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Abstract

We propose a new axiom, Weakest Collective Rationality (WCR) which is weaker than both Weak Pareto Optimality (WPO) in Nash (1950)'s original characterization and Strong Individual Rationality (SIR) in Roth (1977)'s characterization of the Nash bargaining solution. We then characterize the Nash solution by Symmetry (SYM), Scale Invariance (SI), Independence of Irrelevant Alternatives (IIA) and our Weakest Collective Rationality (WCR) axiom.

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1 Introduction

Nash's bargaining problem is a pair which consists of a convex and compact utility possibility set and a disagreement point; the latter is the utility allocation that results if no agreement is reached by both parties.

A bargaining solution selects a unique allocation from any given bargaining problem. Nash (1950) axiomatically characterized the first solution to the bargaining problem, the Nash solution N, by using Scale Invariance (SI), Symmetry (SYM), Independence of Irrelevant Alternatives (IIA), and Weak Pareto Optimality (WPO).

Roth (1977) showed that WPO can be replaced by Strong Individual Rationality (SIR) in the characterization of N. Although neither WPO nor SIR imply each other,

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intuitively Roth's characterization has been considered as a very significant improvement over Nash's original characterization.

We propose a new axiom, Weakest Collective Rationality (WCR), which is weaker than both WPO and SIR. We then show that combining WCR with SI, SYM and IIA uniquely characterizes N.

2 Basic Definitions and Our Result

A two-person *bargaining problem* is a pair (S, d), where $S \subset \mathbb{R}^2$ is the *set of utility possibilities*, and $d \in S$ is the *disagreement point*, which is the utility allocation that results if no agreement is reached by both parties. It is assumed that (1) S is compact and convex, and (2) $x > d$ for some $x \in S$.[1] Let Σ be the class of all two-person problems satisfying (1) and (2) above. Define $IR(S, d) \equiv \{x \in S | x \geq d\}$, and $WPO(S) \equiv \{x \in S | \forall x' \in \mathbb{R}^2 \text{ and } x' > x \Rightarrow x' \notin S\}$. A *solution* is a function $f : \Sigma \to \mathbb{R}^2$ such that for all $(S, d) \in \Sigma$, $f \in S$. The Nash solution, N, is such that its outcome for each $(S, d) \in \Sigma$ is given by $N(S, d) = \arg\max\{(x_1 - d_1)(x_2 - d_2) | x \in IR(S, d)\}$.

Nash (1950) showed that N is the unique solution that satisfies the following four axioms:

Weak Pareto Optimality (WPO): For all $(S, d) \in \Sigma$, $f(S, d) \in WPO(S)$.

Symmetry (SYM): For all $(S, d) \in \Sigma$, if $[d_1 = d_2$, and $(x, y) \in S \Rightarrow (y, x) \in S]$, then $f_1(S, d) = f_2(S, d)$.

Let $T = (T_1, T_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a positive affine transformation if $T(x_1, x_2) = (a_1 x_1 + b_1, a_2 x_2 + b_2)$ for some positive constants a_i and b_i.

Scale Invariance (SI): For all $(S, d) \in \Sigma$ and a positive affine transformation T, $f(T(S, d)) = T(f(S, d))$ holds.

Independence of Irrelevant Alternatives (IIA) For all (S, d), $(T, e) \in \Sigma$ with $d = e$, if $T \supset S$ and $f(T, d) \in S$, then $f(S, d) = f(T, d)$.

Roth (1977) showed that WPO can be replaced by the following axiom:

Strong Individual Rationality (SIR): For all $(S, d) \in \Sigma, f(S, d) > d$.

WPO, as a condition on collective rationality, requires that the compromise reached by parties cannot be strictly improved upon. On the other hand, SIR, as a condition on individual rationality, requires that each party must strictly benefit from bargaining. As mentioned in the Introduction, WPO and SIR do not imply each other.

Next, we will introduce a requirement of collective rationality that is considerably weaker than both WPO and SIR. Denote by $D(a) \equiv \{x \in \mathbb{R}^2 | x \leq a\}$ the set of all points

[1] Given $x, y \in \mathbb{R}^2$, $x > y$ if $x_i > y_i$ for each i, and $x \geq y$ if $x_i \geq y_i$ for each i.

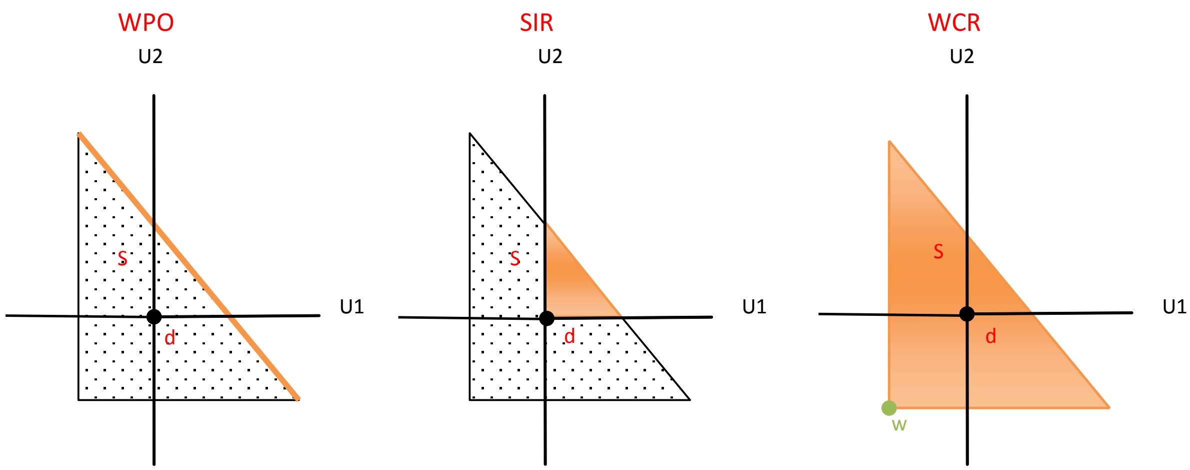

FIGURE 1

dominated by a. For a given problem (S, d), denote by $PI(S) \equiv \{x \in S \backslash WPO(S) | D(x) \cap S = \{x\}\}$ the set of *Pareto inferior* points in S. $PI(S)$ is the collection of non-weak Pareto-optimal compromises such that for each compromise, there is no other feasible compromise that make both parties worse off. For any (S, d) where d is in the interior of S, the points in PI are dominated by either d_1 or d_2 or both and they are the worst possible compromises that the two parties can possibly find in S. Our axiom states that parties should avoid such compromises as a solution outcome.

Weakest Collective Rationality (WCR) For all $(S, d) \in \Sigma$, $f(S, d) \notin PI(S)$.

Figure 1 shows the differences among WPO, SIR and WCR. Suppose the set of utility possibilities S consists of all points in a right-angled triangle (boundary included). WPO then requires the bargaining outcome $f(S, d)$ to be on the hypotenuse. SIR requires $f(S, d)$ to be in the small right triangle above d. In sharp contrast to WPO and SIR, WCR allows $f(S, d)$ to be any point in S except w, as $PI(S) = \{w\}$. Observe that if $f(S, d) \notin PI(S)$, then it is not necessarily the case that $f(S, d) > d$. To see that consider an $f(S, d)$ arbitrarily close to w in Part 3 of Figure 1. Thus, WCR does not imply SIR. Likewise, if $f(S, d) \notin PI(S)$, then it is not necessarily case that $f(S, d) \in WPO(S)$. To see that again consider an $f(S, d)$ arbitrarily close to w in Part 3 of Figure 1. Thus, WCR does not imply WPO either. But clearly, if $f(S, d) > d$, then $f(S, d) \notin PI(S)$. Likewise, if $f(S, d) \in WPO(S)$, then $f(S, d) \notin PI(S)$. Thus, both SIR and WPO imply WCR. Hence the main result of this paper below improves on Nash (1950) and Roth (1977)[2]:

[2] Although our result holds for more than two parties as well, we will only provide it for two parties

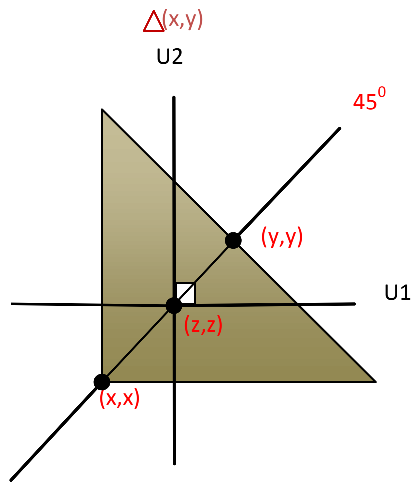

FIGURE 2

Proposition 1 *N is the unique solution satisfying SYM, SI, IIA, and WCR.*

Proof. N satisfies these four axioms. Suppose f satisfies SYM, SI, IIA and WCR. We will show that $f = N$. First we prove a lemma. Given any $x \in \mathbb{R}$ and $y \in \mathbb{R}$ such that $x < y$, denote by $\triangle(x, y) \equiv conv\{(x, x), (x, 2y - x), (2y - x, x)\}$ the symmetric right-angled triangle with (x, x) and (y, y) on the boundary of $\triangle(x, y)$ (see figure 2).[3]

Lemma 1. If f satisfies SYM, SI, IIA and WCR, then for any $x \leq z < y$, $f(\triangle(x, y), (z, z)) = (y, y)$.

Proof. Let $l[x, y]$ ($l(x, y)$) be the closed (open) line segment connecting (x, x) and (y, y). We prove it by establishing the following claims:

(i) $f(\triangle(x, y), (z, z)) \in l[x, y]$ by SYM.

(ii) $f(\triangle(x, y), (z, z)) \neq (x, x)$. Since $(x, x) \in PI(\triangle(x, y))$. By WCR, $f(\triangle(x, y), (z, z)) \neq (x, x)$.

(iii) $f(\triangle(x, y), (z, z)) \notin l(x, y) \backslash \{(z, z)\}$. Simply follow Roth (1977)'s proof.

(iv) $f(\triangle(x, y), (z, z)) \neq (z, z)$. Suppose to the contrary that $f(\triangle(x, y), (z, z)) = (z, z)$. By IIA, $f(\triangle(z, y), (z, z)) = (z, z)$. On the other hand, $f(\triangle(z, y), (z, z)) \neq (z, z)$ by WCR, a contradiction.

Consider now a problem (S, d) and identify its Nash solution outcome, $N(S, d)$. By SI, we can, without loss of generality assume, $d = (0, 0)$ and $N(S, d) = (1, 1)$. By compactness of S and the fact that $(1, 1)$ is the Nash bargaining solution in (S, d), there

here.

[3] "conv" denotes "the convex hull of."

exists $x \in \mathbb{R}$ such that $\triangle(x,1) \supset S$. By Lemma 1, $f(\triangle(x,1),(0,0)) = (1,1)$. By IIA, $f(S,d) = (1,1) = N(S,d)$. ■

Now we provide intuition as to what roles different axioms fulfill in singling out the solution outcome of a given bargaining problem when one combines SI, IIA and SYM with (i) SIR or (ii) WCR. Axioms - sometimes in tandem - can be used in ruling out different parts or points of the utility possibility set until only the solution outcome remains.

With Roth (1977)'s replacement of WPO by SIR, all alternatives but the strongly individually rational ones can be eliminated via SIR. Then, as the proof of Roth's Lemma illustrates, IIA and SI together are sufficient to rule out any interior (inefficient) alternatives from becoming the solution outcome. Then the rest follows as with Nash's original axioms.

Note that if Individual Rationality (for all $(S,d) \in \Sigma, f(S,d) \in IR(S,d)$) is used instead of SIR with SYM, SI and IIA, then both N and d become solution outcomes. If only SYM, SI and IIA are used, then not only N and d but some point in $PI(S)$ too becomes a solution outcome.

With our replacement of SIR (or WPO) by WCR, in a nutshell, the two multi-purpose axioms, IIA and SI, still perform their usual roles, and SYM and WCR provide finishing touches (the above proof is surely more complex than this story, but the intutition here can still paint a sufficient picture): After SI makes a suitable transformation of the utility possibility set, IIA can be used to reduce the utility possibility set to a symmetric one with only individually rational alternatives. Then IIA and SI together are sufficient to eliminate any interior alternatives from consideration (as in Roth). Subsequently, by SYM, only d and the unique symmetric efficient alternative remain eligible to become the solution outcome. Finally, WCR eliminates the worst remaining alternative, d, from consideration making the unique symmetric efficient alternative the solution outcome.

In the following, we provide the independence of axioms by presenting four solutions. Each solution violates one axiom while satisfying the remaining three.

SYM Let $f(S,d) = \arg\max\{(x_1 - d_1)^{\frac{1}{3}}(x_2 - d_2)^{\frac{2}{3}} | x \in IR(S,d)\}$.

SI Let $e(S,d)$ denote $\max\{y \in \mathbb{R} | (y + d_1, y + d_2) \in S\} \geq 0$. Let $f(S,d) = (e(S,d) + d_1, e(S,d) + d_2)$ if $e(S,d) > 0$, and $f(S,d) = N(S,d)$ otherwise.

IIA The Kalai-Smorodinsky solution (Kalai and Smorodinsky (1975)): $K(S,d) = \max\{u \in S|$ there exists $\alpha \in [0,1]$ such that $u = \alpha b(S,d) + (1-\alpha)d\}$, where $b(S,d) \equiv (b_1(S,d), b_2(S,d))$ with $b_i(S,d) = \max\{x_i | x \in IR(S,d)\}$ is the ideal point in (S,d).

WCR For all $(S,d) \in \Sigma$, $f(S,d) \equiv d$.

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