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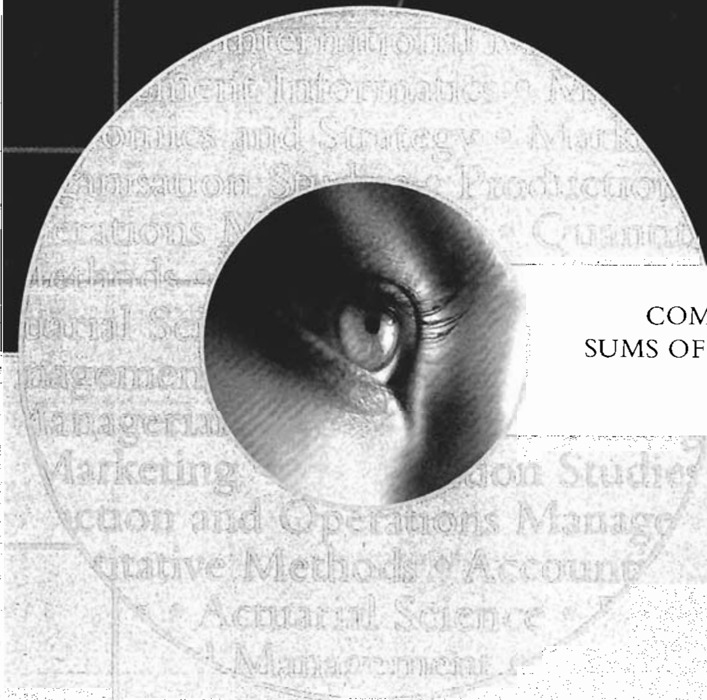

RESEARCH REPORT

COMPARING APPROXIMATIONS FOR RISK MEASURES OF SUMS OF NON-INDEPENDENT LOGNORMAL RANDOM VARIABLES

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Comparing approximations for risk measures of sums of non-independent lognormal random variables

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Abstract

In this paper, we consider different approximations for computing the distribution function or risk measures related to a sum of non-independent lognormal random variables. Approximations for such sums, based on the concept of comonotonicity, have been proposed in Dhaene et al. (2002a,b). These approximations will be compared with two well-known moment matching approximations: the lognormal and the reciprocal Gamma approximation. We find that for a wide range of parameter values the comonotonic lower bound approximation outperforms the two classical approximations.
Keywords: comonotonicity, simulation, lognormal, reciprocal Gamma.

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1 Introduction

In this paper we will consider and compare the performance of approximations for the distribution function (d.f.) and risk measures related to a random variable (r.v.) S given by

$$S = \sum_{i=1}^{n} \alpha_i \, e^{Z_i}. \tag{1}$$

Here, the α_i are non-negative real numbers and $(Z_1, Z_2, ..., Z_n)$ is a multivariate normal distributed random vector.

The accumulated value at time n of a series of future deterministic saving amounts α_i can be written in the form (1), where Z_i denotes the random accumulation factor over the period $[i, n]$. Also the present value of a series of future deterministic payments α_i can be written in the form (1), where now Z_i denotes the random discount factor over the period $[0, i]$. For more details, see Dhaene, Vanduffel, Goovaerts, Kaas & Vyncke (2004). The valuation of Asian or basket options in a Black & Scholes model and the setting of provisions and required capitals in an insurance context boils down to the evaluation of risk measures related to the distribution function of a random variable S as defined in (1).

We will investigate how to (approximately) compute risk measures such as quantiles (Q) and conditional tail expections (CTE) of the r.v. S defined in (1). These risk measures are defined by

$$\mathrm{Q}_p[S] = \inf\{s \in R | F_S(s) \geq p\}, \qquad p \in (0,1) \tag{2}$$

and

$$\mathrm{CTE}_p[S] = E[S | S > Q_p[X]], \qquad p \in (0,1), \tag{3}$$

where $F_S(s) = \Pr[S \leq s]$ and by convention, $\inf\{\phi\} = +\infty$. Notice that the quantile risk measure is often called the Value-at-Risk, whereas the conditional tail expectation coïncides with the Tail-Value-at-Risk. The latter holds true because S is a continuous r.v., see for instance Dhaene, Vanduffel, Tang, Goovaerts, Kaas & Vyncke (2004).

The r.v. S defined in (1) will in general be a sum of non-independent lognormal r.v.'s. Its d.f. cannot be determined analytically and is too cumbersome to work with. In the literature, a variety of approximation techniques for this d.f. has been proposed.

Practioners often use a moment matching *lognormal approximation* for the distribution of S. The lognormal approximation is chosen such that its first two moments are equal to the corresponding moments of S.

The present value of a continuous perpetuity with lognormal return process has a reciprocal Gamma distribution, see for instance Milevsky (1997). This present value can be considered as the limiting case of a random variable S as defined above. Motivated by this observation, Milevsky & Posner (1998) and Milevsky & Robinson (2000) propose a moment matching *reciprocal Gamma approximation* for the d.f. of S such that the first two moments match. They use this technique for deriving closed form approximations for the price of Asian and basket options.

Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2002a,b) derive *comonotonic upperbound and lowerbound approximations* (in the convex order sense) for the d.f. of S. Especially the lower bound approximation, which is given by $\mathrm{E}[S \mid \Lambda]$ for an appropriate choice of the conditioning r.v. Λ is extremely accurate, see for instance Vanduffel, Dhaene, Goovaerts & Kaas (2003).

Huang, Milevsky & Wang (2004) compare the performance of different approximations for the probability that a person outlives his money in case of a lifelong contineous consumption pattern. As a special case, they also consider approximations for such a probability when the consumption period is fixed.

Our paper is related to Huang, Milevsky and Wang (2004). However, we will only consider deterministic sums. Furthermore, instead of comparing the approximations of the ruin probabilities, we will evaluate the performance of the above mentioned techniques by comparing the approximated values of quantiles and conditional tail expectations of r.v.'s S as defined in (1).

The paper is organized as follows. In Section 2, we present the comonotonic approximations. In that section, we also focus on the optimal choice of the conditioning r.v. Λ of the comonotonic lower bound $\mathrm{E}[S \mid \Lambda]$. We propose a new conditioning r.v. which is likely to make the variance of the approximation 'as close as possible' to the exact variance. In Section 3 we will briefly recall the mathematical techniques behind the reciprocal Gamma and lognormal moment matching techniques. Finally, in Section 4 we compare the comontonic approximations with the moment matching techniques, using an extensive Monte Carlo simulation as the benchmark.

2 Comonotonic approximations

2.1 General results

In this section, we briefly repeat some results related to the comonotonic lower and upper bounds for the d.f. of the r.v. S defined in (1). For proofs and more details, we refer to Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2002a,b).

A central concept in the the theory on comonotonic r.v.'s is the concept of convex order. A r.v. X is said to preceede a r.v. Y in the convex order sense, notation $X \leq_{cx} Y$, if their means are equal and if their corresponding stop-loss premia are ordered uniformly for all retentions d, i.e. $E[(X-d)_+] \leq E[(Y-d)_+]$ for all d.

Replacing the copula describing the dependency structure of the terms in the sum (1) by the comonotonic copula yields an convex order upper bound for S. On the other hand, applying Jensen's inequality to S provides us with a lower bound. These results are formalized in the following theorem, which is taken from Kaas, Dhaene & Goovaerts (2000).

Theorem 1 *Let the r.v. S be given by (1), where the α_i are non-negative real numbers and the random vector $(Z_1, Z_2, ..., Z_n)$ has a multivariate normal* *distribution. Consider the conditioning r.v.* Λ*, given by*

$$\Lambda = \sum_{i=1}^{n} \gamma_i Z_i. \tag{4}$$

Also consider r.v.'s S^l *and* S^c *defined by*

$$S^l = \sum_{i=1}^{n} \alpha_i \; e^{E[Z_i]+\frac{1}{2}\left(1-r_i^2\right)\sigma_{Z_i}^2+r_i\sigma_{Z_i}\Phi^{-1}(U)} \tag{5}$$

and

$$S^c = \sum_{i=1}^{n} \alpha_i \; e^{E[Z_i]+\sigma_{Z_i}\Phi^{-1}(U)}, \tag{6}$$

respectively. Here U *is a Uniform*$(0,1)$ *r.v. and* Φ *is the cumulative d.f. of the* $N(0,1)$ *distribution. Further, the coefficients* r_i *are defined by*

$$r_i = \frac{cov\left[Z_i, \Lambda\right]}{\sigma_{Z_i} \; \sigma_\Lambda}. \tag{7}$$

For the r.v.'s S, S^l *and* S^c*, the following convex order relations hold:*

$$S^l \leq_{cx} S \leq_{cx} S^c. \tag{8}$$

The theorem states that (the d.f. of) S^l is a convex order lower bound for (the d.f. of) S, whereas (the d.f. of) S^c is a convex order upper bound for (the d.f. of) S.

The upper bound S^c is obtained by replacing the original copula between the marginals of the sum S by the comonotonic copula, but keeping the marginal distributions unchanged.

One can prove that the d.f. of the lower bound S^l corresponds with the d.f. of $\mathrm{E}[S \mid \Lambda]$. The lower bound is obtained by changing both the copula and the marginals of the original sum. Intuitively, one can expect that an appropriate choice of the conditioning variable Λ will lead to much better approximations than the the upper bound approximations.

A random vector is said to be comonotonic if all its components are non-decreasing functions of the same r.v. Notice that all terms in the sum S^c are non-decreasing functions of the r.v. U. This means that S^c is a comonotonic sum. It implies that the quantiles and conditional tail expectations of S^c are given by the sum of the corresponding risk measures for the marginals involved, see for instance Dhaene, Vanduffel, Tang, Goovaerts, Kaas & Vyncke (2004) :

$$Q_p\left[S^c\right] = \sum_{i=1}^{n} \alpha_i \; e^{E[Z_i]+\sigma_{Z_i}\Phi^{-1}(p)}, \tag{9}$$

$$CTE_p\left[S^c\right] = \sum_{i=1}^{n} \alpha_i \; e^{E[Z_i]+\frac{1}{2}\sigma_{Z_i}^2} \frac{\Phi\left(\sigma_{Z_i} - \Phi^{-1}(p)\right)}{1-p}, \qquad p \in (0,1). \tag{10}$$

Provided all coefficients r_i are positive, the terms in S^l are also non-decreasing functions of the same r.v. U. Hence, S^l will also be a comonotonic sum in this case. This implies that the quantiles and conditional tail expectations related to S^l can be computed by summing the corresponding risk measures for the marginals involved. Hence, assuming that all r_i are positive, we find the following expressions for quantiles and conditional tail expectations of S^l:

$$Q_p\left[S^l\right] = \sum_{i=1}^{n} \alpha_i \; e^{E[Z_i]+\frac{1}{2}\left(1-r_i^2\right)\sigma_{Z_i}^2+r_i\sigma_{Z_i}\Phi^{-1}(p)}, \qquad p \in (0,1), \quad (11)$$

$$CTE_p\left[S^l\right] = \sum_{i=1}^{n} \alpha_i \; e^{E[Z_i]+\frac{1}{2}\sigma_{Z_i}^2} \frac{\Phi\left(r_i \; \sigma_{Z_i} - \Phi^{-1}(p)\right)}{1-p}, \qquad p \in (0,1) . (12)$$

Finally, notice that the expected values of the r.v.'s S, $S^c$and S^l are all equal :

$$E(S) = E(S^l) = E(S^c) = \sum_{i=1}^{n} \alpha_i \; e^{E[Z_i]+\frac{1}{2}\sigma_{Z_i}^2}, \qquad (13)$$

whereas their variances are given by

$$Var(S) = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j \; e^{E[Z_i]+E[Z_j]+\frac{1}{2}(\sigma_{Z_i}^2+\sigma_{Z_j}^2)}(e^{cov(Z_i,Z_j)}-1), \qquad (14)$$

$$Var(S^l) = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j \; e^{E[Z_i]+E[Z_j]+\frac{1}{2}(\sigma_{Z_i}^2+\sigma_{Z_j}^2)}(e^{r_i r_j \sigma_{Z_i}\sigma_{Z_j}}-1) \qquad (15)$$

and

$$Var(S^c) = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j \; e^{E[Z_i]+E[Z_j]+\frac{1}{2}(\sigma_{Z_i}^2+\sigma_{Z_j}^2)}(e^{\sigma_{Z_i}\sigma_{Z_j}}-1), \qquad (16)$$

respectively.

2.2 The 'maximal variance' lowerbound approach

If $X \leq_{cx} Y$ and X and Y are not equal in distribution, then $Var[X] < Var[Y]$ must hold. An equality in variance would imply that $X \stackrel{d}{=} Y$. This indicates that if we want to replace S by the less convex S^l, the best approximations probably will be the ones where the variance of S^l is 'as close as possible' to the variance of S. In other words, we should try to choose the coefficients γ_i of the conditioning variable Λ defined in (4) such that the variance of S^l is maximized.

We will now prove that the first order approximation of the variance of S^l will be maximized for the following choice of the parameters γ_i:

$$\gamma_i = \alpha_i e^{E[Z_i]+\frac{1}{2}\sigma_{Z_i}^2}, \quad i = 1, \ldots, n. \qquad (17)$$

Indeed, from (15) we find that

$$\begin{aligned}
Var(S^l) &\approx \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j \; e^{E[Z_i]+E[Z_j]+\frac{1}{2}(\sigma^2_{Z_i}+\sigma^2_{Z_j})}(r_i r_j \sigma_{Z_i}\sigma_{Z_j}) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j \; e^{E[Z_i]+E[Z_j]+\frac{1}{2}(\sigma^2_{Z_i}+\sigma^2_{Z_j})} \left(\frac{Cov[Z_i,\Lambda]Cov[Z_j,\Lambda]}{Var(\Lambda)}\right) \\
&= \frac{(Cov(\sum_{i=1}^{n} \alpha_i \; e^{E[Z_i]+\frac{1}{2}\sigma^2_{Z_i}} Z_i, \Lambda))^2}{Var(\Lambda)} \\
&= (Corr(\sum_{i=1}^{n} \alpha_i \; e^{E[Z_i]+\frac{1}{2}\sigma^2_{Z_i}} Z_i, \Lambda))^2 \; Var(\sum_{i=1}^{n} \alpha_i \; e^{E[Z_i]+\frac{1}{2}\sigma^2_{Z_i}} Z_i). \quad (18)
\end{aligned}$$

Hence, the first order approximation of $Var(S^l)$ is maximized when Λ is given by

$$\Lambda = \sum_{i=1}^{n} \alpha_i \; e^{E[Z_i]+\frac{1}{2}\sigma^2_{Z_i}} Z_i. \quad (19)$$

In the remainder of this paper, we will always assume that the conditioning r.v. Λ is given by (19). Notice that this optimal choice for Λ is slightly different from the choice that was made for this r.v. in Dhaene, Denuit, Kaas, Goovaerts & Vyncke (2002b). Numerical comparisons reveal that the choice proposed here leads to more accurate approximations.

One can easily prove that the first order approximation for $Var(S^l)$ with Λ given by (19) is equal to the first order aproximation of $Var(S)$. This observation gives an additional indication that this particular choice for Λ will provide a good fit.

We emphasize that the conditioning r.v. Λ defined in (19) does not necessarily maximize the variance of S^l, but has to be understood as an approximation for the optimal Λ. Theoretically, one could use numerical procedures to determine the optimal Λ, but this would outweigh one of the main features of the convex bounds, namely that the quantiles and conditional tail expectations (and also other actuarial quantities such as stop-loss premiums) can easily be determined analytically. Having a ready-to-use approximation that can be implemented easily is important from a practical point of view.

3 Two well-known moment matching approximations

It belongs to the toolkit of any actuary to approximate the d.f. of an unknown r.v. by a known d.f. in such a way that the first moments are preserved. In this section we will briefly describe the reciprocal Gamma and the lognormal moment matching approximations. These two methods are frequently used to approximate the d.f. of the r.v. S defined by (1).

3.1 The Reciprocal Gamma approximation

The r.v. X is said to be Gamma distributed when its probability density function (p.d.f.) is given by

$$f_X(x;\alpha,\beta)=\frac{1}{\beta^{\alpha}\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta},\qquad x>0, \tag{20}$$

where $\alpha>0$, $\beta>0$ and $\Gamma(.)$ denotes the Gamma function:

$$\Gamma(\alpha)=\int_0^{\infty}u^{\alpha-1}e^{-u}du\quad(\alpha>0). \tag{21}$$

Consider now the r.v. $Y=1/X$. This r.v. is said to be reciprocal Gamma distributed. Its p.d.f. is given by

$$f_Y(y;\alpha,\beta)=f_X(1/y;\alpha,\beta)/y^2,\qquad y>0. \tag{22}$$

It is straightforward to prove that the quantiles and conditional tail expectations of Y are given by

$$Q_p\left[Y\right]=\frac{1}{F_X^{-1}(1-p;\alpha,\beta)},\qquad p\in(0,1) \tag{23}$$

and

$$CTE_p\left[Y\right]=\frac{F_X(F_X^{-1}(1-p;\alpha,\beta);\alpha-1,\beta)}{(1-p)(\alpha-1)\beta},\qquad p\in(0,1), \tag{24}$$

where $F_X(.;\alpha,\beta)$ is the cumulative d.f. of the Gamma distribution with parameters α and β. Since the Gamma distribution is readily available in many statistical software packages, these risk measures can easily be determined.

The first two moments of the reciprocal Gamma distributed r.v. Y are given by

$$E[Y]=\frac{1}{\beta(\alpha-1)} \tag{25}$$

and

$$E[Y^2]=\frac{1}{\beta^2(\alpha-1)(\alpha-2)}. \tag{26}$$

Expressing the parameters α and β in terms of $E[Y]$ and $E[Y^2]$ gives

$$\alpha=\frac{2E[Y^2]-E[Y]^2}{E[Y^2]-E[Y]^2} \tag{27}$$

and

$$\beta=\frac{E[Y^2]-E[Y]^2}{E[Y]E[Y^2]}. \tag{28}$$

The d.f. of the r.v. defined in (1) is now approximated by a reciprocal Gamma distribution with first two moments (13) and (14), respectively. The coefficients α and β of the reciprocal Gamma approximation follow from (27) and (28). The reciprocal Gamma approximations for the quantiles and the conditional tail expectations are then given by (23) and (24).

Dufresne (1990) proves that the present value of a continuous perpetuity with a Wiener logreturn process is reciprocal Gamma distributed, under suitable parameter restrictions. An elegant proof for this result can be found in Milevsky (1997).

Intuitively, one expects that the present value of a finite discrete annuity with a normal logreturn process with independent periodic returns, can be approximated by a reciprocal Gamma distribution, provided the time period involved is long enough. This idea was set forward and explored in Milevsky & Posner (1998), Milevsky & Robinson (2000) and Huang, Milevsky & Wang (2004).

3.2 The lognormal approximation

The r.v. X is said to be lognormal distributed if its p.d.f. is given by

$$f_X(x;\mu,\sigma^2) = \frac{1}{x\sigma\sqrt{2\pi}} e^{\frac{-(\log x-\mu)^2}{2\sigma^2}}, \qquad x > 0, \tag{29}$$

where $\sigma > 0$.

The quantiles and conditional tail expectations of X are given by

$$Q_p\left[X\right] = e^{\mu+\sigma\Phi^{-1}(p)}, \qquad p \in (0,1) \tag{30}$$

and

$$CTE_p\left[X\right] = \ e^{\mu+\frac{1}{2}\sigma^2} \frac{\Phi\left(\sigma - \Phi^{-1}(p)\right)}{1-p}, \qquad p \in (0,1)\,. \tag{31}$$

The first two moments of X are given by

$$E[X] = e^{\mu+\frac{1}{2}\sigma^2} \tag{32}$$

and

$$E[X^2] = e^{2\mu+2\sigma^2}. \tag{33}$$

Expressing the parameters μ and σ^2 of the lognormal distribution in terms of $E[X]$ and $E[X^2]$ leads to

$$\mu = \log\left(\frac{E[X]^2}{\sqrt{E[X^2]}}\right) \tag{34}$$

and

$$\sigma^2 = \log\left(\frac{E[X^2]}{E[X]^2}\right). \tag{35}$$

The same procedure as the one explained in the previous subsection can be followed in order to obtain a lognormal approximation for S, with the first two moments matched. Dufresne (2002) obtains a lognormal limit distribution for S as volatility σ tends to zero and this provides a theoretical justification for the use of the lognormal approximation.

4 Comparing the approximations

In order to compare the performance of the different approximations presented above, we consider the r.v. S_n which is defined as the random present value of a series of n deterministic unit payment obligations due at times 1, 2, ..., n, respectively:

$$S_n = \sum_{i=1}^{n} e^{-Y_1-Y_2-\cdots-Y_i} \overset{not}{=} \sum_{i=1}^{n} e^{Z_i}. \tag{36}$$

where the r.v. Y_i is the random return over the year $[i-1, i]$ and $e^{-(Y_1+Y_2+\cdots+Y_i)}$ is the random discount factor over the period $[0, i]$.
We will assume that the yearly returns Y_i are i.i.d. normal distributed with mean $\left(0.075 - \frac{\sigma^2}{2}\right)$ and variance σ^2. Notice that S_n is a r.v. of the general type defined in (1).

The provision or reserve to set up at time 0 for these future unit payment obligations can be determined as $Q_p[S_n]$ or $CTE_p[S_n]$, with p sufficiently large. A provision equal to $Q_{0.95}[S_n]$ for instance, will guarantee that all payments can be made with a probability of 0.95, see for instance Dhaene, Vanduffel, Goovaerts, Kaas & Vyncke (2004).

As the time unit that we consider is long (1 year), assuming a Gaussian model for the returns seems to be appropriate, at least approximately, by the Central Limit Theorem. In order to verify whether our theoretical setup can be approximately compared with the data generating mechanism of real situations, we refer to Cesari & Cremonini (2003). They investigate four well-known stock market indices in US dollars, from Morgan Stanley: MSCI World, North America, Europe and Pacific, covering all major stock markets in industrial as well as emerging countries. For the period 1997-1999, the authors conclude that weekly (and longer period) returns can be considered as normal and independent. Daily returns on the other hand are both non-normal and autocorrelated.

In order to compute the comonotonic approximations for quantiles and conditional tail expectations, notice that $E[Z_i]$, $\sigma^2_{Z_i}$ and r_i are given by

$$E[Z_i] = -i(0.075 - \frac{\sigma^2}{2}), \tag{37}$$

$$\sigma^2_{Z_i} = i\,\sigma^2 \tag{38}$$

and

$$r_i = \frac{\sum_{j=1}^{i}\sum_{k=j}^{n}\gamma_k}{\sqrt{i\sum_{j=1}^{n}\left(\sum_{k=j}^{n}\gamma_k\right)^2}}, \tag{39}$$

with γ_k given by

$$\gamma_k = e^{E[Z_k]+\frac{1}{2}\sigma^2_{Z_k}}, \quad k = 1, \ldots, n.$$

Notice that the correlation coefficients r_i are positive, so that the formulae (11) and (12) can be applied.

n	Method	$\sigma = 0.05$	$\sigma = 0.15$	$\sigma = 0.25$	$\sigma = 0.35$
20	UB	+3.24%	+8.02%	+9.36%	+7.50%
	LB	**-0.01%**	**+0.02%**	**+0.00%**	**+0.35%**
	RECG	+0.07%	-0.15%	-4.28%	-14.27%
	LN	-0.16%	-0.06%	+2.99%	+9.04%
	MC (±s.e)	12.1957 (0.04%)	20.4592 (0.10%)	41.5854 (0.25%)	106.1389 (0.30%)
40	UB	+4.39%	+10.26%	+9.42%	+1.47%
	LB	**+0.00%**	**-0.06%**	**+0.06%**	**-0.83%**
	RECG	+0.06%	-0.55%	-8.52%	-19.70%
	LN	-0.23%	+0.58%	+9.73%	+9.96%
	MC (±s.e)	15.4733 (0.04%)	30.4033 (0.16%)	87.7482 (0.32)	427.0793 (0.49)

Table 1: Approximations for the 0.95-quantile of S_n for different horizons and volatilities.

Now we will compare the performance of the different approximation methods that were presented in Sections 2 and 3: the comonotonic upperbound method (UB), the comonotonic 'maximal variance' lowerbound method (LB), the reciprocal Gamma method (RG) and the lognormal method (LN).

We will compare the different approximations for quantiles and conditional tail expectations with the values obtained by Monte-Carlo simulation. The simulation results are based on generating 500.000 random paths. The estimates obtained from this time-consuming simulation will serve as benchmark. The random paths are based on antithetic variables in order to reduce the variance of the Monte-Carlo estimates.

The tables that we will present display the Monte Carlo simulation result (MC) for the risk measure at hand, as well as the procentual deviations of the different approximation methods, relative to the Monte-Carlo result. For the quantiles and conditional tail expectations, these procentual deviations are defined as follows:

$$\frac{Q_p[S_n^{approx}] - Q_p[S_n^{MC}]}{Q_p[S_n^{MC}]} \times 100\%$$

and

$$\frac{CTE_p[S_n^{approx}] - CTE_p[S_n^{MC}]}{TVaR_p[S_n^{MC}]} \times 100\%,$$

where S_n^{approx} corresponds to one of the approximation methods and S_n^{MC} denotes the Monte Carlo simulation result. The figures displayed in bold in the tables correspond to the best approximations, this means the ones with the smallest procentual deviation compared to the Monte-Carlo results. In the tables, we also present the standard errors of the Monte Carlo estimates. Note that these standard errors are also expressed as a procentual deviation from the Monte Carlo estimate.

Method	p=0.995	p=0.90	p=0.75	p=0.50	p=0.25
UB	+17.41%	+6.11%	+0.32%	-6.15%	-12.55%
LB_v	**-0.65%**	**+0.12%**	**-0.03%**	**-0.10%**	**+0.13%**
RECG	+0.73%	-4.19%	-2.52%	+1.20%	+6.18%
LN	-3.76%	+4.19%	+3.81%	+0.25%	-6.36%
MC (±s.e)	84.0466 (0.51%)	32.0758 (0.25%)	21.2666 (0.12%)	13.8933 (0.04%)	9.3833 (0.09%)

Table 2: Approximations for some selected quantiles of S_{20}. The yearly volatility equals 0.25.

n	Method	$\sigma = 0.05$	$\sigma = 0.15$	$\sigma = 0.25$	$\sigma = 0.35$
20	UB	+4.19%	+10.98%	+14.17%	+12.98%
	LB	**-0.02%**	**-0.14%**	**-0.36%**	**-0.59%**
	RECG	+0.21%	+1.18%	-0.98%	-15.41%
	LN	-0.38%	-1.88%	-0.94%	+4.56%
	MC (±s.e)	12.8231 (1.04%)	24.4591 (2.16%)	59.6646 (2.90%)	198.0164 (3.27%)
40	UB	+5.86%	+15.11%	+16.87%	+10.45%
	LB	**+0.09%**	**-0.25%**	**-0.59%**	**-0.84%**
	RECG	+0.28%	+0.87%	-7.49%	-40.77%
	LN	-0.48%	-2.38%	+4.18%	+12.77%
	MC (±s.e)	16.3994 (1.55%)	38.2515 (2.61%)	149.8569 (3.25%)	1206.0858 (3.59%)

Table 3: Approximations for the 0.95-quantile of S_n for different horizon and volatility levels.

Table 1 summarizes the results for the 0.95-quantiles for different yearly volatilities σ and for a time horizon of $n = 20$ and $n = 40$, respectively. The maximum variance lowerbound approach (LB) turns out to fit the quantiles the best for all values of the parameters. Its approximated quantiles fall almost always in the confidence interval $Q_p[S_n^{MC}] \pm s.e.$. It appears that the performance of the reciprocal Gamma approximations is worse for higher levels of volatility and for longer time horizons. This latter result is surprising, given the convergence of the d.f. of S_n to the reciprocal Gamma distribution. The results indicate that this convergence occurs very slowly.

Table 2 compares the different approximations for some selected quantiles of S_{20}, with a fixed yearly volatility of 25% . The results are in line with the previous ones. The lower bound approach outperforms all the others, for high as well as for low values of p.

Table 3 displays the approximated and simulated 95% conditional tail expectations for the same set of parameters as in Table 1. Again the lowerbound approach approximates the exact conditional tail expectations extremely well.

The same conclusions can be drawn from the results in Table 4. This ta-

Method	p=0.995	p=0.90	p=0.75	p=0.50	p=0.25
UB	+21.00%	+11.71%	+7.87%	+4.34%	+1.89%
LB	**-0.99%**	**-0.21%**	**-0.11%**	**-0.09%**	**-0.10%**
RECG	+7.82%	-2.25%	-2.82%	-2.16%	-1.17%
LN	-7.97%	+0.80%	+2.31%	+2.33%	+1.44%
MC (±s.e)	111.5457 (3.24%)	47.9276 (2.77%)	34.6099 (2.52%)	25.8692 (2.23%)	21.0969 (1.96%)

Table 4: Approximations for $CTE_p[S_{20}]$. The yearly volatility equals 0.25.

ble reports the different approximations for $CTE_p[S_{20}]$ for different probability levels p and a fixed yearly volatility $\sigma = 0.25$.

From the 4 tables, one can observe that both moment matching techniques perform poorly for high levels of p and/or σ. The comonotonic lower bound approach however, remains to produce accurate approximations, also in these extreme cases.

Finally, remark that in general the procentual deviation of the comonotonic upper bound compared to the MC-simulation is relatively high. From the Tables 1 and 3, however, we can conclude that for high volatility levels, the crude comonotonic upper bound approximation often performs better than the reciprocal Gamma approximation.

5 Conclusion

In this paper, we compared some approximation methods for a standard actuarial and financial problem: the determination of quantiles and conditional tail expectations of the present value of a series of cash-flows, when discounting is performed by a Brownian motion process. We tested the accuracy of the comontonic lower and upper bound approximations and two moment matching approximations by comparing these approximations with the estimates obtained from extensive Monte Carlo simulations.

Overall, the recently developed comononotonic maximal variance lower bound approach provides the best fit and leads to accurate approximations under varying parameter assumptions, which are in line with realistic market values.

The comonotonic approach has the additional advantage that it gives rise to easy computable approximations for any risk measure that is additive for comonotonic risks. Examples of such risk measures are the distortion risk measures which were introduced in the actuarial literature by Wang (2000).

Finally notice that the comonotonic lower bound approximation that we presented here can easily be transformed to the case when accumulating saving amounts to a final value, and also to the case where the cash flow payments vary from period to period, see Dhaene, Vanduffel, Goovaerts, Kaas & Vyncke (2004).

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