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Testing the null hypothesis of no regime switching with an application to GDP growth rates

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Abstract

This paper presents tests for the null hypothesis of no regime switching in Hamilton's (1989) regime switching model. The test procedures exploit similarities between regime switching models, autoregressions with measurement errors, and finite mixture models. The proposed tests are computationally simple and, contrary to likelihood based tests, have a standard distribution under the null. When the methodology is applied to US GDP growth rates, no strong evidence of regime switching is found.

1 Introduction

Hamilton's (1989) regime switching framework is a popular approach to modeling macroeconomic and financial data. It has been used to study the behavior of GNP growth rates (Hamilton, 1989), real interest rates (Garcia and Perron, 1996), stock returns (Hamilton and Susmel, 1994), trade-off between inflation and unemployment (Ho, 2000) (for a review of the methods and applications see Kim and Nelson (1999)). In this model, the observed process is a sum of two unobserved components. The first component is a Markov chain with a finite number of states that describes the conditional mean of the process (where conditioning is on the current state of the economy). The second component is a weakly dependent noise responsible for deviations from the conditional mean. The popularity of the regime switching model is due to several reasons. First, it allows one to model structural

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breaks without explicitly specifying the break points. Second, the regime switching formulation has natural interpretations in many cases. For example, in the Hamilton's original paper, the GNP growth rates have two possible regimes: recessions and booms. The model allows one as well to study such important characteristics as durations of states (for example, the duration of recessions).

Estimation and inference in the regime switching framework is not a trivial task due to the fact that the state variable is unobservable. Hamilton (1989, 1990) develops a maximum likelihood (ML) estimation procedure by the means of a nonlinear recursive filter. Given the values of the parameters and conditional on the observed data, the filter allows one to compute the probability of the state variable taking on some particular value. Provided with the filtered probabilities, the likelihood function can be evaluated. One can obtain the ML estimates by numerical maximization of the likelihood function. However, this approach suffers from several drawbacks. First, implementation of the filter requires correct dynamic specification of the noise component. It is not clear how the filter behaves if the noise component is misspecified. The usual practice is to assume an autoregressive (AR) form of finite order. However, computation becomes more and more cumbersome as the number of AR terms increases, which makes manipulation with different dynamic specifications impracticable. Second, the asymptotic likelihood has no unique maximum and is not locally quadratic (Hansen, 1992; Boldin, 1996). As a result, optimization appears to be unstable with respect to starting points. For example, Hamilton (1996) says that an econometrician should investigate hundreds of different starting points before he can achieve reasonable results. In practice, iterations tend to end up very close to starting points, at one of the numerous local maxima. Thus, estimation may indicate presence of regime switching even if there is no such behavior in the data. The third problem concerns testing of the model. Typically, after estimation of the model, econometricians will be interested in testing for regime switching. However, in this framework, likelihood-based statistics have nonstandard distributions, because the transition probabilities are not identified under the null. While there are solutions to the last problem (Hansen, 1992), the first two can create serious obstacles.

This paper presents two different testing procedures that aim to help the econometrician to decide whether there is regime switching in the data prior to ML estimation of the model. Proposed tests are computationally simple and have standard distributions under the null. The first approach that I consider exploits similarities between the regime switching model with AR disturbances and an autoregression with measurement errors. The AR model with measurement errors has been studied by Tanaka (1983, 2002). I argue that Tanaka's test for measurement errors allows for detection of regime switching as well. The major disadvantage of this approach is the fact that it requires correct specification of the order of the AR component. The second class of tests avoids this problem by using the steady-state distribution of a process with regime switching, which has a finite mixture form.

The regime switching models can be viewed as a time-series generalization of mixture models (for a discussion of mixture models see Everitt and Hand (1981) and McLachlan and Peel (2000); see Chen et al. (2004) and papers cited therein for examples of the tests for finite mixture models). A mixture model assumes that the probability density function is of the following form:

$$m(x) = \int f(x, \alpha) dP(\alpha), \tag{1}$$

where α is the vector of parameters and P is a probability measure. One can consider two estimators for the density: a parametric estimator that exploits (1), and a nonparametric estimator that makes no assumptions concerning the mixture form. Large differences between two alternative estimators will indicate that the data may not fit the mixture (or regime switching) framework. Hence, a test of regime switching can be based on discrepancy between the two estimators for the steady state density function. Since some transformations of the density function, such as the moment generating function, have the mixture form as well, similar tests can be developed using those transformations. In this paper, the tests based on the steady state distribution are of the Lagrange Multiplier (LM) type. The tests are derived using a GMM type criterion function and are referred to as GMM tests. This approach is also related to the Empirical Characteristic Function method of Knight and Yu (2002).

The two approaches discussed in this paper rely on different characteristics of the regime switching model. The Tanaka's test is based on the short-run dynamic structure of the model. The rejection occurs, if, in addition to an autoregressive component of the pre-specified order, the test detects presence of another stochastic component. The second approach is based on the long-run properties of the model. The test compares properties of the distribution implied by the data with those of the theoretical steady state distribution specified under the null. In the case of the null hypothesis of no regime switching and under the normality assumption for the errors, the test rejects the null if the steady state distribution of the data deviates significantly from the normal distribution. While the test can be viewed as a normality test, it has been designed specifically for the finite mixture alternative. Further, this approach can be extended to testing, for example, the null of two states against the alternative of at least three states in the steady state distribution. The GMM approach to normality testing has been considered recently by Bontemps and Meddahi (2005).

Naturally, due to their differences, the two testing procedures can lead to different conclusions in practice. I recommend to use the Tanaka's test only when it is desirable to avoid assumptions on the distribution of the errors, while their dynamics is correctly specified. In all other cases, the testing procedure based on the steady-state mixture form appears to be more attractive. It is important to emphasizes that, since the state variable is unobservable, the regime switching model cannot be fully nonparametric in dynamics of the errors and their distribution. Distributional or finite order AR assumptions are required in order to separate between the two unobservable components.

The paper proceeds as follows. Section 2 describes the model and tests. Section 3 presents Monte Carlo size and power results. Section 4 applies the methodology to US GDP growth rates. Section 5 concludes.

2 The model and tests

2.1 The model

This section describes the model and illustrates some problems associated with the ML approach in the regime switching framework. Suppose that the econometrician observes a random process $\{y_t \in R : t = 1, 2, \ldots, n\}$ generated according to the following model:

$$B(L)(y_t - \mu_0 - \mu_1 S_t) = \varepsilon_t, \tag{2}$$

where $B(L) = 1 - \beta_1 L - \ldots - \beta_q L^q$ is a polynomial in lag operator with all roots lying outside the unit circle, $\{\varepsilon_t\}$ are iid mean zero random variables with the finite variance σ^2, and S_t is an unobservable state variable. The state variable S_t takes values in $\{0, 1\}$ and is independent from the errors ε_t. It is assumed that S_n follows an ergodic Markov chain with the transition probability matrix given by

$$\begin{pmatrix} p_{00} & 1 - p_{00} \\ 1 - p_{11} & p_{11} \end{pmatrix}, \text{ where} \tag{3}$$

$$\begin{aligned} p_{ii} &= Pr\{S_{t+1} = i | S_t = i, S_{t-1} = j, \ldots\} \\ &= Pr\{S_{t+1} = i | S_t = i\}, \text{ for } i, j = 0, 1. \end{aligned}$$

The hypothesis of interest is $H_0 : \mu_1 = 0$ (no regime switching). It is usually assumed in the literature that the errors ε_t are normally distributed. This assumption is not required for the test presented in Section 2.2. However, the tests proposed in Section 2.3 are valid only if the errors ε_t are normal.

The ML estimation of the model is usually based on a nonlinear recursive filter developed by Hamilton (1989). The filter allows one to obtain the distribution of the state variable conditional on the observed data and the model parameters. The likelihood function can be computed as a by-product of the filter. However, as it has been mentioned in the introduction, this approach suffers from severe computational problems. In order to illustrate this point, I performed the following simulation exercise. I generated 100 observation using the AR(1) model $y_t = 0.5y_{t-1} + \varepsilon_t$, where the errors ε_t were iid standard normal random variables (in this case, the true value of the regime switching parameter μ_1 is zero). Next, I evaluated the log-likelihood using the Hamilton's filter assuming the AR(1) specification. I fixed $p_{00} = p_{11} = 0.9$, $\mu_0 = 0$, $\sigma = 1$, and constructed the log-likelihood as a function of the regime switching parameter μ_1 for $\beta_1 = 0.5$ and 0.01. Figure 1 plots the log-likelihood function against the values of μ_1. When the AR parameter β_1 equals its true value ($\beta_1 = 0.5$), the log-likelihood function is well behaved and maximized in the neighborhood of the true value of μ_1. However, in the case of $\beta_1 = 0.01$, the log-likelihood has two local maxima away from the true value of μ_1, and a local minimum at $\mu_1 = 0$. As a result, a wrong choice of starting points for numerical optimization of the likelihood may lead to estimates with poor properties. The problem becomes more severe as the number of autoregressive coefficients increases.

[Figure 1 about here.]

2.2 Tanaka's test

This section presents a test of no regime switching based on similarity between the regime switching model and an AR model with measurement errors. Equation (2) implies that y_t can be written as a switching component plus an AR process:

$$y_t = \mu_0 + \mu_1 S_t + u_t, \text{ where} \tag{4}$$

$$u_t = B^{-1}(L)\,\varepsilon_t. \tag{5}$$

The AR model with measurement errors was studied by Tanaka (1983, 2002); it is similar to (4)-(5) with $\mu_1 S_t$ replaced by an iid Gaussian measurement error process:

$$\begin{aligned} y_t &= \mu_0 + \eta_t + u_t, \\ \eta_t &\sim \text{ iid } N(0, \rho), \end{aligned} \tag{6}$$

where $\rho \geq 0$, u_t as in (5) and independent of η_t. Testing for the absence of measurement errors is equivalent to testing $\rho = 0$ in (6). Under the null, the model becomes a simple AR equation:

$$y_t = \mu_0 + \beta_1 y_{t-1} + \ldots + \beta_q y_{t-q} + \varepsilon_t, \tag{7}$$

where ε_t are iid. Tanaka (2002) suggested to base a test of the null of no measurement errors on the following LM type statistic.

$$T_n = \sqrt{n}\,\frac{\hat{\beta}_n'\,\hat{J}_n\,\hat{r}_n}{\left(\hat{\beta}_n'\,\hat{J}_n\,\hat{V}_n\,\hat{J}_n'\,\hat{\beta}_n\right)^{\frac{1}{2}}}, \tag{8}$$

where $\hat{\beta}_n$ is the OLS estimator of $\beta = (\beta_1, \ldots, \beta_q)'$ in (7),

$$\hat{r}_{n,k} = \sum_t \hat{\varepsilon}_t \hat{\varepsilon}_{t-k} / \sum_t \hat{\varepsilon}_t^2,$$

$\hat{r}_n = (\hat{r}_{n,1}, \ldots, \hat{r}_{n,q})'$, $\hat{J}_n$ is the plug-in type estimator of

$$J = \begin{pmatrix} -1 & 0 & \ldots & 0 \\ \beta_1 & -1 & \ldots & 0 \\ \beta_2 & \beta_1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ \beta_{q-1} & \beta_{q-2} & \ldots & -1 \end{pmatrix},$$

and $\hat{V}_n$ is a consistent estimator of $V = I - \sigma^2 \left(J'\Gamma J\right)^{-1}$, with Γ being the variance-covariance matrix of $(y_{t-1}, \ldots, y_{t-q})'$. The important component of (8) is $\hat{r}_n$, the vector of autocorrelations of the fitted regression residuals $\hat{\varepsilon}_t$. Under the null, the OLS estimator of β is consistent. Furthermore, enough correlation is removed from the regression residuals to make them asymptotically uncorrelated in the sense that $\hat{r}_n = O_p\left(\frac{1}{\sqrt{n}}\right)$. In this case, Tanaka (1983) shows that

$$\sqrt{n}\,\hat{r}_n \to_d \; N\left(0, V\right), \tag{9}$$

and, as a result, under the null of no measurement errors,

$$T_n \to_d N(0, 1). \tag{10}$$

Tanaka (1983) shows that T_n tends to be positive in presence of measurement errors. Consequently, a one-sided test should be used when testing for measurement errors.

Although the test discussed above has been derived for the model with measurement errors, it can be used in the regime switching framework as well. The two models, regime switching and measurement errors, share an important feature. In both cases, there exists an additional disturbance component independent of the autoregressive part. Under the null hypothesis of no measurement errors or no regime switching both models have the same form. Therefore, under the null of no regime switching, Tanaka's test statistic (8) has the same asymptotic distribution as in (10).

In presence of regime switching, equations (4) and (7) imply that the regression residuals $\hat{\varepsilon}_t$ are computed from a misspecified model. In this case, the correct model is

$$\begin{aligned} y_t = \mu_0 + \beta_1 y_{t-1} + \ldots + \beta_q y_{t-q} \\ + \mu_1 \left(S_t - \beta_1 S_{t-1} - \ldots - \beta_q S_{t-q}\right) + \varepsilon_t. \end{aligned} \tag{11}$$

In general, omitting relevant variables will cause the OLS estimator of β to be inconsistent. Moreover, the regression residuals will be asymptotically correlated. As a result, $\hat{r}_n$ converges in probability to a non-zero constant almost always, and a test statistic based on $\sqrt{n}\hat{r}_n$ diverges to $-\infty$ or $+\infty$ depending on the model parameters. Hence, I propose to use a two-sided test against the alternative of regime switching, i.e. reject the null if $|T_n| > z_{1-\alpha/2}$.

I argue that Tanaka's test statistic is almost always consistent against the alternative hypothesis of regime switching. It is consistent only "almost always" for the following reason. It is possible to find values for β, p_{00} and p_{11} such that $\hat{r}_n$ is $o_p(1)$ regardless of the value of μ_1. However, the set of such values is negligible with respect to the Lebesgue measure defined on the appropriate measure space for the model parameters. In order to illustrate this point, suppose that $q = 1$, i.e. u_t in (4) follows AR(1). It is known that S_t has the following AR(1) representation (see, for example, Gong and Mariano (1997)):

$$\begin{aligned} S_t &= \varphi_0 + \varphi S_{t-1} + \xi_t, \\ \varphi &= p_{00} + p_{11} - 1, \\ \xi_t &\sim \text{ iid}\left(0, \sigma_\xi^2\right). \end{aligned} \tag{12}$$

Suppose further that $\beta_1 = \varphi$. In this case, equation (11) becomes

$$y_t = (\mu_0 + \mu_1 \varphi_0) + \beta_1 y_{t-1} + (\mu_1 \xi_t + \varepsilon_t),$$

which is a usual AR(1) model. Hence, result (9) holds for all values of the regime switching parameter μ_1. Consequently, in such a situation T_n asymptotically has a standard normal distribution regardless of the value of μ_1. The vector of the coefficients $(\beta_1, p_{00}, p_{11})$ takes values in $\mathcal{W} = (-1,1) \times (0,1) \times (0,1)$. However, the set

$$\{(\beta_1, p_{00}, p_{11}) \in \mathcal{W} : \ \beta_1 = p_{00} + p_{11} - 1\}$$

is negligible with respect to the Lebesgue measure on $(\mathcal{W}, \mathcal{B}(\mathcal{W}))$, where $\mathcal{B}$ denotes the Borel σ-field. I show in the Appendix that $\hat{r}_n = o_p(1)$ if and only if $\beta_1 = p_{00} + p_{11} - 1$, provided that $\mu_1 \neq 0$ and that the errors in (4) follow an AR(1) process. Therefore, Tanaka's test statistic is almost always consistent against the regime switching model with AR(1) disturbances. More details on the behavior of $\hat{r}_n$ in the presence of regime switching can be found in the Appendix.

The procedure proposed above does not require estimation of the regime switching component, since this is an LM test and the model needs to be estimated only under the null. Furthermore, the normality of ε_t in (5) is not required. Instead, the procedure relies heavily on correct specification of the length of the polynomial $B(L)$. For example, regardless of the value of μ_1, the test based on T_n tends to reject the null, if the dimension of $\hat{\beta}_n$ is smaller than the order of $B(L)$. On the other hand, the test may have poor power properties, if the dimension of $\hat{\beta}_n$ is larger than necessary.

2.3 GMM-type tests

This section presents the tests derived using the mixture form of the steady state distribution of the regime switching process. Unlike the test discussed in the previews section, these tests do not require explicit specification of the dynamic structure for the errors. However, it is assumed that the distribution of the errors is known up to the value of its parameters. In what follows, I assume that the errors u_t in (5) are normally distributed and satisfy a strong mixing condition.

In the regime switching framework, the normality assumption may be unavoidable due to identification problems. Let $f(x,\alpha_1), \ldots, f(x,\alpha_k)$ be density functions depending on the parameter vectors α_i, and let $c_1, \ldots, c_k$ be scalars such that $\sum_{i=1}^k c_i = 1$, $c_i \leq 1$ for all i. The weighted sum $h(x) = \sum_{i=1}^k c_i f(x,\alpha_i)$ is called the finite mixture density. The mixture density is said to be identifiable if and only if $\sum_{i=1}^k c_i f(x,\alpha_i) = \sum_{i=1}^k c_i' f(x,\alpha_i')$ implies that for all i there exists j such that $c_i = c_j'$ and $\alpha_i = \alpha_j'$. Unlike many other distributions, normality implies that the parameters of the finite mixture are identified (see Teicher (1963), Everitt and Hand (1981)).

Quandt and Ramsey (1978) acknowledged the shortcomings of likelihood based methods when applied to the mixture model and suggested an alternative route. They studied a simple mixture of normals model with the mean and variance switching independently between two possible values. They proposed to estimate the model through minimization of a function based on the sum of squares of the differences between the theoretical and sample moment generating functions (MGFs). Although the Markov regime switching model has more complex dynamics than that of Quandt and Ramsey, its steady state distribution has a finite mixture form as well, which makes it possible to apply their approach to regime switching.

First, I briefly describe the Quandt and Ramsey's framework. The random sample $\{y_t \in R : t = 1, \ldots n\}$ is such that $y_t \sim N(\mu_0, \sigma^2)$ with probability λ, and $y_t \sim N(\mu_0 + \mu_1, \sigma^2)$ with probability $1 - \lambda$. The MGF of y_t evaluated at x is given by

$$MGF\left(x; \mu_0, \mu_1, \sigma^2, \lambda\right) = \lambda \exp\left(\mu_0 x + \frac{\sigma^2 x^2}{2}\right) + (1 - \lambda) \exp\left((\mu_0 + \mu_1) x + \frac{\sigma^2 x^2}{2}\right), \tag{13}$$

and its natural estimator is $n^{-1} \sum_{t=1}^{n} \exp(x y_t)$. Quandt and Ramsey (1978) proposed the following estimation procedure. One selects the constants $x_1, \ldots, x_k$, where $k \geq 4$, and minimizes

$$Q_n\left(\mu_0, \mu_1, \sigma^2, \lambda\right) = \sum_{j=1}^{k} \left(\frac{1}{n} \sum_{t=1}^{n} \exp(x_j y_t) - MGF\left(x; \mu_0, \mu_1, \sigma^2, \lambda\right)\right)^2$$

with respect to μ_0, μ_1, σ^2 and λ to obtain their estimates. They show that this method leads to a consistent and asymptotically normal estimator of the parameters, provided that the true data generating process (DGP) is a mixture of normals; they did not address the problem of testing the hypothesis that the true model consists of only one component, i.e. $\mu_1 = 0$. In this case, the usual t-test based on $\hat{\mu}_1$ and its asymptotic standard error leads to a statistic with a non-standard distribution, since the nuisance parameter λ is not identified under the null. Alternatively, one can use the LM principle to derive a test statistic with a standard distribution. I follow this approach for testing of the regime switching model (4). The tests discussed in this section are derived using variations of the function Q_n, which is a GMM-type function. I refer to such tests as GMM-type tests.

In the steady state, we have that

$$y_t|\{S_t = 0\} \sim N(\mu_0, \sigma^2),$$
$$y_t|\{S_t = 1\} \sim N(\mu_0 + \mu_1, \sigma^2),$$

where the event $\{S_t = 0\}$ occurs with the probability

$$\lambda = P(S_t = 0) = \frac{1 - p_{11}}{2 - p_{00} - p_{11}}.$$

Therefore, the steady state MGF of a process generated according to (4) with normal errors has exactly the same form as that in (13). I assume that S_0 was drawn from the steady state distribution of the state variable. Next, I define a k-vector of the normal MGFs evaluated at the points $x_1, \ldots, x_k$:

$$m(\mu, \sigma^2) = \left(\exp\left(\mu x_1 + \frac{\sigma^2 x_1^2}{2}\right), \ldots, \exp\left(\mu x_k + \frac{\sigma^2 x_k^2}{2}\right)\right)',$$

and its nonparametric estimator:

$$m_n = \left(\frac{1}{n}\sum_{t=1}^{n} \exp(y_t x_1), \ldots, \frac{1}{n}\sum_{t=1}^{n} \exp(y_t x_k)\right)'.$$

Lastly, I define

$$Q_n^{MGF}(\mu_0, \mu_1, \sigma_u^2, \lambda) = \left\| A_n \left(m_n - \lambda m(\mu_0, \sigma_u^2) - (1-\lambda) m(\mu_0 + \mu_1, \sigma_u^2)\right)\right\|^2,$$

where σ_u^2 is the variance of u_t, and A_n is a possibly random $k \times k$ weight matrix. I assume that $A_n \to_p A$, where A is some non-random matrix of rank k. Ignoring the constants, the derivative of Q_n^{MGF} with respect to μ_1 evaluated at $\mu_1 = 0$ is given by

$$\frac{\partial Q_n^{MGF}(\mu_0, 0, \sigma_u^2, \lambda)}{\partial \mu_1} \sim \frac{\partial m(\mu_0, \sigma_u^2)'}{\partial \mu} A_n' A_n \left(m_n - m(\mu_0, \sigma_u^2)\right). \qquad (14)$$

Let $\hat{\mu}_0$ and $\hat{\sigma}_u^2$ be the sample mean and variance of y_t (note that, under the null, the variances of y_t and u_t are equal). Equation (14) implies that the

LM test statistic is based on $m_n - m\left(\hat{\mu}_0, \hat{\sigma}_u^2\right)$. Applying the mean value theorem element-by-element to $m_n - m\left(\hat{\mu}_0, \hat{\sigma}_u^2\right)$, one obtains

$$\begin{aligned}\sqrt{n}\left(m_n - m\left(\hat{\mu}_0, \hat{\sigma}_u^2\right)\right) &= \sqrt{n}\left(m_n - m\left(\mu_0, \sigma_u^2\right)\right) \\ &- \left(\frac{\partial m\left(\mu_0, \sigma_u^2\right)}{\partial \mu}, \frac{\partial m\left(\mu_0, \sigma_u^2\right)}{\partial \sigma^2}\right)\sqrt{n}\begin{pmatrix}\hat{\mu}_0 - \mu_0 \\ \hat{\sigma}_u^2 - \sigma_u^2\end{pmatrix} + o_p(1). \qquad (15)\end{aligned}$$

Under the null hypothesis,

$$\sqrt{n}\left(\left(m_n - m\left(\mu_0, \sigma_u^2\right)\right)', \hat{\mu}_0 - \mu_0, \hat{\sigma}_u^2 - \sigma_u^2\right) \rightarrow_d N(0, \Sigma), \qquad (16)$$

where Σ is the long-run covariance matrix of $\exp\left(y_t x_1\right) - \exp\left(\mu x_1 + \frac{\sigma^2 x_1^2}{2}\right)$, $\ldots$, $\exp\left(y_t x_k\right) - \exp\left(\mu x_k + \frac{\sigma^2 x_k^2}{2}\right)$, u_t, and $u_t^2 - \sigma_u^2$, and is assumed to be positive definite. Equations (15) and (16) imply that under the null

$$\sqrt{n}\left(m_n - m\left(\hat{\mu}_0, \hat{\sigma}_u^2\right)\right) \rightarrow_d N(0, W), \qquad (17)$$

where

$$W = \Lambda \Sigma \Lambda', \qquad (18)$$

$$\Lambda = \left(I_k, -\frac{\partial m\left(\mu_0, \sigma_u^2\right)}{\partial \mu}, -\frac{\partial m\left(\mu_0, \sigma_u^2\right)}{\partial \sigma^2}\right). \qquad (19)$$

In view of the above results, I define the MGF-based test statistic as follows:

$$QR_n^{MGF} = \sqrt{n}\frac{\frac{\partial m\left(\hat{\mu}_0, \hat{\sigma}_u^2\right)'}{\partial \mu} A_n' A_n \left(m_n - m\left(\hat{\mu}_0, \hat{\sigma}_u^2\right)\right)}{\left(\frac{\partial m\left(\hat{\mu}_0, \hat{\sigma}_u^2\right)'}{\partial \mu} A_n' A_n \hat{W}_n A_n' A_n \frac{\partial m\left(\hat{\mu}_0, \hat{\sigma}_u^2\right)}{\partial \mu}\right)^{\frac{1}{2}}}, \qquad (20)$$

where $\hat{W}_n$ is a consistent estimator of W and can be constructed from (18) and (19). The covariance matrix Σ can be estimated using the HAC approach (see Andrews (1991)). A consistent estimator of Λ can be obtained by replacing the unknown parameters in (19) by their null-restricted estimators. The results from the GMM literature (see Hansen (1982)) suggest that the optimal choice of A_n is the one that satisfies

$$A_n' A_n = \hat{W}_n^{-1}.$$

In this case, (20) becomes

$$QR_n^{MGF} = \sqrt{n}\frac{\frac{\partial m\left(\hat{\mu}_0, \hat{\sigma}_u^2\right)'}{\partial \mu} \hat{W}_n^{-1} \left(m_n - m\left(\hat{\mu}_0, \hat{\sigma}_u^2\right)\right)}{\left(\frac{\partial m\left(\hat{\mu}_0, \hat{\sigma}_u^2\right)'}{\partial \mu} \hat{W}_n^{-1} \frac{\partial m\left(\hat{\mu}_0, \hat{\sigma}_u^2\right)}{\partial \mu}\right)^{\frac{1}{2}}}.$$

The result in (17) implies that, under the null, QR_n^{MGF} converges in distribution to a standard normal random variable. Thus, the proposed test statistic asymptotically has a standard distribution. Furthermore, the statistic is computationally simple, since it does not require estimation of the regime switching model. It only requires computation of the sample mean, variance and the estimates of the MGF evaluated at a finite number of points.

The QR_n^{MGF} statistic has been derived using the fact that the MGF of the regime switching process has a mixture form in the steady state. In addition to the MGF, other transformations of the probability density function (PDF) and the PDF itself can be used to construct a test. Next, I present a test based on the steady state PDF. I define a k-vector of the normal PDFs corresponding to the mean μ and variance σ^2 and evaluated at the fixed points $x_1, \ldots, x_k$:

$$f\left(\mu, \sigma^2\right) = \left(\phi\left(\frac{x_1 - \mu}{\sigma}\right), \ldots, \phi\left(\frac{x_k - \mu}{\sigma}\right)\right)',$$

where $\phi(\cdot)$ denotes the standard normal PDF. The density at the point x can be estimated from the data by a nonparametric estimator of the form

$$(nh_n)^{-1} \sum_{t=1}^{n} K\left(\frac{y_t - x}{h_n}\right),$$

where $K(\cdot)$ is a kernel function such that $\int_{-\infty}^{\infty} K(x)\, dx = 1$, and h_n, the bandwidth, is a function of n and goes to zero as $n \to \infty$. Let f_n be a k-vector of the nonparametric estimates of the PDF at the points $x_1, \ldots, x_k$:

$$f_n = \left((nh_n)^{-1} \sum_{t=1}^{n} K\left(\frac{y_t - x_1}{h_n}\right), \ldots, (nh_n)^{-1} \sum_{t=1}^{n} K\left(\frac{y_t - x_k}{h_n}\right)\right)'.$$

The PDF-based criterion function is defined as follows:

$$Q_n^{PDF}\left(\mu_0, \mu_1, \sigma_u^2, \lambda\right) = \\ = \left\| A_n \left(f_n - \lambda f\left(\mu_0, \sigma_u^2\right) - (1-\lambda) f\left(\mu_0 + \mu_1, \sigma_u^2\right)\right)\right\|^2.$$

The PDF-based statistic can be derived along the same lines as QR_n^{MGF}. First,

$$\frac{\partial Q_n^{PDF}\left(\mu_0, 0, \sigma_u^2, \lambda\right)}{\partial \mu_1} \sim \frac{\partial f'\left(\mu_0, \sigma_u^2\right)}{\partial \mu} A_n' A_n \left(f_n - f\left(\mu_0, \sigma_u^2\right)\right).$$

Second, provided that $h_n = o\left(n^{-1/5}\right)$, and that some additional technical conditions found in Robinson (1983) (see also Pagan and Ullah (1999)) hold, under the null, the asymptotic distribution of f_n is given by

$$\sqrt{nh_n}\left(f_n - f\left(\mu_0, \sigma_u^2\right)\right) \rightarrow_d N(0, V), \tag{21}$$

where $V = K_2 \text{diag}\big(f\left(\mu_0, \sigma_u^2\right)\big)$, and $K_2 = \int K^2(x)dx$. Third,

$$\sqrt{n}\left(f\left(\mu_0, \sigma_u^2\right) - f\left(\hat{\mu}_0, \hat{\sigma}_u^2\right)\right) = O_p(1),$$

and, since $h_n = o(1)$, it follows that

$$\sqrt{nh_n}\left(f_n - f\left(\hat{\mu}_0, \hat{\sigma}_u^2\right)\right) = \sqrt{nh_n}\left(f_n - f\left(\mu_0, \sigma_u^2\right)\right) + o_p(1). \tag{22}$$

Thus, replacement of unknown parameters μ_0 and σ_u^2 by their estimators does not affect the asymptotic distribution of $f_n - f\left(\mu_0, \sigma_u^2\right)$. I define the PDF-based test statistic as follows

$$QR_n^{PDF} = \sqrt{nh_n} \frac{\frac{\partial f'\left(\hat{\mu}_0, \hat{\sigma}_u^2\right)}{\partial \mu} A_n' A_n \left(f_n - f\left(\hat{\mu}_0, \hat{\sigma}_u^2\right)\right)}{\left(\frac{\partial f'\left(\hat{\mu}_0, \hat{\sigma}_u^2\right)}{\partial \mu} A_n' A_n \hat{V}_n A_n' A_n \frac{\partial f\left(\hat{\mu}_0, \hat{\sigma}_u^2\right)}{\partial \mu}\right)^{\frac{1}{2}}},$$

where $\hat{V}_n = K_2 \text{diag}\big(f\left(\hat{\mu}_0, \hat{\sigma}_u^2\right)\big)$. Note that since $\hat{V}_n$ is diagonal, if for example A_n is an identity matrix, the statistic QR_n^{PDF} takes the following form:

$$\sqrt{nh_n} \frac{\sum_{j=1}^{k} \left(x_j - \hat{\mu}_0\right) \phi\left(\frac{x_j - \hat{\mu}_0}{\hat{\sigma}_u}\right) \left(\phi_n\left(x_j\right) - \phi\left(\frac{x_j - \hat{\mu}_0}{\hat{\sigma}_u}\right)\right)}{\left(K_2 \sum_{j=1}^{k} \left(x_j - \hat{\mu}_0\right)^2 \phi\left(\frac{x_j - \hat{\mu}_0}{\hat{\sigma}_u}\right)^3\right)^{\frac{1}{2}}}.$$

The optimal choice of A_n is given by $A_n' A_n = K_2^{-1} \text{diag}\left(1/f\left(\hat{\mu}_0, \hat{\sigma}_u^2\right)\right)$. In this case we have that QR_n^{PDF} is given by

$$\sqrt{nh_n} \frac{\sum_{j=1}^{k} \left(x_j - \hat{\mu}_0\right) \left(\phi_n\left(x_j\right) - \phi\left(\frac{x_j - \hat{\mu}_0}{\hat{\sigma}_u}\right)\right)}{\left(K_2 \sum_{j=1}^{k} \left(x_j - \hat{\mu}_0\right)^2 \phi\left(\frac{x_j - \hat{\mu}_0}{\hat{\sigma}_u}\right)\right)^{\frac{1}{2}}}.$$

Similarly to the MGF-based test, under the null of no regime switching $QR_n^{PDF} \rightarrow_d N(0,1)$, as implied by the results in (21) and (22).

In the case of regime switching, $\mu_1 \neq 0$ and, consequently, $m_n - m\left(\hat{\mu}_0, \hat{\sigma}_u^2\right)$ and $f_n - f\left(\hat{\mu}_0, \hat{\sigma}_u^2\right)$ converge in probability to nonzero vectors. As a result,

QR_n^{MGF} and QR_n^{PDF} diverge to $\pm\infty$ depending on the values of the parameters and the choice of $x_1, \ldots x_k$. Therefore, size α tests of the null of no regime switching are given by $\left|QR_n^{MGF}\right| > z_{1-\alpha/2}$ for the MGF-based statistic and $\left|QR_n^{PDF}\right| > z_{1-\alpha/2}$ for the PDF-based statistic, where z_α denotes the α-quantile of the standard normal distribution.

The asymptotic variance of the term $\sqrt{nh_n}\left(f_n - f\left(\hat{\mu}_0, \hat{\sigma}_u^2\right)\right)$ takes a simpler form than that of $\sqrt{n}\left(m_n - m\left(\hat{\mu}_0, \hat{\sigma}_u^2\right)\right)$. It is due to the fact that the kernel density estimators at different points are asymptotically independent, and because the reminder term in (22) is negligible. As a result, the PDF-based statistic appears computationally more appealing. On the other hand, the PDF-based statistic may be less reliable, since, in addition to choosing the points for the density evaluation, a researcher has freedom to select the value of h_n. The PDF-based test is also expected to be less powerful due to the nonparametric rate of convergence.

For the purpose of testing, one can select the value of k smaller than the number of the parameters, including $k = 1$. In this case, the test is based on discrepancy between parametric and nonparametric estimators of the MGF or PDF around a single point. Naturally, inference is affected by the choice of the evaluation points. I postpone investigation of the optimal choice of k and the location of $x_1, \ldots, x_k$ for further study.

3 Simulations results

In this section, I investigate small sample size and power properties of the above tests in a Monte Carlo (MC) study. The data is simulated according to the DGP described by (2) and (3) with normally distributed errors ε_t. Since in Section 4 the tests are applied to US GDP growth rates data, for the power results the data is simulated using the Hamilton (1989) estimates for US GNP growth rates: $\mu_0 = -0.358$, $\mu_1 = 1.522$, $\sigma = 0.769$, $p_{00} = 0.755$, $p_{11} = 0.905$, $q = 4$ and $\beta = (0.014, -0.058, 0.245, 0.213)$. For the size results I use $\mu_1 = 0$, $\mu_0 = 0.739$ (the steady-state mean computed from the Hamilton's estimates), and the same values for σ, q and β as for the power results. In all MC experiments reported here, the number of simulation repetitions is 2,000.

Since the DGP includes an AR(4) component, I use four lags to compute T_n unless noted otherwise. Computation of the GMM-type statistics requires a selection of the evaluation points for the MGF and PDF. It also requires a choice of the weighting matrix A_n, kernel functions and bandwidth parameters for density estimation and HAC covariance matrix estimation. In order to compute QR_n^{PDF}, the density was evaluated at a series of points equally spaced on the interval $[-2, 2]$ in steps of 0.1. For computation of QR_n^{MGF}, the MGF was evaluated at the points -0.7, -0.6, -0.4, -0.1, 0.1, 0.4, 0.6 and 0.7. I set $A_n = I_k$ for all GMM-type tests, and use the normal kernel for the PDF-based statistic. The bandwidth for the density estimation was chosen in the following way. The Mean Integrated Square Error (MISE) optimal value of the bandwidth for density estimation is of the form $cn^{-1/5}$, where c depends on the second derivative of the PDF. It simplifies to $c = 1.06\sigma$, provided that the normal kernel function is used and that the data is distributed according to $N(\mu, \sigma^2)$ (see, for example, Pagan and Ullah (1999)). However, Silverman (1986) reports that the choice $c = 0.9\min(\sigma, R/1.34)$, where R is the interquartile range, performs better in the case of a mixture of normals. In addition, the rate of convergence of the MISE optimal bandwidth is not fast enough to guarantee (21). After some simulations, I concluded that the rate $n^{-0.35}$ performs reasonably well in this case. I use $h_n = 0.9\min\left(\hat{\sigma}, \hat{R}/1.34\right) n^{-0.35}$, where $\hat{\sigma}$ is the sample standard deviation and $\hat{R}$ is the sample interquartile range. I used the Quadratic Spectral kernel with the bandwidth equal to $n^{1/5}$ for HAC covariance matrix estimation.

Table 1 reports the size results for the nominal size $\alpha = 0.01$, 0.05, 0.10, and the sample size $n = 200$, 300 and 500. The finite sample distributions of T_n and QR_n^{MGF} are approximated reasonably well by the standard normal distribution. However, the PDF-based test tends to over reject especially for $n = 200$ and 300. The power results are reported in Table 2. All three statistics have power even in relatively small samples. For example, for $\alpha = 0.05$ and $n = 200$ the obtained rejection rates for T_n, QR_n^{MGF} and QR_n^{PDF} are 34%, 45% and 56% respectively (note that these results are not size corrected). The rejection rates increase with the sample size for all three tests.

[Table 1 about here.]

[Table 2 about here.]

The MGF-based test is the most appealing among the three considered procedures. The test based on T_n and Q_n^{MGF} have good size properties, however, the MGF-based test is more powerful than the T_n test. Furthermore, misspecification of the AR component may have an adverse effect on the power and size properties of the T_n test as illustrated below.

Table 3 reports the size results when the number of lags used to compute T_n is less than 4, the number of lags in the true DGP. As expected, misspecification distorts the size of the test. The test rejects the null too often, and rejection rates increase with the sample size. Table 4 reports the power results when the number of lags used in computation of T_n exceeds the number of lags in the true DGP. The rejection rates for 5, 6 and 7 lags are lower than in the case of the correctly specified model, and actually may decrease with the sample size.

[Table 3 about here.]

[Table 4 about here.]

Lastly, I compare the MGF-based test with the likelihood ratio test suggested by Hansen (1992). He treats the likelihood function as an empirical process in the nuisance parameters and parameters of interest. This approach allows one to obtain bounds on the asymptotic distribution of the likelihood ratio (LR) statistic. The test requires constrained optimization of the likelihood function with respect to μ_0, the variance of ε_t and autoregressive parameters for each combination of μ_1, p_{00} and p_{11} using some grid of values. The distribution of the resulting statistic is nonstandard, and the critical values must be simulated. The CPU requirements for a MC study of the Hansen's LR test applied to the AR(4) two-state regime-switching model are enormous. Therefore, for power and size comparisons I consider the iid errors case by setting $\beta = (0, 0, 0, 0)$. The sample size is 100, and the number of replications is 500. For each replication, the critical values for the LR test were simulated using 1,000 simulation replications. I used the same grid as in Hansen (1992): the range [0.1,2] in steps of 0.1 for μ_1, and [0.2, 0.8] in steps of 0.2 for p_{00} and p_{11}.

Second and third columns of Table 5 report the size and power results respectively for the Hansen's likelihood ratio and MGF-based test statistics. Similarly to the results reported in Table 1, the MGF-based test shows good size properties. The test based on the LR statistic under rejects the null in the case of no regime switching, which reflects the fact that the Hansen's LR test is conservative. Nevertheless, the LR test is more powerful than the test based on QR_n^{MGF}. However, the advantage of the GMM-type tests suggested in this paper is that, contrary to the Hansen's LR test, they do not require correct specification of the dynamic component. Last column of Table 5 illustrates this point. In this case, the error component was simulated as AR(1), with the first-order autoregressive coefficient equal to 0.5, while the LR statistic was constructed under the assumptions that the errors are iid. As a result, the LR tests strongly over rejects the null, with the rejection rates achieving 92% for the significance level of 5%. At the same time, the rejection rates of the MGF-based test are very close to assumed nominal.

[Table 5 about here.]

4 Application to GDP growth rates

In this section, I apply the test proposed in Section 2 to US GDP growth rate data. Hamilton (1989) used the regime switching framework to study business cycle patterns. He estimated the regime switching model for US GNP growth rates for the sample period 1952:II-1984:IV. His estimates suggest two regimes with recession and boom periods. Kim and Nelson (1999, Chapter 4) re-estimated the model on the US GDP growth rates data for the same sample period and found parameter estimates very close to those reported in Hamilton (1989). However, the model failed when they extended the sample to 1952:II-1995:III. Hansen (1992) investigated the same problem using a likelihood approach and concluded that the Hamilton (1989) specification did not fit the data. He decided in favor of a simple switching model with states arriving independently over time. Since simple switching fits in the mixture class as well, the tests presented in Section 2 are consistent against such an alternative.

I apply the T_n, QR_n^{MGF} and QR_n^{PDF} tests to the first differences of the log of quarterly GDP. I compute the test statistics for the three sample periods, 1952:II-1984:IV, 1952:II-1995:III and 1947:I-2001:IV. The choice of the testing periods was determined by the availability of the data and the results in the literature. Note that in the case of the period 1952:II-1984:IV, Hamilton (1989) and Kim and Nelson (1999) found regime switching. I compute the Tanaka's statistic using four lags. Construction of the GMM-type tests is as described in Section 3 unless noted otherwise.

Table 6 reports the test statistics and their p-values. The Tanaka's test has a low p-value (0.0745) for the period 1952:II-1984:IV, strongly rejects the null hypothesis for the period 1952:II-1995:III, however, it is unable to reject the null for the period 1947:I-2001:IV. Neither version of the GMM-type tests reject the null for any of the considered periods.

[Table 6 about here.]

The sample period 1947:I-2001:IV consists of 219 observations, and, therefore, one can learn about the power properties of the tests from the first rows of Table 2. The table shows that even in fairly small samples, the probability to reject the null attains 45%-63% for the GMM-type tests, depending on the nominal size. In order to evaluate the likelihood of the results reported in Table 6 in presence of regime switching, I performed the following exercise. I simulated 1,000 samples of size 219 using the Hamilton's estimates as the DGP. In each simulated sample, I computed T_n, QR_n^{PDF} and QR_n^{MGF}. Figure 2 shows the estimated densities of the three statistics. The vertical lines indicate the values of the statistics obtained for the sample period 1947:I-2001:IV. Note that the figure plots the estimated distributions of the statistics, provided that the Hamilton's results accurately describe the true DGP. In this case, T_n and QR_n^{MGF} tend to be negative, while QR_n^{PDF} tends to be positive. The T_n statistic exceeds its value found for the GDP growth rates only in about 10% of the simulation repetitions. The simulated QR_n^{PDF} was smaller than its value obtained for the GDP growth rates in approximately 29% of the simulation replications. As it appears, the test based on QR_n^{PDF} is rather inconclusive. However, in the case of QR_n^{MGF}, only in about 3% of the simulation replications the statistic was on the right of its value obtained for the GDP growth rates. While these result evaluate only one specific alternative and do not take into account uncertainty associated with the Hamilton's estimates, nevertheless, they suggest that the regime switching model does not provide an accurate description of the behavior of the GDP growth rates. The results for the Tanaka's test in Table 6 can be explained by misspecification of the AR component. It is likely that the number of lags the true DGP exceeds four.

[Figure 2 about here.]

Lastly, Tables 7 reports the result for the GMM-type tests for different choices of the evaluation points. This is to see whether the choice of evaluation points affects the results obtained for the QR_n^{MGF} and QR_n^{PDF} tests. In all cases, the test statistics are not significant at 5% level. Only in two cases small p-values (0.0561 and 0.0738) were obtained for the QR_n^{PDF} test for the period 1952:II-1984:IV. In all cases, the p-values are large for the full sample period.

[Table 7 about here.]

I offer the following explanation to the Kim and Nelson (1999) findings. The estimates of the regime switching model for the period 1952:II-1984:IV imply that recessions last on average four quarters, and booms last ten quarters. It is likely that low and high points of the GDP growth rate process follow this pattern only during the period 1952:II-1984:IV. Since in general the US GDP growth rate does not follow the regime switching model, the evidence of switching behavior disappears when more data becomes available.

5 Conclusion

This paper presents a number of tests for the hypothesis of no regime switching. Contrary to the tests based on the ML principle, the proposed tests enjoy such attractive properties as computational simplicity and standard distribution under the null. The MC study shows that the tests have good size and power, especially in moderately large samples. When the methodology is applied to US GDP growth rates, no strong evidence in support of regime switching is found. The direction for future work include investigation of the optimal choice of the points for evaluation of the MGFs and PDFs. The GMM-type procedure can be extended as well to estimation of the number of states in the regime switching model.

6 Appendix

The Tanaka's statistic is based on $\hat{r}_n$, the autocorrelation coefficient of the fitted regression residuals $\hat{\varepsilon}_n$. This appendix discusses properties of $\hat{r}_n$ in the case of regime switching. I define

$$\begin{aligned} \gamma(k) &= \operatorname{cov}\left(y_t, y_{t-k}\right), \\ \gamma_k &= \left(\gamma(k), \gamma(k+1), \ldots, \gamma(k+q-1)\right)', \\ \Gamma_k &= \left(\gamma_k, \gamma_{k-1}, \ldots, \gamma_{k-q+1}\right), \\ \theta(k) &= \operatorname{cov}\left(S_t, S_{t-k}\right), \\ \psi(k) &= \operatorname{cov}\left(u_t, u_{t-k}\right). \end{aligned}$$

Equation (4) and independence of S_t and u_t imply that

$$\gamma(k) = \mu_1^2 \theta(k) + \psi(k). \tag{23}$$

Equation (12) implies that

$$\theta(k) = \varphi \theta(k-1), \tag{24}$$

It follows from equation (5) that

$$\psi(k) = \beta_1 \psi(k-1) + \ldots + \beta_q \psi(k-q). \tag{25}$$

Next, under the usual regularity conditions (see, for example, White (2001, Chapter 3))

$$\frac{1}{n}\sum_{t=k+1}^{n}\hat{\varepsilon}_t\hat{\varepsilon}_{t-k} = \gamma_1'\Gamma_0^{-1}\left(\Gamma_k\Gamma_0^{-1}\gamma_1 - \gamma_{k+1}\right) + o_p(1). \tag{26}$$

Under the null hypothesis $\mu_1 = 0$, and therefore $\gamma(k) = \psi(k)$, $\gamma_{k+1} = \Gamma_k\beta$, and $\beta = \Gamma_0^{-1}\gamma_1$. Consequently, the first term on the right-hand side of (26) is zero, and the regression residuals $\hat{\varepsilon}_t$ are asymptotically uncorrelated. Under the alternative, the first term on the right-hand side of (26) is different from zero, unless some particular combination of the model parameters is chosen. Consider the case $q = 1$. Equations (23)-(25) imply that

$$\gamma_1'\Gamma_0^{-1}\left(\Gamma_k\Gamma_0^{-1}\gamma_1 - \gamma_{k+1}\right) = \frac{\gamma(1)\theta(0)\psi(0)}{\gamma^2(0)}\mu_1^2\left(\varphi - \beta_1\right)\left(\varphi^k - \beta_1^k\right).$$

The above result implies that the AR residuals $\hat{\varepsilon}_t$ are asymptotically uncorrelated if and only if $\mu_1 = 0$ or $\beta_1 = \varphi$. As a result, T_n is almost always consistent with respect to the Lebesgue measure on $(\mathcal{W}, \mathcal{B}(\mathcal{W}))$, where $\mathcal{W} = (-1,1)\times(0,1)\times(0,1)$, the space of possible values for $(\beta_1, p_{00}, p_{11})$ and $\mathcal{B}$, denotes the Borel σ-field.

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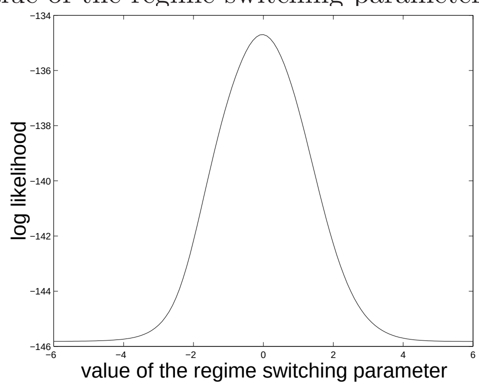
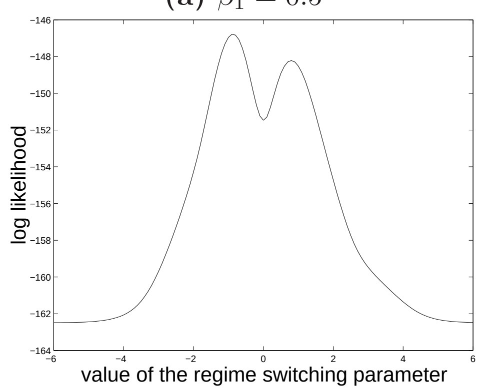
Figure 1: Log-likelihood function for different values of the autoregressive parameter β_1: (a) β_1 equals to its true value, (b) β_1 is different from its true value. The true value of the regime switching parameter is zero.

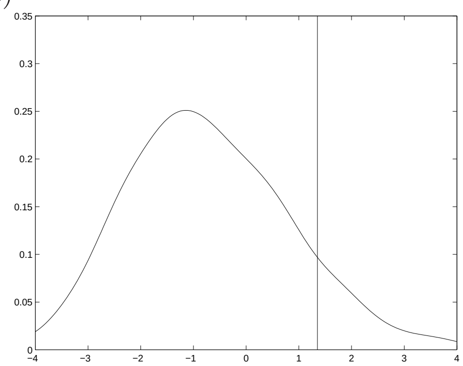


(a) $\beta_1 = 0.5$

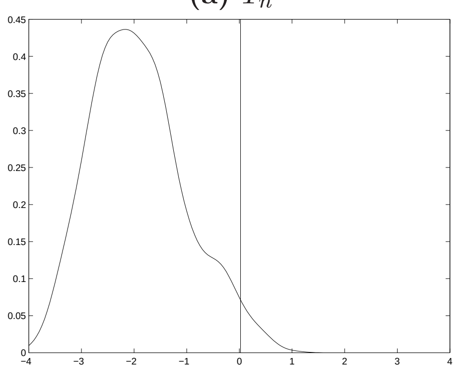


(b) $\beta_1 = 0.01$

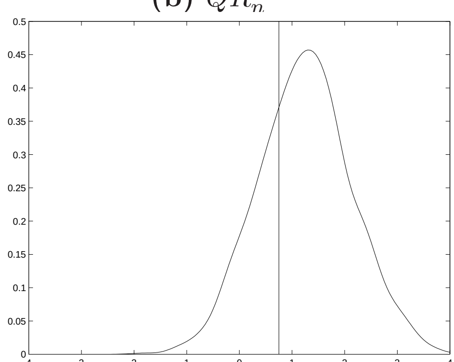
Figure 2: Estimated distributions of (a) T_n, (b) QR_n^{MGF} and (c) QR_n^{PDF} in the case of regime switching and their values obtained for US GDP growth rates (vertical lines)



(a) T_n



(b) QR_n^{MGF}



(c) QR_n^{PDF}

Table 1: Finite sample size of T_n, QR_n^{MGF} and QR_n^{PDF} tests for different values of nominal size (α) and sample size (n)

α	T_n	QR_n^{MGF}	QR_n^{PDF}
		$n = 200$	
0.01	0.013	0.007	0.034
0.05	0.064	0.043	0.112
0.10	0.120	0.102	0.182
		$n = 300$	
0.01	0.011	0.008	0.019
0.05	0.055	0.047	0.078
0.10	0.112	0.098	0.143
		$n = 500$	
0.01	0.011	0.008	0.010
0.05	0.050	0.051	0.058
0.10	0.097	0.104	0.120

Table 2: Finite sample power of T_n, QR_n^{MGF} and QR_n^{PDF} tests for different values of nominal size (α) and sample size (n)

α	T_n	QR_n^{MGF}	QR_n^{PDF}
		$n = 200$	
0.01	0.208	0.193	0.406
0.05	0.338	0.449	0.557
0.10	0.403	0.592	0.630
		$n = 300$	
0.01	0.289	0.358	0.504
0.05	0.443	0.632	0.669
0.10	0.527	0.751	0.744
		$n = 500$	
0.01	0.352	0.630	0.667
0.05	0.520	0.828	0.801
0.10	0.606	0.897	0.866

Table 3: Finite sample size of T_n test for different values of nominal size (α) and sample size (n), when the number of lags (q) used in computation of T_n is smaller than 4, the number of lags in the true DGP

α	$q = 1$	$q = 2$	$q = 3$
	$n = 200$		
0.01	0.326	0.488	0.150
0.05	0.434	0.613	0.309
0.10	0.513	0.698	0.698
	$n = 300$		
0.01	0.281	0.627	0.627
0.05	0.405	0.733	0.359
0.10	0.500	0.789	0.445
	$n = 500$		
0.01	0.319	0.734	0.307
0.05	0.491	0.824	0.466
0.10	0.580	0.873	0.566

Table 4: Finite sample power of T_n test for different values of nominal size (α) and sample size (n), when the number of lags (q) used in computation of T_n exceeds 4, its value in the true DGP

α	$q = 5$	$q = 6$	$q = 7$
	$n = 200$		
0.01	0.079	0.112	0.148
0.05	0.143	0.186	0.202
0.10	0.216	0.248	0.251
	$n = 300$		
0.01	0.074	0.066	0.129
0.05	0.171	0.126	0.192
0.10	0.238	0.186	0.256
	$n = 500$		
0.01	0.075	0.043	0.095
0.05	0.197	0.101	0.166
0.10	0.301	0.156	0.227

Table 5: Rejection rates for Hansen (1992) LR and QR_n^{MGF} tests for different DGPs: no switching with iid errors, switching with iid errors and no switching with AR(1) errors, and significance levels (α).

α	no switching iid errors	switching iid errors	no switching AR(1) errors
		LR	
0.01	0.006	0.702	0.844
0.05	0.022	0.838	0.920
0.10	0.048	0.912	0.956
		QR_n^{MGF}	
0.01	0.002	0.198	0.002
0.05	0.048	0.550	0.032
0.10	0.104	0.686	0.008

Table 6: T_n, QR_n^{MGF} and QR_n^{PDF} test statistics and corresponding p-values for US GDP growth rates for different sampling periods

	1952:II-1984:IV		1952:II-1995:III		1947:I-2001:IV	
	statistic	p-value	statistic	p-value	statistic	p-value
T_n	1.7834	0.0745	2.4050	0.0162	1.3528	0.1761
QR_n^{MGF}	-1.3672	0.1715	-1.0155	0.3099	0.0217	0.9827
QR_n^{PDF}	1.2910	0.1967	0.7767	0.4374	0.7511	0.4526

Table 7: The values QR_n^{MGF} and QR_n^{PDF} test statistics for US GDP growth rates for different sampling periods and values of the evaluation points for the MGF and PDF (p-values in parenthesis)

	1952:II-1984:IV	1952:II-1995:III	1947:I-2001:IV
		QR_n^{MGF}	
$-0.9, -0.8, \ldots, 0.9$	-1.1728	-0.8189	0.2814
	(0.2409)	(0.4129)	(0.7784)
$-1.9, -1.7, \ldots, 1.9$	-0.7723	-0.4321	0.8015
	(0.4399)	(0.6656)	(0.4228)
$-0.5, -0.45, \ldots, 0.5$	-1.3613	-1.0543	-0.1089
	(0.1734)	(0.2918)	(0.9133)
		QR_n^{PDF}	
$-3.0, -2.9, \ldots, 3.0$	1.9104	0.9735	0.2767
	(0.0561)	(0.3303)	(0.7820)
$-5.0, -4.9, \ldots, 5.0$	1.7876	0.9383	0.2449
	(0.0738)	(0.3481)	(0.8065)
$-6.0, -5.9, \ldots, 6.0$	1.2640	0.6639	0.1764
	(0.2062)	(0.5067)	(0.8600)