_{i_{k+1}}, e_\alpha)$ as follows,

$$e_\alpha(\varnothing) := 0,$$

$$e_\alpha(R) := \max_{Q \subseteq P^\theta_{i_{k+1}}} \{0, v(R \cup Q) - \alpha(R \cup Q)\}, \quad \text{for all } \varnothing \neq R \subseteq F^\theta_{i_{k+1}},$$

and consider its monotonic cover $(F^\theta_{i_{k+1}}, \hat{e}_\alpha)$. Notice

$$\hat{e}_\alpha(R) \in \{0, \varepsilon\}, \text{ for all } R \subseteq F^\theta_{i_{k+1}}. \tag{2}$$

To see this, recall that $\hat{e}_\alpha(R) = v(R' \cup Q) - \alpha(R' \cup Q)$, for some $R' \subseteq R$ and $Q \subseteq P^\theta_{i_{k+1}}$. If $i_k \notin Q$ then $v(R' \cup Q) - \alpha(R' \cup Q) = v(R' \cup Q) - \ell^\theta(R' \cup Q) \leq 0$. If $i_k \in Q$ and $v(R' \cup Q) - \ell^\theta(R' \cup Q) < 0$ then, by definition of ε, $v(R' \cup Q) - \alpha(R' \cup Q) < 0$. Finally, if $i_k \in Q$ and $v(R' \cup Q) - \ell^\theta(R' \cup Q) = 0$ then $v(R' \cup Q) - \alpha(R' \cup Q) = \varepsilon$. Moreover, it holds that $\hat{e}_\alpha(F^\theta_{i_{k+1}}) = v(N) - \alpha(N) = \varepsilon$, just taking $Q = P^\theta_{i_{k+1}}$ in its definition. Hence, let us define

$$\mathcal{W} := \{R \subseteq F^\theta_{i_{k+1}} \mid \hat{e}_\alpha(R) = \varepsilon \text{ and } \hat{e}_\alpha(R') = 0 \text{ for all } R' \subsetneq R\}.$$

Notice, since $(F^\theta_{i_{k+1}}, \hat{e}_\alpha)$ is a monotonic game, $\hat{e}_\alpha(R \cup \{i\}) - \hat{e}_\alpha(R) \geq 0$, for all $i \in F^\theta_{i_{k+1}}$ and $R \subseteq F^\theta_{i_{k+1}} \setminus \{i\}$. Now, choose an element of the prekernel of this game, let us say $\delta \in \mathcal{PK}(\hat{e}_\alpha)$. By Theorem 5.6.1. in Peleg and Südholter (2007) we know that, for all player $i \in F^\theta_{i_{k+1}}$ and for all element in the prekernel of $(F^\theta_{i_{k+1}}, \hat{e}_\alpha)$, his payoff is bounded below by $m_i(\hat{e}_\alpha) = \min_{S \subseteq F^\theta_{i_{k+1}} \setminus \{i\}} \{\hat{e}_\alpha(S \cup \{i\}) - \hat{e}_\alpha(S)\}$ and bounded above by $M_i(\hat{e}_\alpha) = \max_{S \subseteq F^\theta_{i_{k+1}} \setminus \{i\}} \{\hat{e}_\alpha(S \cup \{i\}) - \hat{e}_\alpha(S)\}$. Therefore, for all $\delta \in \mathcal{PK}(\hat{e}_\alpha)$,

$$0 \leq m_i(\hat{e}_\alpha) \leq \delta_i \leq M_i(\hat{e}_\alpha) \leq \varepsilon, \text{ for all } i \in F^\theta_{i_{k+1}}.$$

It is easy to check that $\hat{e}_\alpha(M \cup \{i\}) - \hat{e}_\alpha(M) = 0$, for all $i \in F^\theta_{i_{k+1}} \setminus \bigcup_{R \in \mathcal{W}} R$ and all $M \subseteq F^\theta_{i_{k+1}} \setminus \{i\}$. To prove it, if $\hat{e}_\alpha(M) = \varepsilon$, by monotonicity of the game $(F^\theta_{i_{k+1}}, \hat{e}_\alpha)$, we get $\hat{e}_\alpha(M \cup \{i\}) = \varepsilon$. If $\hat{e}_\alpha(M) = 0$, then $R \not\subseteq M$, for all $R \in \mathcal{W}$. But then $\hat{e}_\alpha(M \cup \{i\}) = 0$, since otherwise $M' \cup \{i\} \in \mathcal{W}$, for some $M' \subseteq M$, and this contradicts $i \in F^\theta_{i_{k+1}} \setminus \bigcup_{R \in \mathcal{W}} R$. As a consequence,

$$\delta_i = 0, \text{ for all } i \in F^\theta_{i_{k+1}} \setminus \bigcup_{R \in \mathcal{W}} R. \tag{3}$$

Now, define the payoff-vector $x \in \mathbb{R}^N$ as $x_{i_r} = \alpha_{i_r} = \ell^\theta_{i_r}$, for all $r \in \{1, \ldots, k-1\}$, $x_{i_k} = \alpha_{i_k} = \ell^\theta_{i_k} - \varepsilon$ and $x_{i_r} = \alpha_{i_r} + \delta_{i_r} = \ell^\theta_{i_r} + \delta_{i_r}$, $r \in \{k+1, \ldots, n\}$. It holds $x(N) = v(N)$ and $x_i \geq v(\{i\})$, for all $i \in N$. However, $x \notin C(v)$ since otherwise $x \prec^\theta_\ell \ell^\theta(v)$ as $x_{i_r} = \ell^\theta_{i_r}$ for all $r \in \{1, \ldots, k-1\}$ and $x_{i_k} < \ell^\theta_{i_k}$. We want to prove that $x \in \mathcal{Z}_{Sh}(v)$, and in the first place we prove some basic properties of x:

(a) If $S \subseteq N$ is such that $v(S) - x(S) > 0$, then $i_k \in S$. Otherwise, $v(S) - x(S) = v(S) - \ell^\theta(S) - \delta(S \cap F^\theta_{i_{k+1}}) \leq v(S) - \ell^\theta(S) \leq 0$, where the last inequality follows from, $\ell^\theta(v) \in C(v)$.

(b) If $S \subseteq N$ and $v(S) - x(S) > 0$, then $S \cap F^\theta_{i_{k+1}} \neq \varnothing$. Otherwise, by (a), $x(S) = \ell^\theta_{i_k} - \varepsilon + x(S \setminus \{i_k\}) > x^\theta_{i_k} + x^\theta(S \setminus \{i_k\}) = x^\theta(S) \geq v(S)$, where the strict inequality follows from the definition of ε and the last inequality by the definition of $x^\theta(v)$.

(c) If $S \subseteq N$ and $v(S) - x(S) > 0$, then $\hat{e}_\alpha(S \cap F^\theta_{i_{k+1}}) = \varepsilon$. Otherwise, by (2), $\hat{e}_\alpha(S \cap F^\theta_{i_{k+1}}) = 0$ and so $0 < v(S) - x(S) \leq v(S) - \alpha(S) \leq \hat{e}_\alpha(S \cap F^\theta_{i_{k+1}}) = 0$, where the second inequality holds since $x \geq \alpha$ and the third one just by definition of the monotonic cover of e_α.

Let $S \subseteq N$ be an arbitrary coalition with positive excess, i.e. $v(S) - x(S) > 0$. By property (b), $S \cap F^\theta_{i_{k+1}} \neq \varnothing$. Let us take now $R_S \in \mathcal{W}$ such that

$$\delta(R_S \setminus (S \cap F^\theta_{i_{k+1}})) \neq 0. \tag{4}$$

Such a coalition R_S exists, since otherwise $\delta_r = 0$ for all $r \in R \setminus (S \cap F^\theta_{i_{k+1}})$ and all $R \in \mathcal{W}$, and so $\delta(\bigcup_{R \in \mathcal{W}}(R \setminus (S \cap F^\theta_{i_{k+1}}))) = \delta(\bigcup_{R \in \mathcal{W}} R \setminus (S \cap F^\theta_{i_{k+1}})) = 0$. But then

$$\begin{aligned}\varepsilon = \quad & \delta(\textstyle\bigcup_{R \in \mathcal{W}} R) = \delta((\bigcup_{R \in \mathcal{W}} R) \setminus (S \cap F^\theta_{i_{k+1}})) + \delta((\bigcup_{R \in \mathcal{W}} R) \cap (S \cap F^\theta_{i_{k+1}})) \\ & = \delta((\textstyle\bigcup_{R \in \mathcal{W}} R) \cap (S \cap F^\theta_{i_{k+1}})) \leq \delta(S \cap F^\theta_{i_{k+1}}) \leq \varepsilon,\end{aligned}$$

where the first equality follows from (3) and the last inequality follows from $\delta_i \geq 0$, for all $i \in F^\theta_{i_{k+1}}$, and $\delta(F^\theta_{i_{k+1}}) = \varepsilon$. Therefore, we conclude $\delta(S \cap F^\theta_{i_{k+1}}) = \varepsilon$, but then $v(S) - x(S) = v(S) - \alpha(S) - \delta(S \cap F^\theta_{i_{k+1}}) = v(S) - \ell^\theta(S) + \varepsilon - \varepsilon = v(S) - \ell^\theta(S) \leq 0$, contradicting S to be a coalition with strictly positive excess with respect to v at x.

Now, let us choose a coalition $S' \in \mathcal{W}$, $S' \subseteq S \cap F^\theta_{i_{k+1}}$, with the largest excess at δ , i.e.

$$\hat{e}_\alpha(S') - \delta(S') \geq \hat{e}_\alpha(S'') - \delta(S''), \tag{5}$$

for all $S'' \in \mathcal{W}$ with $S'' \subseteq S \cap F^{\theta}_{i_{k+1}}$. Notice that the existence of S' is guaranteed by property (c).

Since S' and R_S are in $\mathcal{W}$, $S' \setminus R_S \neq \varnothing$ and $R_S \setminus S' \neq \varnothing$. Taking into account $\delta(R_S \setminus (S \cap F^{\theta}_{i_{k+1}})) \neq 0$, let us now select

$$i \in S' \setminus R_S \text{ and } j \in R_S \setminus (S \cap F^{\theta}_{i_{k+1}}) \text{ such that } \delta_j > 0. \tag{6}$$

Then, since $\hat{e}_\alpha(S \cap F^{\theta}_{i_{k+1}}) = \hat{e}_\alpha(S') = \varepsilon$ and $\delta(S') \leq \delta(S \cap F^{\theta}_{i_{k+1}})$, we have

$$\begin{aligned} 0 < v(S) - x(S) &= v(S) - \alpha(S) - \delta(S \cap F^{\theta}_{i_{k+1}}) \\ &\leq \hat{e}_\alpha(S \cap F^{\theta}_{i_{k+1}}) - \delta(S \cap F^{\theta}_{i_{k+1}}) \\ &\leq \hat{e}_\alpha(S') - \delta(S') \leq s^{\hat{e}_\alpha}_{ij}(\delta) = s^{\hat{e}_\alpha}_{ji}(\delta) \\ &= \hat{e}_\alpha(R') - \delta(R'), \end{aligned} \tag{7}$$

where $R' \subseteq F^{\theta}_{i_{k+1}}$, $j \in R'$ but $i \notin R'$. As a consequence of (7) and $\delta_i \geq 0$, for all $i \in F^{\theta}_{i_{k+1}}$ we obtain $\hat{e}_\alpha(R') > 0$ which implies

$$\hat{e}_\alpha(R') = e_\alpha(R'') = \varepsilon > 0, \text{ for some } R'' \subseteq R'. \tag{8}$$

Notice that $i \notin R''$, since $i \notin R'$ and $R'' \subseteq R'$. Moreover, by (8),

$$\begin{aligned} \hat{e}_\alpha(R') - \delta(R') &= e_\alpha(R'') - \delta(R') \\ &= v(R'' \cup Q'') - \alpha(R'' \cup Q'') - \delta(R'), \end{aligned} \tag{9}$$

for some $Q'' \subseteq P^{\theta}_{i_{k+1}}$.

If we set $T = R'' \cup Q''$ we obtain, by (7) and (9),

$$\begin{aligned} 0 &< v(S) - x(S) \leq v(T) - \alpha(T) - \delta(R') \leq v(T) - \alpha(T) - \delta(R'') \\ &= v(T) - x(T), \end{aligned} \tag{10}$$

where the last inequality follows from $R'' \subseteq R'$ and $\delta_i \geq 0$, for all $i \in F^{\theta}_{i_{k+1}}$. Moreover, notice that $i \in S \setminus T \neq \varnothing$ since, by (6), $i \in S' \subseteq S \cap F^{\theta}_{i_{k+1}}$ and $i \notin R''$.

On the other hand, we claim $T \setminus S \neq \varnothing$. To check this, we first prove that $R'' \setminus (S \cap F^{\theta}_{i_{k+1}}) \neq \varnothing$. If $j \in R''$, then the statement is proved since, by (6), $j \notin S \cap F^{\theta}_{i_{k+1}}$. Let us suppose $j \notin R''$ and $R'' \setminus (S \cap F^{\theta}_{i_{k+1}}) = \varnothing$ and so $R'' \subseteq S \cap F^{\theta}_{i_{k+1}}$. By (7) and (9), we have

$$\begin{aligned} 0 < \hat{e}_{\alpha}(S') - \delta(S') &\leq v(R'' \cup Q'') - \alpha(R'' \cup Q'') - \delta(R') \\ &< v(R'' \cup Q'') - \alpha(R'' \cup Q'') - \delta(R'') \\ &\leq \hat{e}_{\alpha}(R'') - \delta(R''), \end{aligned}$$

where the second strict inequality follows from the fact that $j \in R'$, $\delta_j > 0$ and we are supposing $j \notin R''$. But this contradicts (5). Hence, $\varnothing \neq R'' \setminus (S \cap F^{\theta}_{i_{k+1}}) \subseteq T \setminus S$.

Finally, by property (a), $S \cap T \neq \varnothing$ since $v(S) - x(S) > 0$ and $v(T) - x(T) > 0$. Therefore, since $S \setminus T \neq \varnothing$, $T \setminus S \neq \varnothing$, $S \cap T \neq \varnothing$ and (10) we have proved that x is in the steady bargaining set and so in the Zhou bargaining set, $x \in \mathcal{SB}(v) \subseteq \mathcal{Z}_{Sh}(v)$.

To sum up, $x \in \mathcal{Z}_{Sh}(v) \setminus C(v)$ which involves a contradiction with the initial hypothesis. The proof of item 2. is straightforward since $C(v) \subseteq \mathcal{Z}_{Sh}(v) \subseteq \mathcal{MB}_{Sh}(v)$. $\square$

The condition stated in the theorem is necessary for the coincidence of the core and the bargaining sets, but not sufficient as we will see in the next final section.

4 An application to assignment games

We end the paper with an application of Theorem 1 to the case of assignment games (Shapley and Shubik, 1972). Two-sided assignment games represent two-sided markets (buyers and sellers) where each buyer-seller pair obtains a non-negative gain of trading. Assuming that each agent of one side can only trade with one agent of the opposite side, the problem at issue is firstly to find an optimal matching between buyers and sellers such that the joint profit is maximized; and secondly to allocate this profit among agents taking into account the joint optimal profit every submarket can obtain.

For example, consider the following market of 3 buyers (players 1, 2 and 3) and 3 sellers (players 4, 5 and 6). Matrix A summarizes the gain of each pair of agents:

$$
\begin{array}{c|ccc|}
 & 4 & 5 & 6 \\
\hline
1 & 3 & 1 & 4 \\
2 & 1 & 6 & 1 \\
3 & 1 & 1 & 3 \\
\hline
\end{array}
$$

Notice the only optimal matching is located along the main diagonal: player 1 trades with player 4, 2 trades with 5 and 3 trades with 6, with a total gain of $a_{14}+a_{25}+a_{36}=12$.

A cooperative game (N, w_A) can be associated by defining $w_A(N) = 12$ and computing, for every subcoalition of agents $S \subseteq N$, the profit of an optimal matching of the submarket restricted to agents in S. Notice that $w_A(S) = 0$, if S consists either only of buyers or only of sellers, since no trade is possible in these cases. The reader may easily check, for instance, that $w_A(\{2,6\}) = a_{26} = 1$ and $w_A(\{1,3,6\}) = a_{16} = 4$.

Any efficient allocation $(x_1, x_2, x_3, x_4, x_5, x_6)$ of the total profit $w_A(N) = 12$, assigns $x_1 + x_4 = 3$, $x_2 + x_5 = 6$ and $x_3 + x_6 = 3$. However, notice that if we want x to be in the core of the assignment game w_A, $x_6 \leq 3$, since $x_3 \geq 0$ and $x_3 + x_6 = 3$. But then, if $x_6 \leq 3$, $x_1 \geq 1$ since players 1 and 6 can obtain $a_{16} = 4$. This remark is crucial since any max-payoff vector associated to an ordering starting with player 1 will not be in the core of the game. To check this, take $\theta = (i_1, \ldots, i_6) \in \Theta_N$ such that $i_1 = 1$ and notice $x^\theta_{i_1} = w_A(\{1\}) = 0$ which is smaller than the minimum core payoff to player 1. Hence, $x^\theta(v) \notin C(v)$ and, by Theorem 1, $C(v) \subsetneq \mathcal{Z}_{Sh}(v) \subseteq \mathcal{MB}_{Sh}(v)$.

Note that buyer 1 is optimally matched with seller 4, but the element a_{14} is not the maximum of row 1; it is said this matrix is not dominant diagonal. On the contrary, a matrix is *dominant diagonal* (see Solymosi and Raghavan, 2001) if, provided the optimal matching is placed on the main diagonal, each of its elements is the maximum of the corresponding row and column. They prove that all players achieve a zero payoff in the core of the assignment game if and only if the corresponding matrix is dominant diagonal. Hence, following the same reasoning as in the previous example, we can state a general result for the case of markets with the same number of buyers and sellers[1]:

$$\text{if } A \in M_n^+ \text{ is not dominant diagonal, then } C(w_A) \subsetneq \mathcal{Z}_{Sh}(w_A).$$

On the other hand, if A is dominant diagonal the max-payoff vectors might be or not core elements. But even in the case that all the max-payoffs vectors are in the core of the game, this is not sufficient to guarantee the coincidence between the core and the bargaining sets. To check this, consider the assignment game corresponding to the matrix B:

$$\begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{4} \\ \cline{2-3} 1 & 1 & 1 \\ \cline{2-3} 2 & 1 & 1 \\ \cline{2-3} \end{array}\ .$$

The associated cooperative game corresponds to the 2×2 glove market game. It can be checked that $x^\theta(w_B) \in C(w_B)$, for all $\theta \in \Theta_N$. However the core and the Zhou bargaining set *à la* Shimomura do not coincide. The core is $C(w_B) = \{(\alpha, \alpha, 1-\alpha, 1-\alpha) \mid 0 \le \alpha \le 1\}$ while the Mas-Colell and Zhou bargaining sets *à la* Shimomura are

$$\left\{ (\alpha, \beta, 1 - \tfrac{\alpha+\beta}{2}, 1 - \tfrac{\alpha+\beta}{2}) \,\middle|\, \begin{array}{c} 0 \le \alpha \le 1 \\ 0 \le \beta \le 1 \end{array} \right\} \cup \left\{ (1 - \tfrac{\gamma+\delta}{2}, 1 - \tfrac{\gamma+\delta}{2}, \gamma, \delta) \,\middle|\, \begin{array}{c} 0 \le \gamma \le 1 \\ 0 \le \delta \le 1 \end{array} \right\}.$$

This example proves that the condition stated in Theorem 1 is just necessary but not sufficient for the coincidence of the core and the Shimomura's bargaining sets. Furthermore, let us remark that the Davis-Maschler bargaining set of an assignment game always coincides with its core (see Solymosi, 1999), as this is the case of the above examples.

To end the paper, let us mention that Shapley and Shubik (1972) already point out in their seminal paper some weaknesses of the concept of the core applied to an assignment game and the necessity to explore the behavior of other solution concepts. This work is a contribution to this task, but it remains open a complete description and interpretation of the bargaining sets of Shimomura for this class of games.

[1] We denote by M_n^+ the set of non-negative square matrices.

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