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RESEARCH REPORT 0229

TRANSITION PROBABILITIES FOR DIFFUSION EQUATIONS BY MEANS OF PATH INTEGRALS

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D/2002/2376/29

Transition probabilities for diffusion equations by means of path integrals

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Abstract

In this paper, we investigate the transition probabilities for diffusion processes. In a first part, we show how transition probabilities for rather general diffusion processes can always be expressed by means of a path integral. For several classical models, an exact calculation is possible, leading to analytical expressions for the transition probabilities and for the maximum probability paths. A second part consists of the derivation of an analytical approximation for the transition probability, which is useful in case the path integral is too complex to be calculated. The approximation we present, is based on a convex combination of a new analytical upper and lower bound for the transition probabilities. The fact that the approximation is analytical has some important advantages, e.g. for the investigation of Asian options. Finally, we demonstrate the accuracy of the approximation by means of a few graphical illustrations.

Keywords : diffusion processes, transition probability, path integral, comonotonicity.

1 Introduction

Dynamic models, and more specifically continuous-time models, are widely used and studied nowadays in pricing and investment theories. Most of the existing one-factor models refer to the general diffusion equations, which are stochastic differential equations in the form

$$dY(t) = \mu(Y(t), t)\ dt + \sigma(Y(t), t)\ dW(t)\ . \tag{1}$$

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This equation defines a stochastic process $Y = \{Y(s), s \in [0,t]\}$, reflecting e.g. the price process in time. In this equation, $W = \{W(s), s \in [0,t]\}$ is a standard Brownian motion, $\mu(y,t)$ is the drift of the process Y, and $\sigma^2(y,t)$ is the diffusion of Y. Both μ and σ can contain one or more parameters.

In this contribution, we will assume that the drift μ and the diffusion σ^2 do not depend explicitly on time t. Thus, we consider stochastic differential equations of the form

$$dY(t) = A(Y(t))\, dt + B(Y(t))\, dW(t)\ , \tag{2}$$

where as in the general diffusion model, the functions $A(y)$ and $B(y)$ can contain parameters. Fortunately, this time-independence is only a minor restriction, since most of the classical models e.g. for interest rates are members of this class of processes (see also section 8).

One of the questions in this context is to find a closed-form expression for the probability of the process Y reaching the value y_t at time t given the value y_s at a former point in time $s \leq t$. We will use the notation

$$p(t_o, y_o; t_e, y_e) = \frac{d}{dy_e} Prob\left[Y(t_e) \leq y_e | Y(t_o) = y_o\right]$$

for the transition density of the process Y. The knowledge of this density is important for instance in the framework of derivative pricing, where the stochastic process Y then reflects the price process.
Contrary to the rather simple form of the diffusion equation (2), such a closed-form is only known for a few cases, e.g. the Wiener model, the geometric Wiener model, the Vasicek model and the Cox-Ingersol-Ross model.

In a paper of 1999 (see [1]), Ait-Sahalia presented a method leading to a closed-form approximation for the exact transition density. In this case the advantages for derivative pricing remain, be it that the accuracy diminishes. The method Ait-Sahalia proposes, converges for $\Delta t = t_e - t_o$ going to zero, but may lead to bad approximations when the time horizon increases. For financial applications, the author says that Δt is never bigger than three or six months, so this will not cause any problems. However, in actuarial applications, we may need a much larger horizon.

In the present paper, we want to give an answer towards the solution of the problems sketched above. We show how for general types of diffusion processes, whether the time interval is small or big, the transition density $p(t_o, y_o; t_e, y_e)$ can be expressed by means of a Feynman path integral. This is a powerful concept borrowed from quantum mechanics used to describe the amplitude to move between two points if each possible path is given a certain probability.

Making use of specific properties and calculation techniques on path integrals, we show that an exact calculation is possible for the four models mentioned earlier, but also for some more types of processes.

Starting from the path integral expression for the transition density, we also show how it is possible to find in any case a closed-form approximation for the transition density, with very high accuracy.

The paper is organized as follows. We start with a brief description of the concepts and notations about stochastic differential equations and Feynman path integrals in section 2. Section 3 contains the first important result, expressing the transition densities for general diffusion processes by means of a path integral. In section 4, we show how the modal path or maximum probability path can be determined. Section 5 is meant to prove how the famous Itô lemma can be translated into the path integral formalism. Afterwards in section 6, we mention the limiting case of long term probabilities. Section 7 –together with section 3 of course– constitutes the "body" of this paper. Here we show, based on the path integral expression, how the transition density for general diffusion processes can be approximated with a closed-form formula. In section 8 we present examples of the methodology, for common models in the financial theory. We give an expression for the transition probability in each case, together with an explicit calculation if possible. Section 9 demonstrates the accuracy of our new approximation by means of a few graphical illustrations.
The proofs of the theorems and some explicit computational results about path integrals are brought together in the appendix.

2 Definitions

2.1 Stochastic differential equations

In order to explain the similarities and dissimilarities between Itô integrals and path integrals, we briefly introduce the concept of a general stochastic differential equation.

A θ-stochastic differential equation is defined as

$$dY(t) = a(Y(t),t)dt + b(Y(t),t)_\theta \, dW(t) \tag{3}$$

with solution

$$Y(t) = Y(0) + \int_0^t a(Y(s),s)ds + \int_0^t b(Y(s),s)_\theta \, dW(s) \, , \tag{4}$$

where $W(t)$ is a standard Brownian motion.
The first integral in (4) is a Riemann-integral, the second one is a θ-stochastic integral.

If $X = \{X(s), s \in [0, t]\}$ is a process adapted to the natural Brownian filtration, the θ-stochastic integral

$$I_t^{(\theta)}(X) = \int_0^t X(s)_\theta \, dW(s) \tag{5}$$

is defined by

$$I_t^{(\theta)}(X) = \lim_{n \to \infty} \sum_{i=0}^{n-1} X(t_i^\theta) \left(W(t_{i+1}) - W(t_i)\right) \tag{6}$$

for any partition

$$0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = t \tag{7}$$

and with t_i^θ equal to

$$t_i^\theta = t_i + \theta(t_{i+1} - t_i). \tag{8}$$

We want to draw attention to three special choices of θ.

- When θ is equal to zero, the values of X are chosen at the left points, and the θ-stochastic integral coincides with an Itô stochastic integral. We will use the notation

$$I_t^{(0)}(f(W)) = \int_0^t f(W(s)_L) \, dW(s) \tag{9}$$

  when we use this type of integration. The most important advantage of this choice is the fact that Itô stochastic integrals satisfy the martingale property. A disadvantage however is that the chain rule of classical calculus is not valid.

- When θ is equal to $1/2$, the values of X are chosen at the mid points, and the θ-stochastic integral reduces to a Stratonovich stochastic integral. We will use the notation

$$I_t^{(1/2)}(f(W)) = \int_0^t f(W(s)) \, dW(s) \tag{10}$$

  (without index) when we use this type of integration. The stochastic integral no longer satisfies the martingale property, but now the classical chain rule is formally satisfied, or

$$\int_0^t f'(W(s))_{\theta=1/2} \, dW(s) = f(W(t)) - f(W(0)). \tag{11}$$

  Since stochastic integrals with $\theta = 1/2$ behave like Riemann integrals (to a certain extent), the omittance of an index seems acceptable.

- The situation with θ equal to 1 corresponds to a choice for the right points. We will denote this kind of integration as

$$I_t^{(1)}(f(W)) = \int_0^t f(W(s)_R) \, dW(s). \tag{12}$$

The following relation between general θ-stochastic integrals and Stratonovich stochastic integrals will be very helpful in the development of our methodology :

$$\int_0^t f(W(s)_\theta)\, dW(s) = \int_0^t f(W(s))\, dW(s) + (\theta - \frac{1}{2}) \int_0^t f'(W(s))\, ds. \qquad (13)$$

A proof can be found in an easy way using Taylor expansions.

2.2 Path Integrals

Feynman path integrals originate from quantum mechanics, where they are used to describe the amplitude to go from one point to another point, where each possible trajectory is given a certain probability. When using imaginary times, a Feynman path integral provides a very efficient tool in the derivation of transition probabilities.

A Feynman path integral

$$K(t_o, x_o; t_e, x_e) = \int_{(t_o,x_o)}^{(t_e,x_e)} Dx(s)\, e^{-\int_{t_o}^{t_e} L(\dot{x}(s),x(s),s)\, ds} \qquad (14)$$

where $L(\dot{x}(s), x(s), s)$ is called the Lagrangian, is defined by

$$\begin{aligned} K(t_o, x_o; t_e, x_e) &= lim_{n\to\infty} \frac{1}{\sqrt{2\pi\varepsilon}^n} \int dx_1 \int dx_2 ... \int dx_{n-1} \\ &\quad \cdot\, e^{-\varepsilon \sum_{i=0}^{n-1} L\left(\frac{x_{i+1}-x_i}{\varepsilon}, \frac{x_i + x_{i+1}}{2}, \frac{t_i + t_{i+1}}{2}\right)} \end{aligned} \qquad (15)$$

for a partition

$$t_o = t_0 < t_1 < ... < t_n = t_e \qquad (16)$$

where $\varepsilon = (t_e - t_o)/n$ and $t_{i+1} = t_i + \varepsilon$ and where we used the notation $x_i = x(t_i)$.

It is important to note that in fact this definition makes use of a midpoint choice as it was the case for the partition in a Stratonovich stochastic integral. As a consequence, one has to be very careful when comparing or mixing Itô calculus and Feynman path integrals.

As an example, we consider a Brownian motion, for which the Lagrangian is equal to $L(\dot{x}, x, s) = \frac{\dot{x}^2}{2}$. In that case the multiple integration can be worked out in a straightforward way, resulting in

$$\int_{(t_o,x_o)}^{(t_e,x_e)} Dx(s)\, e^{-\frac{1}{2}\int_{t_o}^{t_e} \dot{x}(s)^2\, ds}$$

$$= lim_{n\to\infty} \frac{1}{\sqrt{2\pi\varepsilon}^n} \int dx_1 ... \int dx_{n-1}\; e^{-\frac{\varepsilon}{2}\sum_{i=0}^{n-1}\left(\frac{x_{i+1}-x_i}{\varepsilon}\right)^2}$$

$$= \frac{1}{\sqrt{2\pi(t_e - t_o)}}\; e^{-\frac{(x_e - x_o)^2}{2(t_e - t_o)}} . \tag{17}$$

This well known result can be read as the transition probability to go from the starting point x_o at time t_o to the final point x_e at time t_e when the underlying process is a standard Brownian motion.

A handsome result about Feynman path integrals can be found in the so called Kolmogorov property. It shows how to write a path integral as a combination of successive events :

$$K(t_o, x_o; t_e, x_e) = \int_{-\infty}^{+\infty} dx_s\; K(t_o, x_o; t_s, x_s) \cdot K(t_s, x_s; t_e, x_e), \tag{18}$$

where t_s is any time between t_o and t_e.

Proofs, applications and more details about this powerful concept can be found e.g. in [3, 6, 5]. Important computational results are summarized in appendix A.

3 Transition Probabilities

In this section, we show how the transition probability for stochastic processes defined by means of a stochastic differential equation, can be expressed by means of a Feynman path integral. We start with a diffusion equation with unit diffusion, and we generalize the result for equations where the diffusion is a function of the stochastic process. Proofs are provided in appendix B.

Theorem 3.1 *Consider a θ-stochastic differential equation*

$$dY(t) = A(Y(t))\; dt + dW(t) \tag{19}$$

where $W(t)$ is a standard Brownian motion.
The transition probability for the stochastic process $Y = \{Y(s), s \in [0,t]\}$ can be written by means of a path integral as

$$p(0, y_0; t, y_t) = \frac{d}{dy_t} Prob\,[Y(t) \leq y_t | Y(0) = y_0] \tag{20}$$

$$= \int_{(0,y_0)}^{(t,y_t)} Dy(s)\; e^{-\frac{1}{2}\int_0^t \dot{y}^2 ds - \frac{1}{2}\int_0^t \left(A(y)^2 + \frac{\partial A}{\partial y}\right) ds + \int_0^t A(y) dy} .$$

This result is independent of the choice of θ in the definition of the stochastic integral.

Remark 1. If the domain of the stochastic process Y is $(0, +\infty)$ instead of $(-\infty, +\infty)$, the differential part $Dy(s)$ has to be replaced by $D_+y(s)$.

Remark 2. The last integral in the exponent of (20) behaves as a Stratonovich integral. A transformation into an Itô integral as mentioned in (13), enables us to write the short time transition probability as

$$\begin{aligned} &p(0, y_0; \Delta t, y_\Delta) \\ &= \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{(y_\Delta - y_0)^2}{2\Delta t} - \frac{\Delta t}{2} A^2(y_0) + A(y_0)(y_\Delta - y_0)} \\ &= \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{(y_\Delta - y_0 - \Delta t A(y_0))^2}{2\Delta t}}, \end{aligned} \tag{21}$$

which coincides with the classical expression for measures associated with diffusion processes with unit volatility.

Remark 3. The constitution of the path integral (20) nicely illustrates Girsanov's theorem (see e.g. [12]).

Indeed, the process

$$M(t) = \exp\left\{\int_0^t A(W(s)_L) dW(s) - \frac{1}{2}\int_0^t A^2(W(s)) ds\right\} \tag{22}$$

is the Radon-Nikodym derivative of the measure in (20) to the Wiener measure. As a consequence, the diffusion process defined by (19) is a Brownian motion with respect to the measure defined by the transition probability in (20).

Note that in case the stochastic process Y has domain $(0, +\infty)$ instead of $(-\infty, +\infty)$, the process $M(t)$ of (22) is the Radon-Nikodym like derivative with respect to an absorbed Brownian motion. In the latter case, it could be a problem that both measures are not equivalent, since the probability of staying in 0 is different from zero.

Proposition 3.1 *The transition probability in (20) also satisfies the forward Fokker Planck equation*

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial y_t^2} - \frac{\partial}{\partial y_t}\left(A(y_t) \cdot p\right), \tag{23}$$

where p is used as a short hand notation for the probability $p(0, y_0; t, y_t)$.

Theorem 3.2 *Consider a θ-stochastic differential equation*

$$dY(t) = A(Y(t))\, dt + B(Y(t))_\theta\, dW(t) \tag{24}$$

where $W(t)$ is a standard Brownian motion.
A change of variables

$$X(t) = \int_0^t \frac{dY(s)}{B(Y(s))_{\theta=1/2}} = \psi(Y(t)) \tag{25}$$

results in the new stochastic differential equation

$$dX(t) = \left(\frac{A(\psi^{-1}(X(t)))}{B(\psi^{-1}(X(t)))} + (\theta - \frac{1}{2}) \frac{\partial B}{\partial y} (\psi^{-1}(X(t))) \right) dt + dW(t). \tag{26}$$

This result is dependent on the choice of θ in the definition of the stochastic integral.

Theorem 3.3 *Consider an Itô stochastic differential equation*

$$dY(t) = A(Y(t))\, dt + B(Y(t)_L)\, dW(t) \tag{27}$$

where $W(t)$ is a standard Brownian motion, and where

$$\psi(y) = \int^y \frac{dz}{B(z)} \tag{28}$$

defines a non-decreasing function.
The transition probability for the stochastic process $Y = \{Y(s), s \in [0,t]\}$ can be written by means of a path integral as

$$p(0, y_0; t, y_t) = \frac{d}{dy_t} Prob\left[Y(t) \leq y_t | Y(0) = y_0\right] \tag{29}$$

$$= \frac{1}{B(y_t)} \cdot \int_{(0,\psi(y_0))}^{(t,\psi(y_t))} Dy(s)\, e^{-\frac{1}{2} \int_0^t \dot{y}^2 ds}$$

$$\cdot\, e^{-\frac{1}{2} \int_0^t \left(T(y)^2 + \frac{\partial T}{\partial y} \right) ds + \int_0^t T(y) dy},$$

where the function T is defined by

$$T(z) = \frac{A(\psi^{-1}(z))}{B(\psi^{-1}(z))} - \frac{1}{2} \frac{\partial B}{\partial y} (\psi^{-1}(z)). \tag{30}$$

Remark 1. If the domain of the stochastic process $\psi(Y) = \{\psi(Y(s)),\ s \in [0,t]\}$ is $(0,+\infty)$ instead of $(-\infty,+\infty)$, the differential part $Dy(s)$ as before has to be replaced by $D_+y(s)$.

Remark 2. For the stochastic process of theorem 3.3, the short time transition probability is equal to

$$\begin{aligned} & p(0,y_0;\Delta t,y_\Delta) \\ & = \frac{1}{B(y_\Delta)}\frac{1}{\sqrt{2\pi\Delta t}}\, e^{-\dfrac{(\psi(y_\Delta)-\psi(y_0)-\Delta t\, T(\psi(y_0)))^2}{2\Delta t}} . \end{aligned} \tag{31}$$

Since we are dealing with infinitesimal time periods, we can write

$$\psi(y_\Delta)-\psi(y_0) = (y_\Delta - y_0)\cdot\psi'(y_0) = (y_\Delta - y_0)\cdot\frac{1}{B(y_0)}\,, \tag{32}$$

and with an explicitation of T, we obtain the classical expression

$$p(0,y_0;\Delta t,y_\Delta) = \frac{1}{\sqrt{2\pi\Delta t B^2(y_0)}}\, e^{-\dfrac{(y_\Delta - y_0 - \Delta t A(y_0))^2}{2\Delta t B^2(y_0)}} . \tag{33}$$

Proposition 3.2 *The transition probability in (29) now satisfies the forward Fokker Planck equation*

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial y_t^2}\left(B(y_t)\cdot p\right) - \frac{\partial}{\partial y_t}\left(A(y_t)\cdot p\right), \tag{34}$$

where p is used as a short hand notation for the probability $p(0,y_0;t,y_t)$.

For $\theta = 0$, the previous results were already derived by the same authors earlier (see [4]) ; however, in that contribution the path integrals were found in a completely different way, without making use of Itô calculus. The result of theorem 3.2 for $\theta = 0$ is also mentioned in [1].

4 Maximal probability path

As mentioned before, a Feynman path integral $K(t_o,x_o;t_e,x_e)$ as in (14) describes the amplitude to go from the point x_o at time t_o to the point x_e at time t_e, where each trajectory is given a certain probability according to the stochastic process related to the Lagrangian. In fact, in the whole set of trajectories connecting the two points, only paths in the vicinity of the classical or modal path provide important contributions to $K(t_o, x_o; t_e, x_e)$. Indeed, for other paths, there are always neighbouring trajectories that cancel out their contribution.

This modal path, or maximum probability path, can be determined (see e.g. [6]) as the solution of the ordinary second order differential equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \tag{35}$$

subject to the boundary conditions

$$\begin{cases} x(t_o) &= x_o \\ x(t_e) &= x_e. \end{cases} \tag{36}$$

As an example, if we consider the Brownian motion, the maximum probability path is given by

$$x_{mod}(s|t_o, x_o; t_e, x_e) = \frac{t_e - s}{t_e - t_o} x_o + \frac{s - t_o}{t_e - t_o} x_e. \tag{37}$$

Looking for the modal path for stochastic processes defined by stochastic differential equations in a form as in section 3, the following nice result appears.

Theorem 4.1 *Consider a stochastic process $Y = \{Y(s), s \in [0,t]\}$ defined by a diffusion equation with unit diffusion (19). The maximal probability path $y_{mod}(s)$ for this process when starting at point y_0 at time 0 and arriving at y_t at time t, can be determined implicitly by*

$$\int_{y_0}^{y_{mod}(s)} \frac{dy}{\sqrt{A(y)^2 + \frac{\partial A}{\partial y} + C(y_0, y_t)}} = \pm s\ , \tag{38}$$

where $C(y_0, y_t)$ is fixed by the condition $y_{mod}(t) = y_t$. The sign in the right hand side is equal to the sign of the difference $y_{mod}(s) - y_0$.

Theorem 4.2 *Consider a stochastic process $Y = \{Y(s), s \in [0,t]\}$ defined by a diffusion equation with unit diffusion (27). The maximal probability path $y_{mod}(s)$ for this process when starting at point y_0 at time 0 and arriving at y_t at time t, can be determined implicitly by*

$$\int_{\psi(y_0)}^{\psi(y_{mod}(s))} \frac{dx}{\sqrt{T(x)^2 + \frac{\partial T}{\partial x} + C(y_0, y_t)}} = \pm s\ , \tag{39}$$

where ψ is defined in (28), T is defined in (30), and where $C(y_0, y_t)$ is fixed by the condition $y_{mod}(t) = y_t$. The sign in the right hand side is equal to the sign of the difference $\psi(y_{mod}(s)) - \psi(y_0)$.

5 The Itô lemma in the path integral formalism

Consider again a stochastic process $Y = \{Y(s), s \in [0,t]\}$ determined by the stochastic differential equation

$$dY(t) = A(Y(t))\, dt + dW(t), \tag{40}$$

where $W(t)$ is a standard Brownian motion. From theorem 3.1, we know that the transition probability can be written by means of the path integral

$$\begin{aligned} &p_Y(0, y_0; t, y_t) \\ &= \int_{(0,y_0)}^{(t,y_t)} Dy(s)\, e^{-\frac{1}{2}\int_0^t \dot{y}^2 ds - \frac{1}{2}\int_0^t \left(A(y)^2 + \frac{\partial A}{\partial y}\right) ds + \int_0^t A(y) dy} . \end{aligned} \tag{41}$$

Following Itô's lemma, the stochastic differential equation for the process $X = \{X(s), s \in [0,t]\}$ when

$$Y(t) = f(X(t)), \tag{42}$$

is given by

$$\begin{aligned} dX(t) &= \frac{1}{2}\frac{\partial^2}{\partial y^2}[f^{-1}(Y(t)_L)]\, dt + \frac{\partial}{\partial y}[f^{-1}(Y(t)_L)]\, dY(t) \\ &= -\frac{1}{2}\frac{f''(Y(t)_L)}{f'(Y(t)_L)^3}\, dt + \frac{1}{f'(Y(t)_L)}\, dY(t), \end{aligned} \tag{43}$$

or in the other direction

$$dY(t) = \frac{1}{2}\frac{f''(X(t)_L)}{f'(X(t)_L)^2}\, dt + f'(X(t)_L)\, dX(t). \tag{44}$$

The question that arises is : how can this transformation be extended into the path integral (41) ? Note that we have to take into account the difficulty that, contrary to the Itô lemma, the integrations in the path integral are of the Stratonovich type.

Making use of a stochastic time change in the path integral (41), one can prove the following result (see appendix B) :

Theorem 5.1 *Consider the stochastic differential equation*

$$dY(t) = A(Y(t))\, dt + dW(t) \tag{45}$$

where $W(t)$ is a standard Brownian motion, and a transformation

$$Y(t) = f(X(t)), \tag{46}$$

for which the inverse function is well-defined.

Starting from the path integral expression for the transition probability for the process $Y = \{Y(s), s \in [0,t]\}$, the transition probability for the process $X = \{X(s), s \in [0,t]\}$ can be found as

$$
\begin{aligned}
& p_X(0, x_0 = f^{-1}(y_0); t, x_t = f^{-1}(y_t)) \\
& \quad = \frac{1}{\sqrt{f'(f^{-1}(y_0)) \cdot f'(f^{-1}(y_t))}} \cdot \int_{-\infty}^{+\infty} d\beta\, e^{i\beta t} \int_0^{+\infty} dt^* \\
& \qquad \cdot \int_{(0,f^{-1}(y_0))}^{(t^*,f^{-1}(y_t))} Dx(\sigma)\, e^{-\frac{1}{2}\int_0^{t^*} \dot{x}^2 d\sigma - i\beta \int_0^{t^*} f'(x)^2 d\sigma} \\
& \qquad \cdot e^{-\frac{1}{2}\int_0^{t^*} \left(A[f(x)]^2 + \frac{\partial A}{\partial y}[f(x)]\right) f'(x)^2 d\sigma} \\
& \qquad \cdot e^{+\int_0^{t^*} A[f(x)] f'(x) dx - \frac{1}{8}\int_0^{t^*} \left[3\frac{f''(x)^2}{f'(x)^2} - 2\frac{f'''(x)}{f'(x)}\right] d\sigma} .
\end{aligned}
\tag{47}
$$

As an example, consider the transformation

$$Y(t) = f(X(t)) = g^{-1}(X(t)) \tag{48}$$

where the function g is chosen in such a way that

$$g'(y) = e^{-2\int_{y_0}^{y} A(z)\, dz} . \tag{49}$$

Following Itô's lemma, for this choice we know that

$$dX(t) = g'(Y(t)_L)\, dW(t) = \frac{1}{f'(X(t)_L)}\, dW(t) , \tag{50}$$

or

$$dW(t) = f'(X(t)_L)\, dX(t) . \tag{51}$$

If we apply theorem 5.1, due to (49) which enables us to simplify and eliminate the integrations over derivatives of f, a straightforward calculation leads to the result

$$
\begin{aligned}
p_X(0, x_0; t, x_t) = & \int_0^{+\infty} dt^*\ \delta\left(t - \int_0^{t^*} f'(x)^2 d\sigma\right) \\
& \cdot \int_{(0,x_0)}^{(t^*,x_t)} Dx(\sigma)\, e^{-\frac{1}{2}\int_0^{t^*} \dot{x}^2 d\sigma} ,
\end{aligned}
\tag{52}
$$

which nicely fits with (51).

6 Long term probabilities

An interesting limiting case is the long term probability or stationary probability for a stochastic process defined by means of a stochastic differential equation (19) or (27).

For the transition probability

$$p(0, y_0; t, y_t) = \frac{d}{dy_t} Prob\left[Y(t) \leq y_t | Y(0) = y_0\right] \tag{53}$$

we will denote the long term probability as

$$\overline{p}(\overline{y}) = \lim_{t \to \infty} p(0, y_0; t, \overline{y}), \tag{54}$$

which is independent of the starting point.

Theorem 6.1 *Consider a stochastic process $Y = \{Y(s), s \in [0, t]\}$ defined by a diffusion equation with unit diffusion (19). The long term probability for the process Y can be calculated as*

$$\overline{p}(\overline{y}) = C(y_0)\, e^{2 \int_{y_0}^{\overline{y}} A(z) dz}, \tag{55}$$

where the constant $C(y_0)$ is determined by the condition of a total mass equal to one.

Theorem 6.2 *Consider a stochastic process $Y = \{Y(s), s \in [0, t]\}$ defined by a general Itô diffusion equation (27). The long term probability for the process Y can be calculated as*

$$\overline{p}(\overline{y}) = \frac{C(y_0)}{B(\overline{y})}\, e^{2 \int_{\psi(y_0)}^{\psi(\overline{y})} T(z) dz}, \tag{56}$$

where ψ is defined in (28), T is defined in (30), and where the constant $C(y_0)$ is determined by the condition of a total mass equal to one.

Both results immediately follow from the forward Fokker Planck equations when choosing $\frac{\partial p}{\partial t} = 0$.

7 Calculation of the Transition Probabilities

7.1 Exact results for path integrals

In the previous sections, we showed how to find analytical expressions for the transition probability of diffusion processes by means of path integrals. For the computation of these functional integrations, we can rely on some methods and calculations from quantum mechanics. In appendix A we summarize a few important and useful exact computational results for common classes of path integrals, some of which were derived in the framework of earlier research on annuities with stochastic interest rates.

However, when the Lagrangian appearing in the path integral becomes too complicated, we will have to use approximations instead of exact results. In the following subsections, we will show how to find an approximation based on properties that hold for general path integrals.

In order to make things clear, we will use the notation $K(t_o, x_o; t_e, x_e)$ for Wiener integrals

$$\begin{aligned} K(t_o, x_o; t_e, x_e) &= \int_{(t_o,x_o)}^{(t_e,x_e)} Dx(s)\, e^{-\frac{1}{2}\int_{t_o}^{t_e} \dot{x}(s)^2\, ds} \\ &= \frac{1}{\sqrt{2\pi(t_e - t_o)}}\, e^{-\frac{(x_e - x_o)^2}{2(t_e - t_o)}} \end{aligned} \tag{57}$$

and the notation $I(t_o, x_o; t_e, x_e)$ for path integrals which are related but more general than Wiener integrals :

$$I(t_o, x_o; t_e, x_e) = \int_{(t_o,x_o)}^{(t_e,x_e)} Dx(s)\, e^{-\frac{1}{2}\int_{t_o}^{t_e} \dot{x}(s)^2\, ds - \int_{t_o}^{t_e} V[x(s)]\, ds}. \tag{58}$$

As a consequence, the transition probabilities for stochastic processes as derived in section 3 can be expressed as

$$p(t_o, x_o; t_e, x_e) = C(t_o, x_o; t_e, x_e) \cdot I(t_o, x_o; t_e, x_e), \tag{59}$$

for adequate choices of the function V.

Furthermore, we will make use of the expected value over Wiener paths with known starting point and known final point, which can be written as

$$E_W\left[\!\left[e^{-\int_{t_o}^{t_e} V[X(s)]\, ds} \right]\!\right] = \frac{I(t_o, x_o; t_e, x_e)}{K(t_o, x_o; t_e, x_e)}. \tag{60}$$

If we are dealing with the absorbed Wiener process due to the restriction of the diffusion process to the domain $(0, +\infty)$, in each of the equations (57) and (58), the differential part $Dx(s)$ has to be replaced by $D_+x(s)$. More specifically, equation (57) has to be changed into (see [17])

$$\begin{aligned} K(t_o, x_o; t_e, x_e) &= \int_{(t_o,x_o)}^{(t_e,x_e)} D_+x(s)\, e^{-\frac{1}{2}\int_{t_o}^{t_e} \dot{x}(s)^2\, ds} \qquad (61) \\ &= \frac{1}{\sqrt{2\pi(t_e - t_o)}}\, e^{-\frac{(x_e - x_o)^2}{2(t_e - t_o)}} - \frac{1}{\sqrt{2\pi(t_e - t_o)}}\, e^{-\frac{(x_e + x_o)^2}{2(t_e - t_o)}} . \end{aligned}$$

Note that the expectation in (60) can also easily be extended for path integrals different from Wiener path integrals.

Finally, with the same notations, we will write the distribution for Wiener paths with fixed starting and final points as

$$F_s(x) = Prob[X(s) \leq x \mid X(t_o) = x_o, X(t_e) = x_e] \qquad (62)$$

and

$$f_s(x) = \frac{d}{dx}F_s(x) = \frac{K(t_o, x_o; s, x) \cdot K(s, x; t_e, x_e)}{K(t_o, x_o; t_e, x_e)}. \qquad (63)$$

A straightforward calculation leads to

$$F_s(x) = \Phi\left(\sqrt{\frac{(t_e - t_o)}{(s - t_o)(t_e - s)}}\left(x - \frac{(t_e - s)x_o + (s - t_o)x_e}{t_e - t_o}\right)\right), \qquad (64)$$

where $\Phi(x)$ denotes the standard normal cumulative probability, or

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt. \qquad (65)$$

7.2 Approximation of path integrals

In case the path integral can not be calculated in an exact way with as closed form, as mentioned before the only alternative consists of an approximation. Making use of techniques from quantum mechanics on the one hand and of the properties of convex ordered risks on the other hand, we can find a lower and upper bound for the path integrals. A combination of both expressions together with a correct scaling will enable us to find an approximation that seems to be very accurate.

The most important advantage of our methodology has to be found in the fact that it results in an analytical expression for the approximation, whereas most techniques that are presented in the literature lead to exclusively numerical approximations. Consequently, this new methodology is very interesting in the framework of the pricing of Asian options.

7.2.1 Upperbound

The method we propose to find accurate upperbounds for the transition probabilities, makes use of convex order. We briefly recall the most important concepts and necessary results ; we refer to [9, 10] for proofs and more details.

A variable A is said to be smaller than B in convex ordering,

$$A \leq_{cx} B, \tag{66}$$

if for each convex function $u : \Re \to \Re : x \mapsto u(x)$ the expected values (provided they exist) are ordered as

$$E[u(A)] \leq E[u(B)]. \tag{67}$$

As a consequence, $E[A] = E[B]$ and

$$E[(A-k)_+] \leq E[(B-k)_+] \qquad \text{for all } k, \tag{68}$$

with $(x)_+ = \max(0, x)$.

If an expression is known for the stop-loss premium $E[(B-k)_+]$, the distribution of the variable B can be found as

$$Prob[B \leq k] = 1 + \frac{d}{dk} E[(B-k)_+]. \tag{69}$$

The notion of convex ordering can be extended from two single variables to two sums of variables, discrete or continuous. The results are summarized in the following two propositions (for a proof see [9, 10]). For the distributions, we make use of the notation

$$F_X(x) = Prob[X \leq x] \; ; \tag{70}$$

the inverse distributions are defined in the classical way as

$$F_X^{-1}(p) = \inf\{x \in \Re : F_X(x) \geq p\}. \tag{71}$$

Proposition 7.1 *Consider a sum of functions of random variables*

$$A = g_1(X_1) + g_2(X_2) + ... + g_n(X_n) \tag{72}$$

and for U an arbitrary random variable that is uniformly distributed on $[0, 1]$, define the related stochastic quantity

$$B = F^{-1}_{g_1(X_1)}(U) + F^{-1}_{g_2(X_2)}(U) + ... + F^{-1}_{g_n(X_n)}(U) \; . \tag{73}$$

Then

$$A \leq_{cx} B. \tag{74}$$

Remark. The corresponding terms in the sums A and B are all mutually identically distributed, e.g.

$$g_j(X_j) \stackrel{d}{=} F^{-1}_{g_j(X_j)}(U) \, . \tag{75}$$

Proposition 7.2 *Consider a functional integration*

$$A = \int_{t_o}^{t_e} g(X(s))ds \, , \tag{76}$$

and for U an arbitrary random variable that is uniformly distributed on $[0,1]$, define the related stochastic quantity

$$B = \int_{t_o}^{t_e} F^{-1}_{g(X(s))}(U))ds \, . \tag{77}$$

Then

$$A \leq_{cx} B. \tag{78}$$

An application of the method of convex upperbounds to the transition probabilities of diffusion processes, brings us to the following result :

Theorem 7.1 *For a path integral with a structure as mentioned in equation (58), an upperbound can be found as*

$$I(t_o, x_o; t_e, x_e) \leq I^{upp}(t_o, x_o; t_e, x_e) \tag{79}$$

where

$$I^{upp}(t_o, x_o; t_e, x_e) = K(t_o, x_o; t_e, x_e) \cdot E_U \left[e^{-\int_{t_o}^{t_e} F^{-1}_{V(X(\tau))}(U) \, d\tau} \right] , \tag{80}$$

with U uniformly distributed on $[0,1]$.
The expectation can also be written as

$$I^{upp}(t_o, x_o; t_e, x_e) = K(t_o, x_o; t_e, x_e) \cdot \int_{-\infty}^{+\infty} e^{-k} \frac{\partial^2}{\partial k^2} G(k|t_o, x_o; t_e, x_e) dk \tag{81}$$

with

$$G(k|t_o, x_o; t_e, x_e) = E_U \left[\left(\int_{t_o}^{t_e} F^{-1}_{V(X(\tau))}(U) \, d\tau - k \right)_+ \right] . \tag{82}$$

7.2.2 Lowerbound

In order to improve the upperbound, we also need to derive a lowerbound for the transition probability of diffusion processes. The present result mainly originates from an application of the well-known inequality of Jensen.

Theorem 7.2 *For a path integral with a structure as mentioned in equation (58), a lower bound can be found as*

$$I(t_o, x_o; t_e, x_e) \geq I^{low}(t_o, x_o; t_e, x_e) \tag{83}$$

where

$$\begin{aligned} &I^{low}(t_o, x_o; t_e, x_e) \\ &= K(t_o, x_o; t_e, x_e) \cdot E_{X(t_s)}\left[e^{-\int_{t_o}^{t_s} E_W[V(X(\tau))]d\tau} \cdot e^{-\int_{t_s}^{t_e} E_W[V(X(\tau))]d\tau} \right], \end{aligned} \tag{84}$$

with t_s any point in time between t_o and t_e .

Remark. The notation $E_W[...]$ means an expectation over Wiener paths (or absorbed Wiener paths if the domain of the stochastic process is $(0, +\infty)$) between the two boundary time points.

7.2.3 Approximation

Consider a stochastic process $Y = \{Y(s), s \in [0, t]\}$ for which the transition probability can be expressed as in (59).

From theorems 7.1 and 7.2, we know that

$$p(t_o, x_o; t_e, x_e) \leq C(t_o, x_o; t_e, x_e) \cdot I^{upp}(t_o, x_o; t_e, x_e), \tag{85}$$

and

$$p(t_o, x_o; t_e, x_e) \geq C(t_o, x_o; t_e, x_e) \cdot I^{low}(t_o, x_o; t_e, x_e), \tag{86}$$

for specific choices of the function V.

The problem with both bounds is the fact that we are no longer dealing with density functions. Therefore, we suggest to use of a convex combination

$$\begin{aligned} \tilde{p}(t_o, x_o; t_e, x_e) &= C(t_o, x_o; t_e, x_e) \cdot \{z(t_o, x_o; t_e) I^{low}(t_o, x_o; t_e, x_e) \\ &\quad + (1 - z(t_o, x_o; t_e))\, I^{upp}(t_o, x_o; t_e, x_e)\} \end{aligned} \tag{87}$$

resulting in an analytical approximation (which is a density) for the transition probability.

The factor $z(t_o, x_o; t_e)$ can be determined by the condition of a total mass equal to one, or

$$z(t_o, x_o; t_e) \qquad (88)$$

$$= \frac{\int_{-\infty}^{+\infty} C(t_o, x_o; t_e, x_e) \cdot I^{upp}(t_o, x_o; t_e, x_e) dx_e \quad - \quad 1}{\int_{-\infty}^{+\infty} C(t_o, x_o; t_e, x_e) \cdot [I^{upp}(t_o, x_o; t_e, x_e) - I^{low}(t_o, x_o; t_e, x_e)]\, dx_e}.$$

There is an extra advantage when working with this convex combination, due to the factor $z(t_o, x_o; t_e)$. Indeed, in the situation where one of the bounds turns out to be not that accurate, the contribution of that bound will have a less important impact on the approximation. Either of the bounds has an influence on the final approximation, but the closer the bound to the exact transition density, the larger the impact of that bound.

8 Examples

In the following examples, we show how the transition probability for some common classes of diffusion processes can be expressed by means of a Feynman path integral, as explained in the previous section. If possible, we give the exact computational result for the path integral. We also mention expressions for the long term probability and for the maximal probability path between two fixed points.

8.1 Wiener Model

a. **Model :**
The Wiener model is defined by the SDE

$$dY(t) = \mu dt + \sigma dW(t) \; ; \qquad (89)$$

Y is distributed on $(-\infty, +\infty)$.

b. **Transformation to unit diffusion :**
For $X(t) = \frac{1}{\sigma} Y(t)$, the SDE transforms into

$$dX(t) = \frac{\mu}{\sigma} dt + dW(t) \; ; \qquad (90)$$

X is distributed on $(-\infty, +\infty)$.

c. **Maximal probability path :**
The classical trajectory is given by

$$y_{mod}(s|0, y_0; t, y_t) = \frac{(t-s)y_0 + s y_t}{t}. \qquad (91)$$

d. **Long term probability :**

When t tends to infinity, we get the instable probability density

$$\overline{p}(\overline{y}) = C \cdot e^{\frac{2\mu}{\sigma^2}\overline{y}} . \tag{92}$$

e. **Transition probability :**

In a first step, the transition probability can be expressed by means of a path integral as

$$p(0, y_0; t, y_t) = \frac{1}{\sigma}\, e^{-\frac{\mu^2 t}{2\sigma^2} + \frac{\mu}{\sigma^2}(y_t - y_o)} \int_{(0, \frac{1}{\sigma} y_o)}^{(t, \frac{1}{\sigma} y_t)} Dx(s)\, e^{-\frac{1}{2}\int_0^t \dot{x}^2 ds} . \tag{93}$$

This path integral can be calculated exactly, making use of the result for the Wiener integral in (149). This results in

$$p(0, y_0; t, y_t) = \frac{1}{\sqrt{2\pi\sigma^2 t}}\, e^{-\frac{(y_t - y_0 - \mu t)^2}{2\sigma^2 t}} . \tag{94}$$

8.2 Geometric Wiener Model

a. **Model :**

The Geometric Wiener model is defined by the SDE

$$dY(t) = \left(\mu + \frac{\sigma^2}{2}\right) Y(t)dt + \sigma Y(t) dW(t) \; ; \tag{95}$$

Y is distributed on $(0, +\infty)$.

b. **Transformation to unit diffusion :**

For $X(t) = \frac{1}{\sigma} \ln Y(t)$, the SDE transforms into

$$dX(t) = \frac{\mu}{\sigma} dt + dW(t) \; ; \tag{96}$$

X is distributed on $(-\infty, +\infty)$.

c. **Maximal probability path :**

The classical trajectory is given by

$$y_{mod}(s|0, y_0; t, y_t) = y_0^{(t-s)/t}\, y_t^{s/t} . \tag{97}$$

d. **Long term probability :**
When t tends to infinity, we get the instable probability density

$$\overline{p}(\overline{y}) = C \cdot \overline{y}^{\frac{2\mu}{\sigma^2} - 1} . \tag{98}$$

e. **Transition probability :**
In a first step, the transition probability can be expressed by means of a path integral as

$$p(0, y_0; t, y_t) = \frac{1}{\sigma\, y_t} \left(\frac{y_t}{y_o}\right)^{\frac{\mu}{\sigma^2}} \cdot e^{-\frac{\mu^2 t}{2\sigma^2}} \int_{(0, \frac{1}{\sigma}\ln y_o)}^{(t, \frac{1}{\sigma}\ln y_t)} Dx(s)\, e^{-\frac{1}{2}\int_0^t \dot{x}^2 ds} . \tag{99}$$

This path integral can be calculated exactly, when making use of the result for the Wiener integral in (149). This results in

$$p(0, y_0; t, y_t) = \frac{1}{\sqrt{2\pi\sigma^2 t}\, y_t}\, e^{-\frac{1}{2\sigma^2 t}\left(\ln \frac{y_t}{y_o} - \mu t\right)^2} . \tag{100}$$

8.3 Vasicek Model

a. **Model :**
The Vasicek model is defined by the SDE

$$dY(t) = \kappa(\alpha - Y(t))dt + \sigma dW(t) \; ; \tag{101}$$

Y is distributed on $(-\infty, +\infty)$.

b. **Transformation to unit diffusion :**
For $X(t) = \frac{1}{\sigma} Y(t)$, the SDE transforms into

$$dX(t) = \kappa \left(\frac{\alpha}{\sigma} - X(t)\right) dt + dW(t) \; ; \tag{102}$$

X is distributed on $(-\infty, +\infty)$.

c. **Maximal probability path :**
The classical trajectory is given by

$$y_{mod}(s|0, y_0; t, y_t) = \alpha + \frac{(y_t - \alpha)\sinh(ks) + (y_0 - \alpha)\sinh(k(t-s))}{\sinh(kt)} . \tag{103}$$

d. **Long term probability :**
When t tends to infinity, we get the probability density

$$\overline{p}(\overline{y}) = C \cdot e^{-\frac{\kappa}{\sigma^2}(\overline{y}-\alpha)^2}, \tag{104}$$

with $C = \sqrt{\frac{\kappa}{\pi\sigma^2}}$.

e. **Transition probability :**
In a first step, the transition probability can be expressed by means of a path integral as

$$\begin{aligned} & p(0,y_0;t,y_t) \\ & = \frac{1}{\sigma} e^{-\frac{\kappa^2\alpha^2 t}{2\sigma^2}+\frac{\kappa t}{2}} \cdot e^{\frac{\kappa\alpha}{\sigma^2}(y_t-y_o)-\frac{\kappa}{2\sigma^2}\left(y_t^2-y_o^2\right)} \\ & \cdot \int_{(0,\frac{1}{\sigma}y_0)}^{(t,\frac{1}{\sigma}y_t)} Dx(s)\, e^{-\frac{1}{2}\int_0^t \dot{x}^2 ds - \frac{\kappa^2}{2}\int_0^t x^2 ds + \frac{\kappa^2\alpha}{\sigma}\int_0^t x ds} . \end{aligned} \tag{105}$$

This path integral can be calculated exactly, making use of the result for the Gaussian integral in (150). This results in

$$\begin{aligned} p(0,y_0;t,y_t) = & \sqrt{\frac{\kappa}{\pi\sigma^2(1-e^{-2\kappa t})}} \\ & \cdot \exp\left\{-\frac{\kappa}{\sigma^2(1-e^{-2\kappa t})}\left((y_t-\alpha)-(y_0-\alpha)e^{-\kappa t}\right)^2\right\} . \end{aligned} \tag{106}$$

8.4 Cox-Ingersoll-Ross Model

a. **Model :**
The Cox-Ingersoll-Ross model is defined by the SDE

$$dY(t) = \kappa(\alpha - Y(t))dt + \sigma\sqrt{Y(t)}dW(t) , \tag{107}$$

where it is assumed that $2\kappa\alpha/\sigma^2 \geq 1$;
Y is distributed on $(0,+\infty)$.

b. **Transformation to unit diffusion :**
For $X(t) = \frac{2}{\sigma}\sqrt{Y(t)}$, the SDE transforms into

$$dX(t) = \left(\left(\frac{2\kappa\alpha}{\sigma^2}-\frac{1}{2}\right)\frac{1}{X(t)} - \frac{\kappa}{2}X(t)\right)dt + dW(t) ; \tag{108}$$

X is distributed on $(0,+\infty)$.

c. **Maximal probability path :**
The classical trajectory is given by

$$
\begin{aligned}
& y_{mod}(s|0, y_0; t, y_t) \\
& = (y_0 + C)\cosh(\kappa s) - C \qquad (109) \\
& + \sqrt{(y_0 + C)^2 - C^2 + \frac{\sigma^4}{4\kappa^2}\left(\frac{2\kappa\alpha}{\sigma^2} - \frac{1}{2}\right)\left(\frac{2\kappa\alpha}{\sigma^2} - \frac{3}{2}\right)}\,\sinh(\kappa s)\,,
\end{aligned}
$$

with

$$
\begin{aligned}
C &= \frac{1}{\cosh(\kappa t) - 1}(y_0 + y_t) + \frac{\sinh(\kappa t)}{\cosh(\kappa t) - 1} \qquad (110) \\
&\times \sqrt{\frac{\sigma^4}{4\kappa^2}\left(\frac{2\kappa\alpha}{\sigma^2} - \frac{1}{2}\right)\left(\frac{2\kappa\alpha}{\sigma^2} - \frac{3}{2}\right) + y_0 y_t \frac{2}{\cosh(\kappa t) - 1}}\,.
\end{aligned}
$$

d. **Long term probability :**
When t tends to infinity, we get the probability density

$$
\overline{p}(\overline{y}) = C \cdot \overline{y}^{\frac{2\kappa\alpha}{\sigma^2} - 1} \cdot e^{-\frac{2\kappa}{\sigma^2}\overline{y}}, \qquad (111)
$$

with $C = \left(2\kappa/\sigma^2\right)^{2\kappa\alpha/\sigma^2} / \Gamma\left(2\kappa\alpha/\sigma^2\right)$.

e. **Transition probability :**
In a first step, the transition probability can be expressed by means of a path integral as

$$
\begin{aligned}
p(0, y_0; t, y_t) &= \frac{1}{\sigma\sqrt{y_t}}\left(\frac{y_t}{y_0}\right)^{\frac{\kappa\alpha}{\sigma^2} - \frac{1}{4}} \cdot e^{\frac{\kappa^2\alpha t}{\sigma^2}} \cdot e^{-\frac{\kappa}{\sigma^2}(y_t - y_o)} \qquad (112) \\
&\cdot \int_{(0, \frac{2}{\sigma}\sqrt{y_0})}^{(t, \frac{2}{\sigma}\sqrt{y_t})} D_+x(s)\, e^{-\frac{1}{2}\int_0^t \dot{x}^2 ds - \frac{\kappa^2}{8}\int_0^t x^2 ds - \left(\left(\frac{\sqrt{2}\kappa\alpha}{\sigma^2} - \frac{1}{\sqrt{2}}\right)^2 - \frac{1}{8}\right)\int_0^t \frac{1}{x^2} ds}\,.
\end{aligned}
$$

This path integral can be calculated exactly, making use of the result for the Calogero integral in (151). This results in

$$
p(0, y_0; t, y_t) = \frac{2\kappa e^{-\kappa t/2}}{\sigma^2(1 - e^{-\kappa t})} \cdot e^{\kappa^2\alpha t/\sigma^2} \cdot \left(\frac{y_t}{y_o}\right)^{\frac{\kappa\alpha}{\sigma^2} - \frac{1}{2}} \qquad (113)
$$

$$\cdot \exp\left\{-\frac{2\kappa}{\sigma^2(1-e^{-\kappa t})}\left(y_0 e^{-\kappa t} + y_t\right)\right\}$$
$$\cdot I_{2\kappa\alpha/\sigma^2 - 1}\left(\frac{4\kappa e^{-\kappa t/2}}{\sigma^2(1-e^{-\kappa t})}\sqrt{y_0 y_t}\right).$$

8.5 Adapted Geometric Wiener Model

a. **Model :**

The Adapted Geometric Wiener model (see [14]) is defined by the SDE

$$dY(t) = \left((\delta + \frac{\sigma^2}{2})Y(t) - 1\right) dt + \sigma Y(t) dW(t) \; ; \tag{114}$$

Y is distributed on $(0, +\infty)$.

b. **Transformation to unit diffusion :**

For $X(t) = \frac{1}{\sigma}\ln Y(t)$, the SDE transforms into

$$dX(t) = \frac{1}{\sigma}\left(\delta - e^{-\sigma X(t)}\right) dt + dW(t) \; ; \tag{115}$$

X is distributed on $(-\infty, +\infty)$.

c. **Maximal probability path :**

The classical trajectory is given by

$$y_{mod}(s|0, y_0; t, y_t) = \frac{e^{-As}}{4A^2 B}\left((2\delta - \sigma^2)^2 - 4A^2 + 2B(2\delta - \sigma^2)e^{As} + B^2 e^{2As}\right), \tag{116}$$

where the constants A and B have to be determined numerically by the constraints $y_{mod}(0|0, y_0; t, y_t) = y_0$ and $y_{mod}(t|0, y_0; t, y_t) = y_t$.

d. **Long term probability :**

When t tends to infinity, we get the instable probability density

$$\overline{p}(\overline{y}) = C \cdot \overline{y}^{\frac{2\delta}{\sigma^2} - 1} e^{\frac{2}{\sigma^2 \overline{y}}}. \tag{117}$$

e. **Transition probability :**

In a first step, the transition probability can be expressed by means of a path integral as

$$p(0, y_0; t, y_t) = \frac{1}{\sigma\, y_t} \left(\frac{y_t}{y_o}\right)^{\frac{\delta}{\sigma^2}} \cdot e^{-\frac{\delta^2 t}{2\sigma^2}} \cdot e^{\frac{1}{\sigma^2}\left(\frac{1}{y_t} - \frac{1}{y_o}\right)} \tag{118}$$
$$\cdot \int_{(0, \frac{1}{\sigma} \ln y_0)}^{(t, \frac{1}{\sigma} \ln y_t)} Dx(s)\, e^{-\frac{1}{2}\int_0^t \dot{x}^2 ds}$$
$$\cdot e^{-\frac{1}{2}\left(1 - \frac{2\delta}{\sigma^2}\right) \int_0^t e^{-\sigma x} ds - \frac{1}{2\sigma^2} \int_0^t e^{-2\sigma x} ds} .$$

This path integral can be calculated exactly, making use of the result for the exponential path integral in (154). This results in

$$\begin{aligned}
&p(0, y_0; t, y_t) \\
&= \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \frac{1}{\sigma^5} \frac{1}{\sqrt{t}} \frac{1}{y_0^2} \sqrt{\frac{y_0}{y_t}} \cdot \exp\left\{ -\frac{\delta^2 t}{2\sigma^2} - \frac{\sigma^2}{2}\left(1 - \frac{\delta}{\sigma}\right)^2 t + \frac{2\pi^2}{\sigma^2 t} \right\} \\
&\cdot \int_0^\infty ds\, e^{-\frac{4}{\sigma^2}\left(\frac{1}{2} - \frac{\delta}{\sigma^2}\right)} \frac{1}{\sinh^2\left(\frac{2s}{\sigma^2}\right)} \exp\left\{ -\frac{1}{\sigma^2} \frac{1}{\tanh\left(\frac{2s}{\sigma^2}\right)} \left(\frac{1}{y_t} + \frac{1}{y_0}\right) \right\} \\
&\cdot \int_0^\infty dz\, e^{-\frac{2z^2}{\sigma^2 t}} \sinh(z) \sin\left(\frac{4\pi z}{\sigma^2 t}\right) \exp\left\{ -\frac{2}{\sigma^2} \frac{1}{\sqrt{y_0 y_t}} \frac{\cosh(z)}{\sinh\left(\frac{2s}{\sigma^2}\right)} \right\} .
\end{aligned} \tag{119}$$

8.6 Bessel Model with Drift

a. **Model :**

The Bessel model is defined by the SDE

$$dY(t) = \left(\frac{1}{Y(t)} - 2\right) dt + dW(t) \; ; \tag{120}$$

Y is distributed on $(0, +\infty)$.

b. **Maximal probability path :**

The classical trajectory $y_{mod}(t|0, y_0; t, y_t)$ is the solution y of the implicit equation

$$\frac{2}{C^3} \ln\left(\frac{C + \sqrt{C^2 - 4/y}}{C - \sqrt{C^2 - 4/y}}\right) + \frac{y}{C^2}\sqrt{C^2 - 4/y} = s \, , \tag{121}$$

where the constant C follows from the boundary condition $y_{mod}(t|0, y_0; t, y_t) = y_t$.

c. **Long term probability :**

When t tends to infinity, we get the probability density

$$\overline{p}(\overline{y}) = C \cdot \overline{y}^2 \, e^{-4\overline{y}}, \tag{122}$$

with $C = 32$.

d. **Transition probability :**

In a first step, the transition probability can be expressed by means of a path integral as

$$p(0, y_0; t, y_t) = \left(\frac{y_t}{y_o}\right) e^{-2t} \cdot e^{-2(y_t - y_o)} \tag{123}$$
$$\cdot \int_{(0,y_0)}^{(t,y_t)} D_+ y(s) \, e^{-\frac{1}{2}\int_0^t \dot{y}^2 ds + 2\int_0^t \frac{1}{y} ds}.$$

If in this path integral we perform a substitution $y = f(x) = x^2$ in the same way as was developped in the proof of theorem 5.1, the path integral in (123) can be rewritten as a path integral of the Calogero type. Making use of the result of (151), the transition probability can be found as

$$p(0, y_0; t, y_t)$$
$$= \left(\frac{y_t}{y_o}\right) e^{-2t} \, e^{-2(y_t - y_o)} \int_{-\infty}^{+\infty} d\beta \, e^{i\beta t} \int_0^{+\infty} d\vartheta \;\, e^{8\vartheta} \tag{124}$$
$$\cdot \frac{\sqrt{2i\beta}}{\sinh(2\sqrt{2i\beta}\,\vartheta)} \cdot I_1\left(\frac{2\sqrt{2i\beta}\sqrt{y_0 y_t}}{\sinh(2\sqrt{2i\beta}\,\vartheta)}\right) \cdot \exp\left(-\frac{2\sqrt{2i\beta}(y_0 + y_t)}{\tanh(2\sqrt{2i\beta}\,\vartheta)}\right).$$

8.7 Inverse of Feller's Square Root Model

a. **Model :**

The Inverse of Feller's Square Root model is defined by the SDE

$$dY(t) = Y(t)\left(\kappa - (\sigma^2 - \kappa\alpha)Y(t)\right) dt + \sigma Y(t)^{3/2} dW(t)\,, \tag{125}$$

where it is assumed that $2\kappa\alpha/\sigma^2 \geq 1$;

Y is distributed on $(0, +\infty)$.

b. **Transformation to unit diffusion :**

For $X(t) = -\frac{2}{\sigma\sqrt{Y(t)}}$, the SDE transforms into

$$dX(t) = \left(-\frac{\kappa}{2}X(t) + \left(\frac{7}{2} - \frac{2\kappa\alpha}{\sigma^2}\right)\frac{1}{X(t)}\right) dt + dW(t)\;; \tag{126}$$

X is distributed on $(-\infty, 0)$.

c. **Maximal probability path :**
The classical trajectory is given by

$$
\begin{aligned}
&\frac{1}{y_{mod}(s|0,y_0;t,y_t)} \\
&= \cosh(\kappa s)\left(\frac{1}{y_0}+C\right) - C \qquad (127)\\
&\quad + \sinh(\kappa s)\sqrt{\left(\frac{1}{y_0}+C\right)^2 - C^2 + \frac{\sigma^4}{4\kappa^2}\left(\frac{7}{2}-\frac{2\kappa\alpha}{\sigma^2}\right)\left(\frac{5}{2}-\frac{2\kappa\alpha}{\sigma^2}\right)},
\end{aligned}
$$

with

$$
\begin{aligned}
C &= \frac{1}{\cosh(\kappa t)-1}\left(\frac{1}{y_0}+\frac{1}{y_t}\right) + \frac{\sinh(\kappa t)}{\cosh(\kappa t)-1} \qquad (128)\\
&\quad \times\sqrt{\frac{\sigma^4}{4\kappa^2}\left(\frac{7}{2}-\frac{2\kappa\alpha}{\sigma^2}\right)\left(\frac{5}{2}-\frac{2\kappa\alpha}{\sigma^2}\right) + \frac{1}{y_0 y_t}\frac{2}{\cosh(\kappa t)-1}}.
\end{aligned}
$$

d. **Long term probability :**
When t tends to infinity, we get the probability density

$$
\overline{p}(\overline{y}) = C\cdot \overline{y}^{-\left(5-\frac{2\kappa\alpha}{\sigma^2}\right)}\cdot e^{-\frac{2\kappa}{\sigma^2}\frac{1}{\overline{y}}}, \qquad (129)
$$

with $C = \left(\frac{2\kappa}{\sigma^2}\right)^{4-\frac{2\kappa\alpha}{\sigma^2}} \Big/ \Gamma\left(4-\frac{2\kappa\alpha}{\sigma^2}\right)$.

e. **Transition probability :**
In a first step, the transition probability can be expressed by means of a path integral as

$$
\begin{aligned}
&p(0,y_0;t,y_t) \\
&= \frac{1}{\sigma y_t^{3/2}}\left(\frac{y_0}{y_t}\right)^{\frac{1}{2}\left(\frac{7}{2}-\frac{2\kappa\alpha}{\sigma^2}\right)}\cdot e^{\frac{\kappa}{2}\left(4-\frac{2\kappa\alpha}{\sigma}\right)t}\cdot e^{-\frac{\kappa}{\sigma^2}\left(\frac{1}{y_t}-\frac{1}{y_0}\right)} \\
&\cdot\int_{\left(0,-\frac{2}{\sigma\sqrt{y_0}}\right)}^{\left(t,-\frac{2}{\sigma\sqrt{y_t}}\right)} D\_x(s)\, e^{-\frac{1}{2}\int_0^t \dot{x}^2 ds - \frac{\kappa^2}{8}\int_0^t x^2 ds} \qquad (130)\\
&\quad\cdot e^{-\frac{1}{2}\left(\frac{7}{2}-\frac{2\kappa\alpha}{\sigma^2}\right)\left(\frac{5}{2}-\frac{2\kappa\alpha}{\sigma^2}\right)\int_0^t \frac{1}{x^2}ds}.
\end{aligned}
$$

This path integral can be calculated exactly, making use of the result for the Calogero integral in (151). This results in

$$p(0, y_0; t, y_t)$$
$$= \frac{2\kappa}{\sigma^2 y_t^2} \frac{1}{(1-e^{-\kappa t})} \left(\frac{y_0}{y_t}\right)^{\frac{1}{2}\left(3-\frac{2\kappa\alpha}{\sigma^2}\right)} \cdot e^{\frac{\kappa}{2}\left(3-\frac{2\kappa\alpha}{\sigma^2}\right)t} \tag{131}$$
$$\cdot \exp\left\{-\frac{\kappa}{\sigma^2}\left(\frac{1}{y_t}-\frac{1}{y_0}\right) - \frac{4\kappa}{\sigma^2}\left(\frac{1}{y_t}+\frac{1}{y_0}\right)\frac{(1+e^{-\kappa t})}{(1-e^{-\kappa t})}\right\}$$
$$\cdot I_{3-\frac{2\kappa\alpha}{\sigma^2}}\left(\frac{4\kappa\, e^{-\kappa t/2}}{\sigma^2}\frac{1}{\sqrt{y_o y_t}}\frac{1}{(1-e^{-\kappa t})}\right) .$$

8.8 Linear Drift, CEV Diffusion Model

a. **Model :**

The CEV Diffusion model is defined by the SDE

$$dY(t) = \kappa(\alpha - Y(t))dt + \sigma Y(t)^{3/2} dW(t) \; ; \tag{132}$$

Y is distributed on $(0, +\infty)$.

b. **Transformation to unit diffusion :**

For $X(t) = -\frac{2}{\sigma\sqrt{Y(t)}}$, the SDE transforms into

$$dX(t) = \left(\frac{\kappa}{2}X(t) + \frac{3}{2}\frac{1}{X(t)} - \frac{\kappa\alpha\sigma^2}{8}X(t)^3\right) dt + dW(t) \; ; \tag{133}$$

X is distributed on $(-\infty, 0)$.

c. **Maximal probability path :**

The classical trajectory $y_{mod}(t|0, y_0; t, y_t)$ can be determined implicitly by

$$\int_{y_o}^{y_{mod}(t|0,y_0;t,y_t)} \frac{dz}{\sqrt{\frac{3\sigma^4}{16}z^4 + C\sigma^2 z^3 + \kappa^2\left(1 - \frac{3\alpha\sigma^2}{\kappa}\right) z^2 - 2\kappa^2\alpha z + \kappa^2\alpha^2}} = s \; , \tag{134}$$

where the constant C follows from the boundary condition $y_{mod}(t|0, y_0; t, y_t) = y_t$.

d. **Long term probability :**

When t tends to infinity, we get the probability density

$$\overline{p}(\overline{y}) = C \cdot \frac{1}{\overline{y}^3} \cdot \exp\left\{\frac{2\kappa}{\sigma^2}\frac{1}{\overline{y}} - \frac{\kappa\alpha}{\sigma^2}\frac{1}{\overline{y}^2}\right\}, \tag{135}$$

with $C = \frac{\sigma^2}{2\kappa\alpha}\left[1 + \sqrt{\frac{\pi\kappa}{\sigma^2\alpha}}\, e^{\frac{\kappa}{\sigma^2\alpha}}\left(1 + \mathrm{erf}\left(\sqrt{\frac{\kappa}{\sigma^2\alpha}}\right)\right)\right]$.

e. **Transition probability :**
In a first step, the transition probability can be expressed by means of a path integral as

$$\begin{aligned} p(0, y_0; t, y_t) &= \frac{e^{-\kappa t}}{\sigma y_t^{3/2}}\left(\frac{y_0}{y_t}\right)^{\frac{3}{4}} \cdot e^{\frac{\kappa}{\sigma^2}\left(\frac{1}{y_t} - \frac{1}{y_0}\right)} \cdot e^{-\frac{\kappa\alpha}{2\sigma^2}\left(\frac{1}{y_t^2} - \frac{1}{y_0^2}\right)} \\ &\cdot \int_{(0, -\frac{2}{\sigma\sqrt{y_0}})}^{(t, -\frac{2}{\sigma\sqrt{y_t}})} D\_x(s)\, e^{-\frac{1}{2}\int_0^t \dot{x}^2 ds} \\ &\times e^{-\frac{1}{8}\int_0^t \left[\frac{3}{x^2} + \kappa^2\left(1 - \frac{3\alpha\sigma^2}{\kappa}\right)x^2 - \frac{\kappa^2\alpha\sigma^2}{2}x^4 + \frac{\kappa^2\alpha^2\sigma^4}{16}x^6\right] ds} . \end{aligned} \tag{136}$$

This path integral can not be calculated exactly, but we get an approximate result when applying the convex combination of upper and lower bound as mentioned in (87) with $V[z] = \frac{1}{8}\left[\frac{3}{z^2} + \kappa^2\left(1 - \frac{3\alpha\sigma^2}{\kappa}\right)z^2 - \frac{\kappa^2\alpha\sigma^2}{2}z^4 + \frac{\kappa^2\alpha^2\sigma^4}{16}z^6\right]$.

8.9 Nonlinear Mean Reversion Model

a. **Model :**
The Nonlinear Mean Reversion model is defined by the SDE

$$dY(t) = \left(\alpha_{-1}Y(t)^{-1} - \alpha_0 + \alpha_1 Y(t) + \alpha_2 Y(t)^2\right) dt + \sigma Y(t)^{3/2} dW(t) \; ; \tag{137}$$

Y is distributed on $(0, +\infty)$.

b. **Transformation to unit diffusion :**
For $X(t) = -\frac{2}{\sigma\sqrt{Y(t)}}$, the SDE transforms into

$$\begin{aligned} dX(t) &= \left(\left(\frac{3}{2} - \frac{2\alpha_2}{\sigma^2}\right)\frac{1}{X(t)} - \frac{\alpha_1}{2}X(t) + \frac{\alpha_0\sigma^2}{8}X(t)^3 - \frac{\alpha_{-1}\sigma^4}{32}X(t)^5\right) dt \\ &\quad + dW(t) \; ; \end{aligned} \tag{138}$$

X is distributed on $(-\infty, 0)$.

c. **Maximal probability path :**
The classical trajectory $y_{mod}(t|0,y_0;t,y_t)$ can be determined implicitly by

$$\int_{y_o}^{y_{mod}(t|0,y_0;t,y_t)} \frac{z\,dz}{\sqrt{Az^6+Gz^5+Bz^4+Cz^3+Dz^2+Ez+F}} = s\ , \qquad (139)$$

where the constant G follows from the boundary condition $y_{mod}(t|0,y_0;t,y_t) = y_t$.
For the respective coefficients, we have

$$\begin{array}{rclrcl} A &=& \frac{\sigma^4}{4}\left(\frac{3}{2}-\frac{2\alpha_2}{\sigma^2}\right)\left(\frac{1}{2}-\frac{2\alpha_2}{\sigma^2}\right) & D &=& \alpha_0^2+2\alpha_1\alpha_{-1} \\ B &=& \alpha_1^2-\alpha_0\sigma^2\left(3-\frac{2\alpha_2}{\sigma^2}\right) & E &=& 2\alpha_0\alpha_{-1} \\ C &=& 2\alpha_1\alpha_0-\alpha_{-1}\sigma^2\left(4-\frac{2\alpha_2}{\sigma^2}\right) & F &=& \alpha_{-1}^2\ . \end{array} \qquad (140)$$

d. **Long term probability :**
When t tends to infinity, we get the probability density

$$\overline{p}(\overline{y}) = C\cdot\overline{y}^{-\left(3-\frac{2\alpha_2}{\sigma^2}\right)} \cdot \exp\left\{-\frac{2\alpha_1}{\sigma^2}\frac{1}{\overline{y}}-\frac{\alpha_0}{\sigma^2}\frac{1}{\overline{y}^2}-\frac{2\alpha_{-1}}{3\sigma^2}\frac{1}{\overline{y}^3}\right\}\ ; \qquad (141)$$

the constant C depends on the coefficients α_{-1}, α_0, α_1, α_2 and σ, and is determined by the condition of a total mass equal to one.

e. **Transition probability :**
In a first step, the transition probability can be expressed by means of a path integral as

$$\begin{aligned} &p(0,y_0;t,y_t) \\ &= \frac{1}{\sigma y_t^{3/2}}\left(\frac{y_0}{y_t}\right)^{\frac{1}{2}\left(\frac{3}{2}-\frac{2\alpha_2}{\sigma^2}\right)} e^{\frac{\alpha_1}{2}\left(2-\frac{2\alpha_2}{\sigma^2}\right)t} \\ &\cdot e^{-\frac{\alpha_1}{\sigma^2}\left(\frac{1}{y_t}-\frac{1}{y_0}\right)} \cdot e^{-\frac{\alpha_0}{2\sigma^2}\left(\frac{1}{y_t^2}-\frac{1}{y_0^2}\right)} \cdot e^{-\frac{\alpha_{-1}}{3\sigma^2}\left(\frac{1}{y_t^3}-\frac{1}{y_0^3}\right)} \\ &\cdot \int_{(0,-\frac{2}{\sigma\sqrt{y_0}})}^{(t,-\frac{2}{\sigma\sqrt{y_t}})} D\_x(s)\ e^{-\frac{1}{2}\int_0^t \dot{x}^2 ds} \\ &\quad\times e^{-\int_0^t \left[Ax^{-2}+Bx^2+Cx^4+Dx^6+Ex^8+Fx^{10}\right] ds}\ . \end{aligned} \qquad (142)$$

For the respective coefficients, we have

$$
\begin{aligned}
A &= \tfrac{1}{2}\left(\tfrac{3}{2}-\tfrac{2\alpha_2}{\sigma^2}\right)\left(\tfrac{1}{2}-\tfrac{2\alpha_2}{\sigma^2}\right) & D &= \tfrac{\sigma^4}{128}\left(\alpha_0^2+2\alpha_1\alpha_{-1}\right)\\
B &= \tfrac{1}{8}\left[\alpha_1^2-\alpha_0\sigma^2\left(3-\tfrac{2\alpha_2}{\sigma^2}\right)\right] & E &= \tfrac{\sigma^6}{256}\alpha_0\alpha_{-1}\\
C &= \tfrac{\sigma^2}{32}\left[2\alpha_1\alpha_0\sigma^2-\alpha_{-1}\sigma^2\left(4-\tfrac{2\alpha_2}{\sigma^2}\right)\right] & F &= \tfrac{\sigma^8}{2048}\alpha_{-1}^2\,.
\end{aligned}
\tag{143}
$$

This path integral can not be calculated exactly, but we get an approximate result when applying the convex combination of upper and lower bound as mentioned in (87) with $V[z] = Az^{-2} + Bz^2 + Cz^4 + Dz^6 + Ez^8 + Fz^{10}$.

8.10 Double Well Potential Model

a. **Model :**

The Double Well Potential model is defined by the SDE

$$dY(t) = \left(Y(t) - Y(t)^3\right) dt + dW(t)\ ; \tag{144}$$

Y is distributed on $(-\infty, +\infty)$.

b. **Maximal probability path :**

The classical trajectory $y_{mod}(t|0, y_0; t, y_t)$ can be determined implicitly by

$$\int_{y_o}^{y_{mod}(t|0,y_0;t,y_t)} \frac{dz}{\sqrt{z^6 - 2z^4 - 2z^2 + C}} = s\ , \tag{145}$$

where the constant C follows from the boundary condition $y_{mod}(t|0, y_0; t, y_t) = y_t$.

c. **Long term probability :**

When t tends to infinity, we get the probability density

$$\overline{p}(\overline{y}) = C \cdot e^{\overline{y}^2 - \dfrac{\overline{y}^4}{2}}, \tag{146}$$

with $C = \dfrac{2}{\pi} \dfrac{e^{-1/4}}{I_{1/4}(\frac{1}{4}) + I_{-1/4}(\frac{1}{4})}$.

d. **Transition probability :**

In a first step, the transition probability can be expressed by means of a path integral as

$$p(0,y_0;t,y_t) = e^{\frac{1}{2}(y_t^2 - y_0^2) - \frac{1}{4}(y_t^4 - y_0^4) - \frac{1}{2}t} \tag{147}$$
$$\cdot \int_{(0,y_0)}^{(t,y_t)} Dy(s)\, e^{-\frac{1}{2}\int_0^t \dot{y}^2 ds - \frac{1}{2}\int_0^t (y^6 - 2y^4 - 2y^2)ds}.$$

This path integral can not be calculated exactly, but we get an approximate result when applying the convex combination of upper and lower bound as mentioned in (87) with $V[z] = \frac{1}{2}(z^6 - 2z^4 - 2z^2)$.

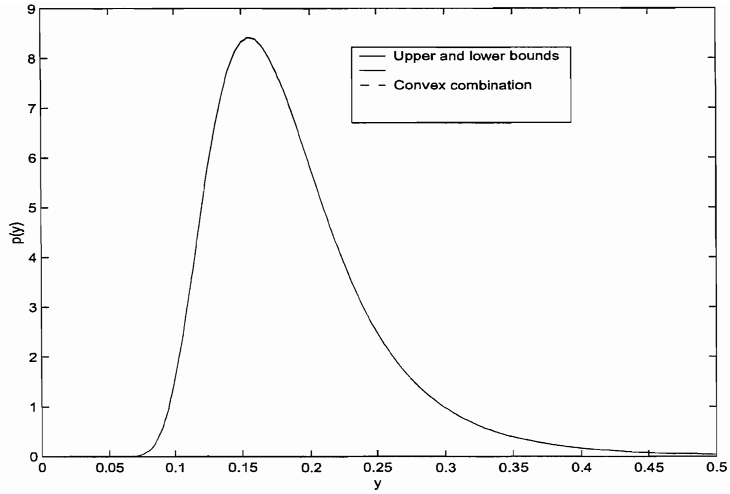
9 Numerical illustration

In this last section, we want to show the high accuracy of our approximations. We present graphs for the CEV Diffusion Model (figure 1), for the double well potential (figures 2 and 3) and for the non-linear mean reversion model (figure 4), three models for which the exact transition probability is not known in a closed form. For the parameters appearing in the models, use has been made of the same values as mentioned in the paper of Ait-Sahalia (see [1]) :

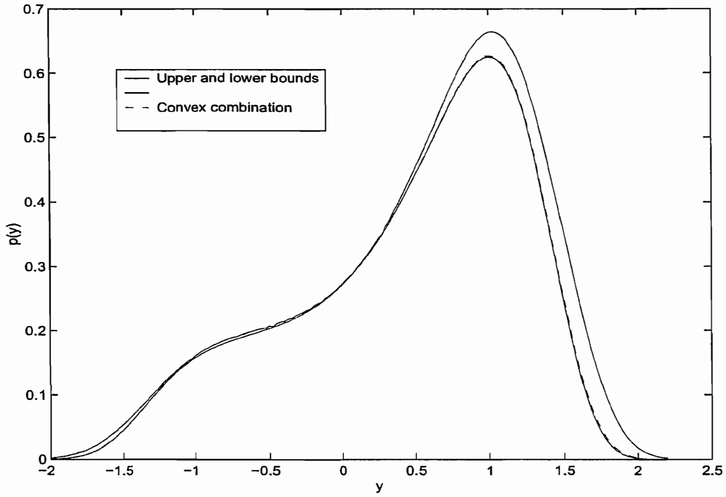
CEV Diffusion model	α	=	0.0808
	κ	=	0.0972
	σ	=	0.7224
Non linear mean reversion model	α_{-1}	=	0.00107
	α_0	=	-0.0517
	α_1	=	0.877
	α_2	=	-4.604
	σ	=	0.8047 .

Each figure contains our upper and lower bound, and the final new approximation which is based on a convex combination of the two bounds.

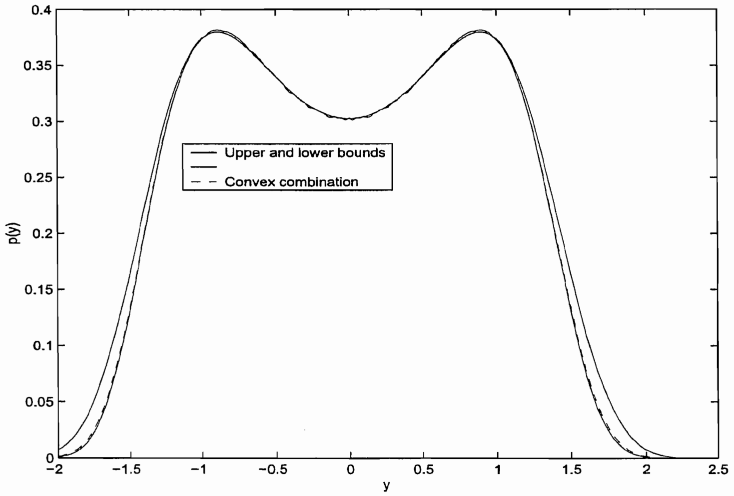
For the CEV diffusion model, upper and lower bounds are almost equal, such that the convex combination provides a very efficient approximation of the exact transition probability density. As can be seen in the graphs for the double well potential, the accuracy of both bounds is still very high, be it that the lower bound performs slightly better. For the non linear mean reversion model, the upper bound is less accurate, but fortunately the lower bound does better. Thanks to the fact that the total mass of the lower bound is only little lower than 1, it seems that also in this case, the final approximation performs very well.



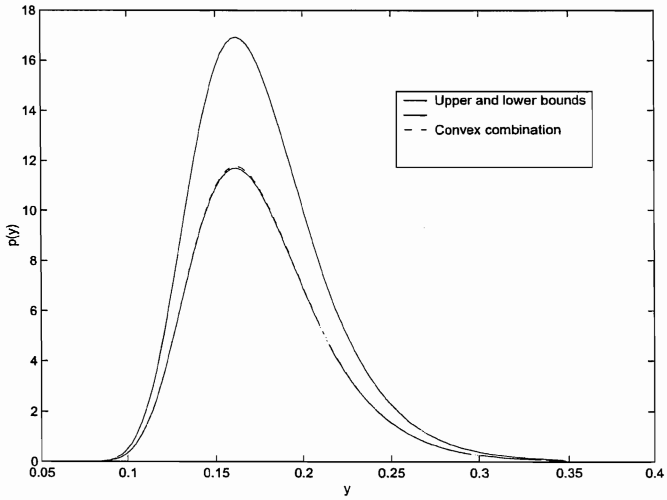
ure 1: Approximation of the density function for the CEV diffusion model, with $t = 1$
l starting point $y_o = 0.2$.



;ure 2: Approximation of the density function for the double well potential model, with
: 1 and starting point $y_o = 0.5$.



ure 3: Approximation of the density function for the double well potential model, with 1 and starting point $y_o = 0$.



;ure 4: Approximation of the density function for the non linear mean reversion model, h $t = 1$ and starting point $y_o = 1$.

Acknowledgement

The authors wish to thank the "Onderzoeksraad K.U.Leuven" for a GOA grant.

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Appendix A : Computational results for path integrals

In this first appendix we mention some useful results about path integrals for which an explicit expression is known. For the calculation of the Wiener and Gaussian integrals, use has been made of the methods of Feynman and Hibbs (see [6]). The results for the Calogero integrals are based on a result of Goovaerts (see [8]) and Vanneste et al. (see [17]). For the calculation of the exponential integral, we used a special coordinate transformation as was done in a similar proof in De Schepper & Goovaerts (see [4]) ; this enables us to rewrite the result of the Calogero integral into a result for the exponential path integral.

A.1 Wiener integrals

Ordinary Wiener process

$$\int_{(t_o,x_o)}^{(t_e,x_e)} Dx(s)\, e^{-\frac{1}{2}\int_{t_o}^{t_e} \dot{x}(s)^2\, ds} = \frac{1}{\sqrt{2\pi(t_e-t_o)}}\, e^{-\frac{(x_e-x_o)^2}{2(t_e-t_o)}}\,. \tag{148}$$

Absorbed Wiener process

$$\int_{(t_o,x_o)}^{(t_e,x_e)} D_+x(s)\; e^{-\frac{1}{2}\int_{t_o}^{t_e} \dot{x}(s)^2\, ds}$$
$$= \frac{1}{\sqrt{2\pi(t_e-t_o)}}\, e^{-\dfrac{(x_e-x_o)^2}{2(t_e-t_o)}} - \frac{1}{\sqrt{2\pi(t_e-t_o)}}\, e^{-\dfrac{(x_e+x_o)^2}{2(t_e-t_o)}} . \quad (149)$$

A.2 Gaussian integrals

$$\int_{(t_o,x_o)}^{(t_e,x_e)} Dx(s)\; e^{-\frac{1}{2}\int_{t_o}^{t_e} \dot{x}(s)^2\, ds - \frac{a^2}{2}\int_{t_o}^{t_e} x(s)^2\, ds + b\int_{t_o}^{t_e} x(s)\, ds}$$
$$= \sqrt{\frac{a}{\pi(1-e^{-2a(t_e-t_o)})}}\; \exp\left\{-\frac{a}{2}(te-to) + \frac{b^2}{2a^2}(t_e-t_o)\right\} \quad (150)$$
$$\cdot\; \exp\left\{\frac{(1+e^{-2a(t_e-t_o)})}{a^3(1-e^{-2a(t_e-t_o)})}\left(\, 2b^2 - 2a^2 b(x_0+x_e) + a^4(x_0^2+x_e^2)\right)\right\}$$
$$\cdot\; \exp\left\{-\frac{2e^{-a(t_e-t_o)}}{a^3(1-e^{-2a(t_e-t_o)})}\left(b-a^2x_o\right)\left(b-a^2x_e\right)\right\}.$$

A.3 Calogero integrals

$$\int_{(t_o,x_o)}^{(t_e,x_e)} Dx(s)\; e^{-\frac{1}{2}\int_{t_o}^{t_e} \dot{x}(s)^2\, ds - \frac{a^2}{2}\int_{t_o}^{t_e} x(s)^2\, ds - b\int_{t_o}^{t_e} \frac{ds}{x(s)^2}}$$
$$= \frac{a\,\sqrt{x_o x_e}}{\sinh\left(a(t_e-t_o)\right)}\; I_{\sqrt{2b+\frac{1}{4}}}\left[\frac{a\; x_o\; x_e}{\sinh\left(a(t_e-t_o)\right)}\right]$$
$$\cdot \exp\left\{-\frac{a\,(x_o^2+x_e^2)}{\tanh\left(a(t_e-t_o)\right)}\right\}, \quad (151)$$

where $I_g[z]$ denotes a modified Bessel function, which can be expressed in a series as

$$I_g[z] = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(g+k+1)} \left(\frac{z}{2}\right)^{g+2k}, \tag{152}$$

and arises as a solution of the differential equation

$$\psi''(z) + \frac{1}{z}\psi'(z) - \left(1 + \frac{g^2}{z^2}\right)\psi(z) = 0. \tag{153}$$

A.4 Exponential path integrals

$$\begin{aligned}
&\int_{(t_o,x_o)}^{(t_e,x_e)} Dx(s)\; e^{-\frac{1}{2}\int_{t_o}^{t_e} \dot{x}(s)^2\, ds - \frac{a^2}{2}\int_{t_o}^{t_e} e^{-2\sigma x}\, ds - b\int_{t_o}^{t_e} e^{-\sigma x}\, ds} \\
&= \frac{2\sqrt{2}\, a^2}{\pi\sqrt{\pi}\,\sigma^2} \frac{1}{\sqrt{t_e - t_o}}\, e^{\frac{2\pi^2}{\sigma^2(t_e - t_o)} - \frac{\sigma}{2}(x_0 + x_e)} \\
&\cdot \int_0^\infty ds\; e^{-\frac{4b}{\sigma^2}s} \frac{1}{\sinh^2\left(\frac{2as}{\sigma}\right)} \exp\left\{-\frac{a}{\sigma}\frac{1}{\tanh\left(\frac{2as}{\sigma}\right)}\left(e^{-\sigma x_o} + e^{-\sigma x_e}\right)\right\} \\
&\cdot \int_0^\infty dy\; e^{-\frac{2y^2}{\sigma^2(t_e - t_o)}} \sinh(y) \sin\left(\frac{4\pi y}{\sigma^2(t_e - t_o)}\right) \\
&\qquad \cdot \exp\left\{-\frac{2a}{\sigma}\frac{\cosh(y)}{\sinh\left(\frac{2as}{\sigma}\right)} e^{-\frac{\sigma}{2}(x_o + x_e)}\right\}.
\end{aligned} \tag{154}$$

Appendix B : Proofs of the theorems

PROOF OF THEOREM 3.1

From probability theory, we know that the solution of the stochastic differential equation (19) is unique. Hence, a solution that is found in another way, automatically leads to the same transition probability.

We start with a discretisation of equation (19),

$$y_{i+1} - y_i = (A(y_i) + \theta(A(y_{i+1}) - A(y_i)))\,\varepsilon + w_{i+1} - w_i \tag{155}$$

for $\varepsilon = t/n$, $t_i = i\varepsilon$ and $y_i = Y(t_i)$. This can be rewritten as

$$y_{i+1} - y_i = A(y_i)\varepsilon + \theta\frac{\partial A(y_i)}{\partial y}(y_{i+1} - y_i)\varepsilon + w_{i+1} - w_i. \tag{156}$$

In order to find a path integral expression for the transition probability of the process $Y = \{Y(s), s \in [0,t]\}$, we will perform a change of variables from w tot y in the Brownian path integral

$$\int_{(0,w_0)}^{(t,w_t)} Dw(s)\; e^{-\frac{1}{2}\int_0^t \dot{w}(s)^2\, ds} \tag{157}$$

when written as the limit of an $(n-1)$–fold integration. Due to the nature of the variables, the Jacobian matrix $J = \dfrac{\partial(w_1, w_2, ..., w_{n-1})}{\partial(y_1, y_2, ..., y_{n-1})}$ is an uppertriangular matrix, and therefore

$$|J| = \prod_{i=1}^{n-1}\left(1 - \theta\varepsilon\frac{\partial A(y_{i-1})}{\partial y}\right) \approx e^{-\theta\varepsilon\sum_{i=1}^{n-1}\frac{\partial A(y_{i-1})}{\partial y}}. \tag{158}$$

This brings us for the path integral to

$$\begin{aligned}
&\int_{(0,w_0)}^{(t,w_t)} Dw(s)\; e^{-\frac{1}{2}\int_0^t \dot{w}(s)^2\, ds} \\
&= lim_{n\to\infty}\frac{1}{\sqrt{2\pi\varepsilon}^{\,n}}\int dy_1 ... \int dy_{n-1}\;\; e^{-\frac{\varepsilon}{2}\sum_{i=0}^{n-1}\left(\frac{y_{i+1}-y_i}{\varepsilon}\right)^2} \\
&\quad\cdot e^{-\theta\varepsilon\sum_{i=1}^{n-1}\frac{\partial A}{\partial y}(y_{i-1})}\;\cdot e^{-\frac{\varepsilon}{2}\sum_{i=0}^{n-1}\left(A^2(y_i) + 2\theta A(y_i)\frac{\partial A}{\partial y}(y_i)(y_{i+1}-y_i)\right)} \\
&\quad\cdot e^{+\varepsilon\sum_{i=0}^{n-1}\left(A(y_i) + \theta\frac{\partial A}{\partial y}(y_i)(y_{i+1}-y_i)\right)\left(\frac{y_{i+1}-y_i}{\varepsilon}\right)}.
\end{aligned} \tag{159}$$

Now, we can return to a Feynman path integral expression by eliminating the limit of the $(n-1)$–fold integration. This results in

$$\begin{aligned}
&\int_{(0,w_0)}^{(t,w_t)} Dw(s)\; e^{-\frac{1}{2}\int_0^t \dot{w}(s)^2\, ds} \\
&= \int_{(0,y_0)}^{(t,y_t)} Dy(s)\;\; e^{-\frac{1}{2}\int_0^t \dot{y}(s)^2\, ds}\;\cdot e^{-\theta\int_0^t \frac{\partial A}{\partial y} ds} \\
&\quad\cdot e^{-\frac{1}{2}\int_0^t A^2(y(s))_\theta\, ds}\;\cdot e^{+\int_0^t A(y(s))_\theta\, dy(s)}.
\end{aligned} \tag{160}$$

Since the first integral in the last line of equation (160) is a Riemann integral, the θ has no influence. The second integral in the last line of (160) behaves as a general stochastic integral in the sence of (5) ; θ can be eliminated when making use of (13) – which completes the proof. Note that this last step is necessary due to the fact that Feynman integrations make use of a midpoint definition.

$\square$

PROOF OF PROPOSITION 3.1

A simple proof can be constructed when the transition probability $p(0, y_0; t+\varepsilon, y_t)$ is expanded.
First, we expand the probability in ε up to order one, giving

$$p(0, y_0; t+\varepsilon, y_t) = p(0, y_0; t, y_t) + \varepsilon \frac{\partial p}{\partial t}(0, y_0; t, y_t). \tag{161}$$

Next, due to the Kolmogorov property and with (21) one has

$$\begin{aligned} p(0, y_0; t+\varepsilon, y_t) &= \int_{-\infty}^{+\infty} dy\, p(0, y_0; t, y)\, p(t, y; t+\varepsilon, y_t) \\ &= \int_{-\infty}^{+\infty} dy\, \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\dfrac{(y_t - y - \varepsilon A(y))^2}{2\varepsilon}} p(0, y_0; t, y). \end{aligned} \tag{162}$$

Changing the integration variable as $y = y_t + \xi$ and expanding in ξ up to order two (which is equivalent with order one for ε), the last expression can be written as

$$\begin{aligned} &p(0, y_0; t+\varepsilon, y_t) \\ &= \int_{-\infty}^{+\infty} d\xi \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\dfrac{\xi^2}{2\varepsilon}} \\ &\quad \cdot \left(1 - \xi\left(A(y_t) + \xi \frac{\partial A}{\partial y_t}(y_t)\right) - \frac{\varepsilon}{2} A^2(y_t)\right) \\ &\quad \cdot \left(p(0, y_0; t, y_t) + \xi \frac{\partial p}{\partial y_t}(0, y_0; t, y_t) + \frac{\xi^2}{2} \frac{\partial^2 p}{\partial y_t^2}(0, y_0; t, y_t)\right) \end{aligned} \tag{163}$$

Performing the integration over ξ and comparing the result with expression (161), the partial differential equation arises.

$\square$

PROOF OF THEOREM 3.2

In a first step, we rewrite the θ-stochastic differential equation into a Stratonovich equation. This is necessary in order to justify the use of the classical chain rule.

Assume that the process $Y = \{Y(s), s \in [0,t]\}$ satisfies equation (24), and define the notations $\varepsilon = t/n$, $t_i = i\varepsilon$, $y_i = Y(t_i)$ and $y_i^\theta = Y(t_i + \theta(t_{i+1} - t_i))$. We then have

$$\begin{aligned}
&\int_0^t dY(s)\\
&= \int_0^t \{A(Y(s))ds + B(Y(s))_\theta \, dW(s)\}\\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \left\{\varepsilon A(y_i) + B(y_i^\theta)\,(w_{i+1} - w_i)\right\}\\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \left\{\varepsilon A(y_i) + \left[B(y_i^{1/2}) + (\theta - \frac{1}{2})\frac{\partial B}{\partial y}(y_i)\,(y_{i+1} - y_i)\right](w_{i+1} - w_i)\right\}\\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \left\{\varepsilon A(y_i) + B(y_i^{1/2})\,(w_{i+1} - w_i) + (\theta - \frac{1}{2})\varepsilon\,\frac{\partial B}{\partial y}(y_i)B(y_i)\right\}\\
&= \int_0^t \left\{\left[A(Y(s)) + (\theta - \frac{1}{2})\,\frac{\partial B}{\partial y}(Y(s))B(Y(s))\right]ds + B(Y(s))\,dW(s)\right\}. \qquad (164)
\end{aligned}$$

Now, if we use a change of variables as suggested in (25), the previous reasoning brings us to

$$\begin{aligned}
&\int_0^t dX(s)\\
&= \int_0^t \frac{dY(s)}{B(Y(s))}\\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \left\{\frac{1}{B(y_i^{1/2})}\,[y_{i+1} - y_i]\right\}\\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \left\{\frac{\varepsilon A(y_i) + B(y_i^{1/2})\,(w_{i+1} - w_i) + (\theta - \frac{1}{2})\varepsilon\,\frac{\partial B}{\partial y}(y_i)B(y_i)}{B(y_i^{1/2})}\right\}\\
&= \int_0^t \left\{\left[\frac{A(Y(s))}{B(Y(s))} + (\theta - \frac{1}{2})\,\frac{\partial B}{\partial y}(Y(s))\right]ds + dW(s)\right\}, \qquad (165)
\end{aligned}$$

which completes the proof.

□

PROOF OF THEOREM 3.3

This immediately follows from the change of variables mentioned in (25). Indeed,

$$\begin{aligned}
p(0, y_0; t, y_t) &= \frac{d}{dy_t} Prob\,[Y(t) \le y_t | Y(0) = y_0]\\
&= \frac{1}{B(y_t)}\frac{d}{dx_t} Prob\,[X(t) \le \psi(y_t) | X(0) = \psi(y_0)]\,. \qquad (166)
\end{aligned}$$

Applying theorem 3.1, the desired result follows.

□

PROOF OF PROPOSITION 3.2

The forward Fokker Planck equation can be derived in the same way as it was done for the result of proposition 3.1, when starting from the transition probability $p(0, y_0; t+\varepsilon, y_t)$ as given in (33).

The expansions now need to be performed up to order 4 for ε, and up to order 8 for ξ (see e.g. [15]).

□

PROOF OF THEOREM 4.1

The Lagrangian for the proces $Y(t)$ equals

$$L(\dot{y}, y, s) = \frac{1}{2}\dot{y}^2 + \frac{1}{2}A(y)^2 + \frac{1}{2}\frac{\partial A(y)}{\partial y} - A(y)\cdot\dot{y}. \tag{167}$$

Therefore, applying (35), the maximal probability path is determined by

$$\ddot{y} = A(y)\frac{\partial A(y)}{\partial y} + \frac{1}{2}\frac{\partial^2 A(y)}{\partial y^2}. \tag{168}$$

After multiplying both sides by $\dot{y}$, two integrations lead to the desired result.

□

PROOF OF THEOREM 5.1

In order to prove (47), we start from the path integral expression for the transition probability for the process Y, as stated in theorem 3.1 :

$$p(0, y_0; t, y_t) = \int_{(0,y_0)}^{(t,y_t)} Dy(s)\, e^{-\frac{1}{2}\int_0^t \dot{y}^2 ds - \frac{1}{2}\int_0^t \left(A(y)^2 + \frac{\partial A}{\partial y}\right) ds + \int_0^t A(y) dy} . \tag{169}$$

As suggested by the stochastic differential equation (44), we will make use of a coordinate transformation, in discretised version

$$y_{i+1} - y_i = \frac{1}{2}\frac{f''(x_i)}{f'(x_i)^2}(t_{i+1} - t_i) + f'(x_i)(x_{i+1} - x_i) , \tag{170}$$

where $\varepsilon = t/n$, $t_i = i\cdot\varepsilon$, $y_i = Y(t_i)$, $x_i = X(t_i)$, for $i = 0, ..., n$.

The transformation is obvious for the second and third integral in the exponent. For the first integral, the kinetic term, we have

$$\dot{y}^2 ds = \left(\frac{1}{2}\frac{f''(x(s)_L)}{f'(x(s)_L)^2} + f'((x(s)_L)\,\dot{x}(s)\right)^2 ds \tag{171}$$

and thus

$$\begin{aligned}&-\frac{1}{2}\int_0^t \dot{y}^2 ds \\ &= -\frac{1}{2}\int_0^t f'(x(s)_L)^2 \left(\frac{dx}{ds}\right)^2 ds - \frac{1}{2}\int_0^t \frac{f''(x(s)_L)}{f'(x(s)_L)} dx(s) - \frac{1}{8}\int_0^t \frac{f''(x(s)_L)^2}{f'(x(s)_L)^4} ds\,.\end{aligned} \tag{172}$$

Due to the conditions of the path integral formalism, the first and second integrations in this expression have to be transformed into mid point integrations (the last integral does not cause any difficulty being a Riemann integral). Making use of the expansion

$$\begin{aligned}\varphi(x_L)dx &= \varphi(x)dx - \frac{1}{2}\varphi'(x)(dx)^2 \\ &= \varphi(x)dx - \frac{1}{2}\varphi'(x)\frac{ds}{f'(x)^2}\end{aligned} \tag{173}$$

which holds for arbitrary functions φ, and where we relied upon (13), (43) and (45), we obtain

$$\begin{aligned}&\int_0^t \frac{f''(x(s)_L)}{f'(x(s)_L)} dx(s) \\ &= \int_0^t \frac{f''(x(s))}{f'(x(s))} dx(s) - \frac{1}{2}\int_0^t \frac{f'''(x(s))}{f'(x(s))^3} ds + \frac{1}{2}\int_0^t \frac{f''(x(s))^2}{f'(x(s))^4} ds\end{aligned} \tag{174}$$

as well as

$$\begin{aligned}&\int_0^t f'(x(s)_L)^2 \left(\frac{dx}{ds}\right)^2 ds \\ &= \int_0^t f'(x(s))^2 \left(\frac{dx}{ds}\right)^2 ds + \frac{1}{4}\int_0^t \frac{f''(x(s))^2}{f'(x(s))^4} ds - \int_0^t \frac{f''(x(s))}{f'(x(s))} dx(s)\,.\end{aligned} \tag{175}$$

As a consequence, the kinetic term can be expressed as

$$\begin{aligned}&-\frac{1}{2}\int_0^t \dot{y}^2 ds \\ &= -\frac{1}{2}\int_0^t f'(x(s))^2 \left(\frac{dx}{ds}\right)^2 ds - \frac{1}{2}\int_0^t \frac{f''(x(s))^2}{f'(x(s))^4} dx + \frac{1}{4}\int_0^t \frac{f'''(x(s))}{f'(x(s))^3} ds\,.\end{aligned} \tag{176}$$

The transformation can be completed by rewriting the path differential measure as

$$
\begin{aligned}
Dy(s) &= lim_{n\to\infty} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\varepsilon}} \prod_{i=1}^{n-1} dy_i \\
&= lim_{n\to\infty} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\varepsilon}} \prod_{i=1}^{n-1} f'(x_i)dx_i = [f'(x(s)_L)Dx(s)] \ . \qquad (177)
\end{aligned}
$$

Summarizing, this brings us to the interim result

$$
\begin{aligned}
p(0,y_0;t,y_t) &= \int_{(0,y_0)}^{(t,y_t)} [f'(x(s)_L)Dx(s)] \ e^{-\frac{1}{2}\int_0^t f'(x(s))^2 \left(\frac{dx}{ds}\right)^2 ds} \qquad (178) \\
&\cdot e^{-\frac{1}{2}\int_0^t \frac{f''(x(s))^2}{f'(x(s))^4} ds + \frac{1}{4}\int_0^t \frac{f'''(x(s))}{f'(x(s))^3} ds} \\
&\cdot e^{-\frac{1}{2}\int_0^t \left(A[f(x(s))]^2 + \frac{\partial A}{\partial y}[f(x(s))]\right) ds} \\
&\cdot e^{+\int_0^t A[f(x(s))] \ f'(x(s)) \ dx(s)} \ .
\end{aligned}
$$

Examining this path integral expression, it is clear that we still need a stochastic time change from t and ds to t^* and $d\sigma$ in order to get both a kinetic term and a differential measure that are independent of the transformation function. This can be done when choosing

$$
t_{i+1} - t_i = f'(x_i) \ f'(x_{i+1}) \ (\sigma_{i+1} - \sigma_i) \qquad (179)
$$

or

$$
ds = f'(x(\sigma)_L) \ f'(x(\sigma)_R) \ d\sigma \ . \qquad (180)
$$

As before, the transformation is obvious for all integrations except the kinetic term in the exponent of (178). For the kinetic term, we use the fact that (see e.g. (173))

$$
\begin{aligned}
f'(x(s))dx &= f'(x(s)_L)dx + \frac{1}{2}\frac{f''(x(s))}{f'(x(s))^2} ds \qquad (181) \\
&= f'(x(s)_R)dx - \frac{1}{2}\frac{f''(x(s))}{f'(x(s))^2} ds. \qquad (182)
\end{aligned}
$$

As a result, the kinetic term can be rewritten as

$$
-\frac{1}{2}\int_0^t f'(x(s))^2 \left(\frac{dx}{ds}\right)^2 ds = -\frac{1}{2}\int_0^{t^*} \left(\frac{dx}{d\sigma}\right)^2 d\sigma + \frac{1}{8}\int_0^{t^*} \frac{f''(x(\sigma))^2}{f'(x(\sigma))^2} d\sigma \ . \qquad (183)
$$

For the path differential measure, the stochastic time change leads to

$$
\begin{aligned}
&[f'(x(s)_L)Dx(s)] \\
&\quad = lim_{n\to\infty} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\ f'(x_{i-1})f'(x_i)\ (\sigma_i - \sigma_{i-1})}} \prod_{i=1}^{n-1} f'(x_i)dx_i \\
&\quad = \frac{1}{\sqrt{f'(x_0)f'(x_t)}}\ Dx(\sigma)\ .
\end{aligned}
\tag{184}
$$

Altogether, this results in the final path integral

$$
\begin{aligned}
&p_X(0, x_0 = f^{-1}(y_0); t, x_t = f^{-1}(y_t)) \\
&= \frac{1}{\sqrt{f'(f^{-1}(y_0)) \cdot f'(f^{-1}(y_t))}} \cdot \int_{(0,f^{-1}(y_0))}^{(t^*,f^{-1}(y_t))} Dx(\sigma)\ e^{-\frac{1}{2}\int_0^{t^*} \dot{x}^2 d\sigma} \\
&\quad \cdot e^{-\frac{1}{2}\int_0^{t^*} \left(A[f(x)]^2 + \frac{\partial A}{\partial y}[f(x)]\right) f'(x)^2 d\sigma} \\
&\quad \cdot e^{+\int_0^{t^*} A[f(x)]f'(x)dx - \frac{1}{8}\int_0^t \left[3\frac{f''(x)^2}{f'(x)^2} - 2\frac{f'''(x)}{f'(x)}\right] d\sigma},
\end{aligned}
\tag{185}
$$

where we still have to impose the condition

$$
t = \int_0^{t^*} f'(x)^2 d\sigma. \tag{186}
$$

This can be done by adding an integration with a Dirac function,

$$
\int_0^{+\infty} dt^*\ \delta\left(t - \int_0^{t^*} f'(x)^2 d\sigma\right), \tag{187}
$$

or

$$
\int_{-\infty}^{+\infty} d\beta \int_0^{+\infty} dt^*\ e^{i\beta\left(t\ -\ \int_0^{t^*} f'(x)^2 d\sigma\right)}, \tag{188}
$$

which completes the proof.

$\square$

Proof of theorem 7.1

The path integral can be written as

$$
I(t_o, x_o; t_e, x_e) = K(t_o, x_o; t_e, x_e) \cdot E_W \left[\!\left[e^{-\int_{t_o}^{t_e} V[X(s)]\, ds} \right]\!\right]. \tag{189}
$$

Applying proposition 7.2, we know that the variable $A = \int_{t_o}^{t_e} V[X(s)]\, ds$ is smaller than $B = \int_{t_o}^{t_e} F_{V(X(s))}^{-1}(U)\, ds$ in convex ordering. Since the exponential function is convex, it follows immediately from the definition of convex ordering (see (67)) that

$$I(t_o, x_o; t_e, x_e) \leq I^{upp}(t_o, x_o; t_e, x_e) \tag{190}$$

with

$$I^{upp}(t_o, x_o; t_e, x_e) = K(t_o, x_o; t_e, x_e) \cdot E_U \left[e^{-\int_{t_o}^{t_e} F_{V(X(s))}^{-1}(U)\, ds} \right]. \tag{191}$$

If we rewrite the expectation in (191) as an expectation over $B = \int_{t_o}^{t_e} F_{V(X(s))}^{-1}(U)\, ds$ instead of over U, an application of (69) leads to the second result, which completes the proof.

$\square$

PROOF OF THEOREM 7.2

We start by writing the path integral as

$$I(t_o, x_o; t_e, x_e) = K(t_o, x_o; t_e, x_e) \cdot E_\Lambda \left[E_W \left[\!\left[e^{-\int_{t_o}^{t_e} V[X(\tau)]\, d\tau} \,|\Lambda \right]\!\right] \right], \tag{192}$$

for an arbitrary stochastic variable Λ.
Applying the inequality of Jensen, it follows that

$$I(t_o, x_o; t_e, x_e) \geq I^{low}(t_o, x_o; t_e, x_e) \tag{193}$$

with

$$I^{low}(t_o, x_o; t_e, x_e) = K(t_o, x_o; t_e, x_e) \cdot E_\Lambda \left[e^{-\int_{t_o}^{t_e} E_W[V[X(\tau)]|\Lambda]\, d\tau} \right]. \tag{194}$$

If we choose $\Lambda = X(t_s)$, with t_s such that $t_o \leq t_s \leq t_e$, the final result immediately follows.

$\square$

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