

Analytic bounds and approximations for annuities and Asian options.

Steven Vanduffel*
Katholieke Universiteit Leuven

Zhaoning Shang
Katholieke Universiteit Leuven

Luc Henrard
Fortis

Jan Dhaene
Katholieke Universiteit Leuven

Emiliano A. Valdez
University of New South Wales

Abstract

Even in case of the Brownian motion as most natural rate of return model it appears too difficult to obtain analytic expressions for most risk measures of constant continuous annuities. In literature so-called comonotonic approximations have been proposed but these still require the evaluation of integrals. In this paper we show that these integrals can sometimes be computed, and we obtain explicit approximations for some popular risk measures for annuities.

Next, we show how these results can be used to obtain fully analytic expressions for lower and upper bounds for the price of a continuously sampled European-style Asian option with fixed exercise price. These analytic lower bound prices are as sharp as those from Rogers & Shi (1995), if not sharper, but in contrast do not require any longer the evaluation of a two-dimensional or a one-dimensional integral.

Keywords: Asian option, Brownian motion, analytic, closed-form, lower bound, annuity, fixed strike, continuously sampled.

*Corresponding author : Steven Vanduffel, Assistant Professor, Katholieke Universiteit Leuven, Naamsestraat 69, 3000 Leuven, Belgium (e-mail: Steven.Vanduffel@econ.kuleuven.be).

1 Introduction

Many risk management problems in finance and insurance involve a random variable (rv) S given by

$$S = \sum_{i=1}^n \alpha_i e^{Z_i}, \quad (1)$$

where the α_i are non-negative real numbers and (Z_1, Z_2, \dots, Z_n) is a random vector of either returns or discount factors. The means of the Z_i will be denoted by $E[Z_i]$ and their variances by $\sigma_{Z_i}^2$. Applications such as the valuation of Asian or basket options, the setting of provisions and optimal portfolio selection problems, amongst others, come down to the evaluation of risk measures related to the cumulative distribution function (cdf) of a random variable S as defined in (1).

We define a risk measure as a mapping from the set of random variables, usually representing the risks at hand, to the set of real numbers \mathbb{R} . Risk measures are a helpful tool for decision making because they summarise the information available about the random variable X into one single number $\rho[X]$.

In this paper we consider the p -quantile risk measure, denoted by $Q_p[X]$, and which for any p in $(0, 1)$ is defined as

$$Q_p[X] = \inf \{x \in \mathbb{R} \mid F_X(x) \geq p\}, \quad p \in (0, 1), \quad (2)$$

where $F_X(x) = \Pr[X \leq x]$. We will also consider the Conditional Tail Expectation, denoted by $CTE_p[X]$, and defined as

$$CTE_p[X] = E[X \mid X > Q_p[X]], \quad p \in (0, 1). \quad (3)$$

In this paper we will only deal with random variables with a continuous and strictly increasing cdf. Then it follows that

$$CTE_p[X] = \frac{\int_p^1 Q_q[X] dq}{1-p}, \quad p \in (0, 1), \quad (4)$$

see e.g. Dhaene *et al.* (2006). Using the notation $(x-d)_+ = \max(x-d, 0)$ we will also define the stop-loss premium with retention d of the random variable X as $E[(X-d)_+]$. It can be easily shown that

$$E[(X-d)_+] = \int_d^\infty (1-F_X(x)) dx, \quad -\infty < d < +\infty. \quad (5)$$

and also that

$$E[(X-d)_+] = (1-p)(CTE_p[X] - d), \quad (6)$$

where $d = Q_p[X]$.

In this paper we will also compare different rv's using the concept of convex orderings. A rv X is convex smaller than the rv Y , denoted by $X \leq_{cx} Y$, if $E[X] = E[Y]$ and $E[(X-d)_+] \leq E[(Y-d)_+]$ for $-\infty < d < +\infty$. The convex ordering reflects the common preferences of all risk averse decision makers when choosing between rv's with equal mean. This holds in both the classical utility theory from von Neuman &

Morgenstern as in Yaari's dual utility theory; see e.g. Denuit *et al.* (1999) or Dhaene *et al.* (2006) for more details.

The discrete case (sums of random variables) has a continuous counterpart (integrals of stochastic processes), and in this paper we mainly focus on developing some explicit results when the stochastic process under consideration is a geometric Brownian motion which is the continuous equivalent of the discrete Gaussian setting. Consequently, in line with Dhaene *et al.* (2002) we will consider in this paper the continuous equivalent of (1), which is the continuous temporary annuity S_t defined by

$$S_t = \int_0^t \alpha(\tau) e^{Z(\tau)} d\tau, \quad (7)$$

where $\{Z(\tau), \tau \geq 0\}$ represents a Brownian motion, i.e. the process has independent and stationary increments, $Z(0) = 0$, and for any $\tau \geq 0$ the random variable $Z(\tau)$ is normally distributed with mean $-\delta\tau$ and variance $\sigma^2\tau$. As some of our findings will also hold for negative discount rates and are useful to derive results for the compounding case, we will not make a general assumption on the sign of δ . Finally, the continuous stream of payments is described by a non-negative function of $\alpha(\tau)$. In the remainder of the paper we tacitly assume all integrals to be well defined.

The random variable S_t can be interpreted as the stochastically discounted value of all future obligations $\alpha(\tau)$. Investing an initial amount equal to $Q_p[S_t]$ according to this return process enables one to meet all future payments with probability p .

Although some analytic results are known for the annuity S_t 's distribution function, no "closed-form" expression appears to exist; see e.g. De Schepper *et al.* (1994, 1995). It was already mentioned in Dhaene *et al.* (2002) that using the so-called comonotonic upper and lower bound approximations for risk measures of constant continuous annuities gives rise to expressions that only involve the evaluation of a single integral. However, Huang *et al.* (2004) mentioned that numerical evaluations of these integrals are sensitive to the granularity that is used to approximate the integral by a sum.

In this paper we show that it is sometimes possible to evaluate these integrals explicitly. First, we show that in case of a constant continuous annuity (7) the comonotonic upper bound approximation gives rise to closed-form results for the quantiles, conditional tail expectation and stop-loss premiums. Next, we demonstrate that for some specific choices of the conditioning random variable Λ explicit results for these risk measures can also be obtained when using the comonotonic lower bound approximations, and we will compare these with available explicit results for perpetuities. Finally, we use these results to obtain fully analytic expressions for a lower and an upper bound for a continuously sampled European-style Asian option with fixed strike. The lower bound prices are as sharp as those from Rogers & Shi (1995) but in contrast do not require any longer the evaluation of integrals.

The remainder of this paper has been organised as follows. In Section 2, as an analogue of the continuous case developed in this paper, we recall from the literature the general results regarding the comonotonic bounds for the discrete case of the sum in (1). In Section 3 we develop explicit expressions for upper and lower bounds of several risk measures for a constant continuous annuity, and in Section 4 we will compare these approximations with available explicit results in the case of constant continuous perpetuities. In Sections 5 we apply our results to the analysis of Asian options. Finally, Section 6 provides some concluding remarks.

2 Comonotonic approximations - the discrete case

2.1 General results

Let the random variable S be given by (1), where the α_i are non-negative real numbers and the random vector (Z_1, Z_2, \dots, Z_n) has a multivariate Normal distribution. Consider the conditioning random variable Λ which is given by

$$\Lambda = \sum_{i=1}^n \gamma_i Z_i, \quad (8)$$

for some γ_i . Following Kaas *et al.* (2000) we define a random variable S^l as

$$S^l \stackrel{\text{def}}{=} E[S|\Lambda] = \sum_{i=1}^n \alpha_i e^{E[Z_i] + \frac{1}{2}(1-r_i^2)\sigma_{Z_i}^2 + r_i\sigma_{Z_i} \frac{\Lambda - E[\Lambda]}{\sigma_\Lambda}}, \quad (9)$$

where r_i denotes the correlation coefficient between Z_i and Λ , for $i = 1, \dots, n$. We find for the cdf of S^l that

$$S^l \stackrel{d}{=} \sum_{i=1}^n \alpha_i e^{E[Z_i] + \frac{1}{2}(1-r_i^2)\sigma_{Z_i}^2 + r_i\sigma_{Z_i} \Phi^{-1}(U)}, \quad (10)$$

with Φ the cdf of a standard normally distributed rv and U a uniformly distributed rv on the unit interval.

Then, provided all coefficients r_i are positive we find that the components in the sum are all increasing in Λ , meaning that they are co-monotonic, so that we can invoke a well-known additivity property for risk measures of sums of comonotonic risks to find that for $p \in (0, 1)$,

$$Q_p[S^l] = \sum_{i=1}^n \alpha_i e^{E[Z_i] + \frac{1}{2}(1-r_i^2)\sigma_{Z_i}^2 + r_i\sigma_{Z_i} \Phi^{-1}(p)}, \quad (11)$$

$$CTE_p[S^l] = \sum_{i=1}^n \alpha_i e^{E[Z_i] + \frac{1}{2}\sigma_{Z_i}^2} \frac{\Phi(r_i \sigma_{Z_i} - \Phi^{-1}(p))}{1-p}, \quad (12)$$

We refer to Kaas *et al.* (2000) or Dhaene *et al.* (2002) for more details.

Finally, from (6) we find that:

$$E[(S^l - d)_+] = \sum_{i=1}^n \alpha_i e^{E[Z_i] + \frac{1}{2}\sigma_{Z_i}^2} \Phi(r_i \sigma_{Z_i} - \Phi^{-1}(p)) - d(1-p), \quad (13)$$

where $Q_p[S^l] = d$. In literature also another rv, denoted by S^c , has been proposed to approximate the risk measures of S ; see e.g. Kaas *et al.* (2000) or Dhaene *et al.* (2002). In our lognormal context S^c can be defined by imposing the correlations in expression (10) to be equal to one. Formally we obtain

$$S^c \stackrel{d}{=} \sum_{i=1}^n \alpha_i e^{E[Z_i] + \sigma_{Z_i} \Phi^{-1}(U)}, \quad (14)$$

and we obtain closed form expressions for $Q_p[S^c]$, $CTE_p[S^c]$ and $E[(S^c - d)_+]$ by applying the substitution $r_i = 1$ in the equivalent expressions (11), (12) and (13) for the rv S^l .

As demonstrated in Kaas *et al.* (2000), it follows that the rv's S , S^l and S^c are convex ordered:

$$S^l \leq_{cx} S \leq_{cx} S^c, \quad (15)$$

and this will imply the following ordering relations:

$$Var [S^l] \leq Var [S] \leq Var [S^c], \quad p \in (0, 1), \quad (16)$$

and

$$\begin{aligned} E [(S^l - d)_+] &\leq E [(S - d)_+] \leq E [(S^c - d)_+], \quad d \in \mathbb{R}. \\ CTE_p [S^l] &\leq CTE_p [S] \leq CTE_p [S^c], \quad p \in (0, 1), \end{aligned} \quad (17)$$

All the above mentioned results can also be found in Dhaene *et al.* (2002).

2.2 The optimal choice for the conditioning random variable

The best closed-form approximations based on the lower bound $E[S|\Lambda]$ will occur when Λ is chosen such that $E[S | \Lambda]$ is sufficiently ‘close’ to the rv S while still finding an explicit expression for it. Note that when we take $\Lambda \equiv S$ we find that $E[S | \Lambda] \equiv S$ but this ideal choice for Λ is not feasible because it does not allow explicit calculations.

From (16) and (17) we find that a good conditioning rv Λ could be defined as one that allows explicit calculations for $E[S | \Lambda]$ while obtaining as large as possible values for $Var[E[S | \Lambda]]$ or, alternatively, for $CTE_p[E[S|\Lambda]]$. Note that the latter approach is more focusing on finding optimal approximations for S in case S takes large values whereas the former intends to provide a global optimal fit to S .

Vanduffel *et al.* (2005a) proved that the first order approximation of the variance of S^l will be maximised when Λ is given by

$$\Lambda = \sum_{i=1}^n \alpha_i e^{E[Z_i] + \frac{1}{2}\sigma_{Z_i}^2} Z_i. \quad (18)$$

This choice for Λ is denoted by Λ^{MV} . When the approximations for the risk measures of S^l are based on Λ^{MV} as conditioning random variable we call these in the paper ‘maximal variance’ approximations.

It was shown in Vanduffel *et al.* (2007) that the following choice for Λ maximises a first-order approximation of $CTE_p[S^l]$:

$$\Lambda = \sum_{i=1}^n \alpha_i e^{E[Z_i] + \frac{1}{2}\sigma_{Z_i}^2} \cdot \Phi'(r_i^{MV} \sigma_{Z_i} - \Phi^{-1}(p)) Z_i. \quad (19)$$

where r_j^{MV} is the correlation between Λ^{MV} and Z_i . This choice for Λ will be denoted by $\Lambda^{(p)}$, and note that $\Lambda^{(p)}$ depends on the probability level p reflecting that it is indeed constructed to be locally optimal in some sense. When $\Lambda^{(p)}$ is used to derive the lower bound approximations we will call these ‘maximal CTE’ approximations. For a detailed account on the topic of appropriately choosing the conditioning rv Λ we refer to Vanduffel *et al.* (2007).

3 Closed-form comonotonic approximations for the continuous case

3.1 General results

We will now focus on the temporary continuous annuity S_t as defined in (7). Analogous to the discrete setting discussed in Kaas *et al.* (2000) it was stated in Dhaene *et al.* (2002) that $S_t^l \leq_{cx} S_t \leq_{cx} S_t^c$, where the random variable S_t^c is now defined by

$$S_t^c \stackrel{d}{=} \int_0^t \alpha(\tau) e^{-\delta\tau + \sigma \sqrt{\tau} \Phi^{-1}(U)} d\tau, \quad (20)$$

with U a uniformly $(0, 1)$ distributed random variable. For S_t^l we have that

$$S_t^l \stackrel{def}{=} E[S_t | \Lambda] = \int_0^t \alpha(\tau) e^{-\delta\tau + \frac{1}{2}\sigma^2\tau(1-r^2(\tau)) + r(\tau)\sigma\sqrt{\tau}\frac{\Lambda - E[\Lambda]}{\sigma\Lambda}} d\tau, \quad (21)$$

where $r(\tau)$ is the correlation between $Z(\tau)$ and the conditioning rv Λ which is defined as

$$\Lambda = \int_0^t \gamma(\tau) Z(\tau) d\tau, \quad (22)$$

for some function $\tau \rightarrow \gamma(\tau)$. The cdf of S_t^l is given by:

$$S_t^l \stackrel{d}{=} \int_0^t \alpha(\tau) e^{-\delta\tau + \frac{1}{2}\sigma^2\tau(1-r^2(\tau)) + r(\tau)\sigma\sqrt{\tau}\Phi^{-1}(V)} d\tau, \quad (23)$$

with V a rv that is uniformly distributed on the unit interval.

Note that S_t^c will be an integral of comonotonic random variables. As a result, its quantiles and conditional tail expectations follow from

$$Q_p[S_t^c] = \int_0^t \alpha(\tau) e^{-\delta\tau + \sigma \sqrt{\tau} \Phi^{-1}(p)} d\tau, \quad (24)$$

$$CTE_p[S_t^c] = \int_0^t \alpha(\tau) e^{-\delta\tau + \sigma^2\tau/2} \frac{\Phi(\sigma\sqrt{\tau} - \Phi^{-1}(p))}{1-p} d\tau, \quad (25)$$

with $0 < p < 1$. The stop-loss premiums can be easily calculated from this using equation (6).

When the function $f(\tau) = Cov[Z(\tau), \Lambda]$ is non-negative we also find that S_t^l will be an integral of comonotonic random variables and we obtain

$$Q_p[S_t^l] = \int_0^t \alpha(\tau) e^{-\delta\tau + \frac{1}{2}\sigma^2\tau(1-r^2(\tau)) + r(\tau)\sigma\sqrt{\tau}\Phi^{-1}(p)} d\tau, \quad (26)$$

$$CTE_p[S_t^l] = \int_0^t \alpha(\tau) e^{-\delta\tau + \sigma^2\tau/2} \frac{\Phi(r(\tau)\sigma\sqrt{\tau} - \Phi^{-1}(p))}{1-p} d\tau, \quad (27)$$

with again $0 < p < 1$. Also in this case we use (6) to derive the lower bound expressions for the stop-loss premiums. The results (24)-(27) can also be found in Dhaene *et al.* (2002).

We note that the formulas (24)-(27) involve the evaluation of an integral and as such may require a numerical evaluation. Dhaene *et al.* (2002) as well as Huang *et al.* (2004) effectively substitute $\alpha(\tau)$ by $\sum_{i=1}^n \alpha(\frac{it}{n}) \Delta_{\tau=\frac{it}{n}}$ where the symbol Δ is here

used to denote Dirac's delta function. This will effectively turn the integral at hand into a discrete sum and it follows that we recover the discrete results as mentioned in Section 2; See also Lord (2006).

However, in case of constant continuous annuities it is sometimes possible to directly evaluate the integrals and this will be discussed in the next subsections. In what follows the results will be valid for $\alpha(\tau)$ being a constant, but for simplicity of the exposition we will consider a continuous stream of unit cash flows, i.e. $\alpha(\tau) = 1$. In the remainder of the paper we will also make use of the quantity δ^* which is defined as:

$$\delta^* = \delta - \frac{1}{2}\sigma^2. \quad (28)$$

3.2 Upper bound approach

In this section we will derive closed-form approximations for several of the risk measures of S_t^c . These explicit formulas will only hold when $\delta > 0$.

The quantiles of S_t^c follow from (24) but now with $\alpha(\tau) = 1$. Then, by substituting $y = \sqrt{\tau}$ and realising that the resulting integral can be rewritten in terms of the standard-normal cdf Φ we find the following analytical expression for $Q_p[S_t^c]$:

$$Q_p[S_t^c] = \frac{1}{\delta} - \frac{1}{\delta}e^{-\delta t + \sigma\sqrt{t}\Phi^{-1}(p)} + \frac{1}{\delta}\sqrt{2\pi}ae^{\frac{a^2}{2}}(\Phi(\sqrt{2\delta t} - a) - \Phi(-a)), \quad (29)$$

with

$$a = \frac{\sigma\Phi^{-1}(p)}{\sqrt{2\delta}}, \quad \delta > 0, \quad (0 < p < 1). \quad (30)$$

A detailed proof for this result can be found in Appendix A.

Furthermore, from (25) and using the same substitution $y = \sqrt{\tau}$ as in the case of the quantiles, we find after some detailed computations the following closed-form expression for the conditional tail expectations $CTE_p[S_t^c]$:

$$\begin{aligned} CTE_p[S_t^c] &= \frac{1}{\delta^*} - \frac{1}{\delta^*}e^{-\delta^*t} \frac{\Phi(\sigma\sqrt{t} - \Phi^{-1}(p))}{1-p} \\ &\quad + \frac{\sigma}{\delta^*}e^{\frac{a^2 - (\Phi^{-1}(p))^2}{2}} \sqrt{\frac{1}{2\delta}} \frac{\Phi(\sqrt{2\delta t} - a) - \Phi(-a)}{1-p}, \quad (0 < p < 1). \end{aligned} \quad (31)$$

Finally, the stop-loss premiums $E[(S^c - d)_+]$ with retentions $d > 0$ can be derived from (6), (31) and (29).

3.3 Lower bound approaches

3.3.1 General results

In order to compute the risk measures of S_t^l we may suggest to use the continuous analogon of (18) namely the conditioning random variable $\Lambda = \int_0^t e^{-\delta^*\tau} Z(\tau) d\tau$. This choice will ensure that Λ is 'close' to S_t because one easily can prove along similar lines as in Vanduffel *et al.* (2005a) it will maximise the first order Taylor approximation

for the variance of S_t^l . In this case we find that Λ is normally distributed with mean 0 and variance given by

$$\begin{aligned}\sigma_\Lambda^2 = Var[\Lambda] &= \int_0^t \int_0^t e^{-\delta^*(\tau+\nu)} \min(\tau, \nu) d\tau d\nu \\ &= \frac{1}{2\delta^{*3}} + \frac{3 + 2\delta^*t - 4e^{\delta^*t}}{2\delta^{*3}e^{2\delta^*t}},\end{aligned}\quad (32)$$

and $r(\tau)$ is given by

$$r(\tau) = \frac{Cov[Z(\tau), \Lambda]}{\sigma_\Lambda \sigma \sqrt{\tau}} = \frac{1}{\sigma_\Lambda \sqrt{\tau}} \left[\frac{1 - e^{-\delta^*\tau}}{\delta^{*2}} - \frac{\tau e^{-\delta^*t}}{\delta^*} \right], \quad \tau \leq t. \quad (33)$$

Since the function $f(\tau) = Cov[Z(\tau), \Lambda]$ is a non-negative function, S_t^l will be an integral of comonotonic random variables. Unfortunately, for finite annuities, it appears that there are no closed-form solutions for the quantiles, conditional tail expectations and stop-loss premiums of S_t^l when using this choice for Λ . We now propose other choices for Λ that will allow to obtain explicit closed-form approximations for these risk measures.

3.3.2 $\Lambda = \int_0^\infty e^{-\delta^*\tau} Z(\tau) d\tau$

Note that this choice for Λ implicitly assumes that $\delta^* > 0$ and the formulas we derive are only valid in this case. Then, the quantiles of S_t^l ($0 < p < 1$) follow from (26) with $\alpha(\tau) = 1$. Then, by substituting $y = \sqrt{\tau}$ and realising that

$$r(\tau) = \sqrt{\frac{2}{\delta^*\tau}} (1 - e^{-\delta^*\tau}). \quad (34)$$

we find that they can be expressed as

$$Q_p[S_t^l] = \int_0^t e^{-\delta^*\tau - \frac{1}{2}c^2(1-e^{-\delta^*\tau})^2 + c\Phi^{-1}(p)(1-e^{-\delta^*\tau})} d\tau, \quad (35)$$

with

$$c = \sigma \sqrt{\frac{2}{\delta^*}}. \quad (36)$$

By making the substitution $y = c(1 - e^{-\delta^*\tau})$ one sees that the integral (35) can be rewritten in terms of the standard-normal cdf. As a result we find the following analytical expression for the quantiles of S_t^l :

$$Q_p[S_t^l] = \frac{1}{c\delta^*} \sqrt{2\pi} e^{\frac{(\Phi^{-1}(p))^2}{2}} (\Phi(k_t) - 1 + p), \quad (37)$$

with

$$k_t = c(1 - e^{-\delta^*t}) - \Phi^{-1}(p). \quad (38)$$

After some tedious algebra we also find that

$$\begin{aligned}CTE_p[S_t^l] &= \frac{1}{\delta^*(1-p)} \Phi(k_t)(1 - e^{-\delta^*t}) \\ &\quad + \frac{1}{\delta^*(1-p)} \left(\frac{\Phi^{-1}(p)}{c} \right) (1 - p - \Phi(k_t)) \\ &\quad - \frac{1}{\sqrt{2\pi}c\delta^*(1-p)} (e^{-\frac{1}{2}(\Phi^{-1}(p))^2} - e^{-\frac{1}{2}k_t^2}),\end{aligned}\quad (39)$$

Finally, invoking (6) again we find from the above expressions also the stop-loss premiums of S_t^l with retentions $d > 0$.

We point out that this specific choice for $\Lambda = \int_0^\infty e^{-\delta^*\tau} Z(\tau) d\tau$ will not always perform very well for finite annuities. However, when t reaches infinity, this choice for Λ leads to the continuous equivalent of the ‘maximal variance’ lower bound approach that we discussed in Section 2. Consequently our specific choice for Λ is likely to yield excellent results when t reaches infinity whilst allowing an analytical expression for the selected risk measures of S_t^l for finite t as well.

3.3.3 $\Lambda = \sum_{i=1}^n \gamma(t_i) Z(t_i)$

The approximations we derive in this subsection will exist for all values for δ . We will assume that $0 < t_1 < t_2 < \dots < t_n = t$ and $\gamma(t_i) > 0$ for $i = 1, 2, \dots, n-1$ whereas $\gamma(t_n) \geq 0$. We will also take $t_0 = 0$ and introduce the notation $S_t^{l,s}$ to denote the convex lower bound for S_t . From (26) we see that the quantiles of $S_t^{l,s}$ ($0 < p < 1$) can also be expressed as

$$Q_p[S_t^{l,s}] = \int_0^t e^{-\delta^*\tau - \frac{\text{Cov}^2[\Lambda, Z(\tau)]}{2\sigma_\Lambda^2} + \frac{\text{Cov}[\Lambda, Z(\tau)]}{\sigma_\Lambda} \Phi^{-1}(p)} d\tau. \quad (40)$$

Here, for $0 \leq \tau < t$ we have that $\text{Cov}[\Lambda, Z(\tau)]$ can be expressed as

$$\text{Cov}[\Lambda, Z(\tau)] = \sum_{j=0}^{n-1} (a_j + b_j \tau) 1_{t_j \leq \tau < t_{j+1}}, \quad (41)$$

where $1_{(\cdot)}$ is the usual notation for the indicator function and

$$\begin{aligned} a_j &= \sigma^2 \sum_{i=1}^j \gamma(t_i) t_i, \\ b_j &= \sigma^2 \sum_{i=j+1}^n \gamma(t_i), \quad j = 0, 1, \dots, n-1, \end{aligned} \quad (42)$$

where we conventionally assumed that $\sum_{i=1}^0 x_i = 0$. Moreover, σ_Λ^2 is given by

$$\sigma_\Lambda^2 = \sigma^2 \sum_{i=1}^n \sum_{j=1}^n \gamma(t_i) \gamma(t_j) \min(t_i, t_j). \quad (43)$$

Then, we find that

$$Q_p[S_t^{l,s}] = \sum_{j=0}^{n-2} c_j e^{\frac{d_j^2}{2e_j}} \sqrt{\frac{2\pi}{e_j}} \left(\Phi\left(t_{j+1}\sqrt{e_j} - \frac{d_j}{\sqrt{e_j}}\right) - \Phi\left(t_j\sqrt{e_j} - \frac{d_j}{\sqrt{e_j}}\right) \right) + f_{n-1}, \quad (44)$$

where in case $\gamma(t_n) \neq 0$ we have that f_{n-1} is given by

$$f_{n-1} = c_{n-1} e^{\frac{d_{n-1}^2}{2e_{n-1}}} \sqrt{\frac{2\pi}{e_{n-1}}} \left(\Phi\left(t_n\sqrt{e_{n-1}} - \frac{d_{n-1}}{\sqrt{e_{n-1}}}\right) - \Phi\left(t_{n-1}\sqrt{e_{n-1}} - \frac{d_{n-1}}{\sqrt{e_{n-1}}}\right) \right), \quad (45)$$

while when $\gamma(t_n) = 0$ we find that

$$f_{n-1} = \frac{c_{n-1}}{\delta^*} (e^{-\delta^* t_{n-1}} - e^{-\delta^* t_n}). \quad (46)$$

Here, for $j = 0, 1, \dots, n-1$ we have that

$$\begin{aligned} c_j &= e^{-\frac{a_j^2}{2\sigma_\Lambda^2} + \frac{a_j}{\sigma_\Lambda} \Phi^{-1}(p)}, \\ d_j &= -\delta^* - \frac{a_j b_j}{\sigma_\Lambda^2} + \frac{b_j}{\sigma_\Lambda} \Phi^{-1}(p), \\ e_j &= \frac{1}{\sigma_\Lambda^2} b_j^2, \end{aligned} \quad (47)$$

For the conditional tail expectations we find after some lengthy computations that

$$\begin{aligned} CTE_p[S_t^{l,s}] &= \frac{1}{1-p} \sum_{j=0}^{n-2} \left(\frac{e^{-\delta^* t_j}}{\delta^*} \Phi(\kappa_j t_j + \lambda_j) - \frac{e^{-\delta^* t_{j+1}}}{\delta^*} \Phi(\kappa_j t_{j+1} + \lambda_j) \right. \\ &\quad \left. + \frac{1}{\delta^*} e^{\frac{1}{2} \left(\frac{\delta^*}{\kappa_j} \right)^2 + \frac{\lambda_j}{\kappa_j} \delta^*} \times \left(\Phi\left(\kappa_j t_{j+1} + \lambda_j + \frac{\delta^*}{\kappa_j}\right) \right. \right. \\ &\quad \left. \left. - \Phi\left(\kappa_j t_j + \lambda_j + \frac{\delta^*}{\kappa_j}\right) \right) \right) + \nu_{n-1}, \end{aligned} \quad (48)$$

where in case $\gamma(t_n) \neq 0$ we have that ν_{n-1} is given by

$$\begin{aligned} \nu_{n-1} &= \frac{1}{(1-p)\delta^*} \left(e^{-\delta^* t_{n-1}} \Phi(\kappa_{n-1} t_{n-1} + \lambda_{n-1}) - e^{-\delta^* t_n} \Phi(\kappa_{n-1} t_n + \lambda_{n-1}) \right. \\ &\quad \left. + e^{\frac{1}{2} \left(\frac{\delta^*}{\kappa_{n-1}} \right)^2 + \frac{\lambda_{n-1}}{\kappa_{n-1}} \delta^*} \times \left(\Phi\left(\kappa_{n-1} t_n + \lambda_{n-1} + \frac{\delta^*}{\kappa_{n-1}}\right) - \Phi\left(\kappa_{n-1} t_{n-1} + \lambda_{n-1} + \frac{\delta^*}{\kappa_{n-1}}\right) \right) \right), \end{aligned} \quad (49)$$

while for $\gamma(t_n) = 0$ it proofs that ν_{n-1} is given by

$$\nu_{n-1} = \frac{\Phi(\lambda_{n-1})}{(1-p)\delta^*} (e^{-\delta^* t_{n-1}} - e^{-\delta^* t_n}). \quad (50)$$

Here, the coefficients κ_j and λ_j ($j = 0, 1, \dots, n-1$) are given by

$$\begin{aligned} \kappa_j &= \frac{b_j}{\sigma_\Lambda}, \\ \lambda_j &= \frac{a_j}{\sigma_\Lambda} - \Phi^{-1}(p). \end{aligned} \quad (51)$$

Finally, the stop-loss premiums for $S_t^{l,s}$ follow from (6), (48) and (44).

4 Perpetuities

Consider the perpetuity S_∞ defined by

$$S_\infty = \int_0^\infty e^{Z(\tau)} d\tau. \quad (52)$$

Dufresne (1990) proved that its reciprocal S_∞^{-1} is Gamma distributed with parameters $\frac{2\delta}{\sigma^2}$ and $\frac{\sigma^2}{2}$ meaning that its probability density function (pdf) is expressed as

$$f_{S_\infty^{-1}}(x; \alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, \quad x > 0, \quad (53)$$

where $\alpha > 0$, $\beta > 0$ and $\Gamma(\cdot)$ denotes the Gamma function:

$$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du \quad (\alpha > 0). \quad (54)$$

It follows that S_∞ has as pdf:

$$f_{S_\infty}(y; \alpha, \beta) = f_{S_\infty}(1/y; \alpha, \beta)/y^2, \quad y > 0, \quad (55)$$

with quantiles and conditional tail expectations given by

$$Q_p[S_\infty] = \frac{1}{F_{S_\infty^{-1}}^{-1}(1-p; \alpha, \beta)}, \quad p \in (0, 1), \quad (56)$$

and

$$CTE_p[S_\infty] = \frac{F_{S_\infty^{-1}}(F_{S_\infty^{-1}}^{-1}(1-p; \alpha, \beta); \alpha-1, \beta)}{(1-p)(\alpha-1)\beta}, \quad p \in (0, 1), \quad (57)$$

where $F_{S_\infty^{-1}}(\cdot; \alpha, \beta)$ is the cdf of the Gamma distribution with parameters α and β . Note that expression (57) implicitly assumes that $\delta > \frac{\sigma^2}{2}$. Since the Gamma distribution is readily available in many statistical software packages these different risk measures can be determined easily.

Hence, we can compare the cumulative distribution functions of the lower bound S_∞^l and the upper bound S_∞^c with the exact cumulative distribution function of S_∞ . For the lower bound approach we take $\Lambda = \int_0^\infty e^{-\delta^* \tau} Z(\tau) d\tau$ as conditioning rv.

From (56), (29) and (37), with $t \rightarrow \infty$, we find for $0 < p < 1$ the following closed-form expressions for the quantiles of S_∞ , S_∞^c and S_∞^l , respectively.

$$Q_p[S_\infty] = \frac{1}{F_{S_\infty^{-1}}^{-1}(1-p; \frac{2\delta}{\sigma^2}, \frac{\sigma^2}{2})}, \quad (58)$$

$$Q_p[S_\infty^c] = \frac{1}{\delta} (1 + a\sqrt{2\pi} e^{\frac{a^2}{2}} \Phi(a)), \quad (59)$$

$$Q_p[S_\infty^l] = \frac{1}{c\delta^*} \sqrt{2\pi} e^{\frac{(\Phi^{-1}(p))^2}{2}} (\Phi(c - \Phi^{-1}(p)) - 1 + p), \quad (60)$$

with F_X the cdf of the Gamma distribution and

$$\begin{aligned} a &= \frac{\sigma \Phi^{-1}(p)}{\sqrt{2\delta}}, \\ c &= \sigma \sqrt{\frac{2}{\delta^*}}. \end{aligned} \quad (61)$$

In Table 1 we show the quantiles of S_∞^l , S_∞^c and S_∞ in the case where $\delta = 0.07$ and $\sigma = 0.1$. These results can be compared with the results reported in Dhaene *et al.* (2002). The small differences we observe, can be explained as follows: Firstly,

p	$Q_p[S_\infty^l]$	$Q_p[S_\infty]$	$Q_p[S_\infty^c]$
0.95	23.62	23.63	25.90
0.975	26.09	26.13	29.34
0.99	29.37	29.49	34.08
0.995	31.90	32.10	37.86
0.999	38.00	38.49	47.38

Table 1: The table compares some selected exact quantiles of the constant perpetuity with the ‘maximal variance’ lower bound and upper bound approximations ($\delta=0.07$, $\sigma=0.1$).

p	$Q_p[S_\infty^l]$	$Q_p[S_\infty]$	$Q_p[S_\infty^c]$
0.25	11.13	11.07	9.34
0.50	15.74	15.76	14.29
0.75	23.51	23.50	23.11
0.95	46.30	46.14	51.84
0.99	79.64	80.71	100.45
0.995	98.35	101.09	130.77

Table 2: The table compares some selected exact quantiles of the constant perpetuity with the ‘maximal variance’ lower bound and upper bound approximations ($\delta=0.07$, $\sigma=0.2$).

these authors computed the quantiles of S_∞^c and S_∞^l by numerical evaluation of the expressions (24) and (26) with $t \rightarrow \infty$ and $\alpha(\tau) = 1$. Secondly, they effectively used a slightly different conditioning variable, i.e. $\Lambda = \int_0^\infty e^{-\delta\tau} Z(\tau) d\tau$, whereas our explicit results rely on $\Lambda = \int_0^\infty e^{-\delta^*\tau} Z(\tau) d\tau$ as conditioning random variable.

In Table 2, we show quantiles of S_∞ , S_∞^c and S_∞^l but now for $\delta = 0.07$ and $\sigma = 0.2$. This example is interesting because it proves that for suitable choices of Λ , the cdf’s of S_∞^l and S_∞ do not necessarily cross only once. In this respect it is worthwhile to mention that Vanduffel *et al.* (2003) showed in a discrete setting that the cdf’s of S_t^l and S_t^c can only cross in the region where their distribution functions take a value that is contained in the interval $[p^-, p^+]$ for some p^- and $p^+ > 0$ leaving it as an open question whether this crossing point is unique or not. Exactly the same result can be drawn in the continuous setting, but despite the explicit expressions for the quantiles, we are still unable to answer satisfactorily the question concerning the uniqueness of the crossing point.

Vanduffel *et al.* (2005b) mentioned that in the discrete setting and using finite horizons, the ‘maximal variance’ comonotonic lower bound approximation provides an excellent fit for several risk measures associated with sums of lognormals. As visually seen from these tables, the same holds true in the continuous setting even when the time horizon goes to infinity. Hence our numerical comparisons provide some more support regarding the conclusions made in Vanduffel *et al.* (2005b).

5 Asian options

Until now we have considered closed-form comonotonic approximations for risk measures of continuous annuities. In this section we will show how these results can be

used to derive closed-form expressions for upper and lower bounds for the price of a continuously sampled European-style arithmetic Asian option with fixed exercise prices.

We are not the first to investigate this problem. Vorst (1992) essentially used the fact that the geometric average is always smaller than the corresponding arithmetic average to derive closed-form expressions for upper and lower bounds. The upper bound has been considerably improved since then. We refer to Rogers & Shi (1995), Nielsen & Sandman (2003) and most recently to Lord (2006) where tight upper bounds have been provided. In this section we derive an upper bound in closed-form but unfortunately this expression is only valid for a restricted choice for the parameters.

As far as the lower bound prices are concerned, using the technique of conditioning, these have been sharpened in a series of papers by Curran (1994), Rogers & Shi (1995) and Thompson (1999). However all these lower bound approximations still involve the evaluation of a two-dimensional or one-dimensional integral. In this section we will derive another lower bound, which is also based on the technique of conditioning but does not require the evaluation of an integral anymore.

The risk neutral price of a European-style Asian option with fixed strike price K and maturity T with averaging starting at $t = 0$ will be denoted by $AC(K, T)$ and is given by

$$AC(K, T) = \frac{e^{-rT}}{T} E^Q \left[\left(\int_0^T A(\tau) d\tau - T \cdot K \right)_+ \right], \quad (62)$$

where Q denotes the equivalent martingale measure, $r > 0$ is the risk-free continuous rate and $A(\tau)$ is the price of the underlying risky asset at time τ for $\tau \in (0, T)$. We will also assume that $A(\tau)$ follows a geometric Brownian motion with drift $\mu - \frac{\sigma^2}{2}$ and volatility σ . It is well known that under the equivalent martingale measure Q we find that $A(\tau)$ is given by

$$A(\tau) = A(0)e^{Z(\tau)}, \quad (63)$$

where $Z(\tau) = (r - \frac{\sigma^2}{2})\tau + \sigma B(\tau)$ is a Brownian motion with drift $r - \frac{\sigma^2}{2}$ and volatility σ .

We can also express $AC(K, T)$ in terms of a stop-loss premium of a continuous annuity as follows. From definition (7) for the continuous temporary annuity S_t we find that

$$\int_0^T A(\tau) d\tau = A(0) \int_0^T e^{Z(\tau)} d\tau = A(0)S_T, \quad (64)$$

but with δ given by

$$\delta = -\left(r - \frac{\sigma^2}{2}\right), \quad (65)$$

so that we can write

$$AC(K, T) = \frac{e^{-rT}}{T} A(0) E^Q \left[\left(S_T - \frac{T \cdot K}{A(0)} \right)_+ \right], \quad (66)$$

Whilst an analytical expression for the call price (66) appears to be out of reach we can apply the bounds that were developed in Section 3.2 and Section 3.3.

The upper bound price will be denoted by $AC^c(K, T)$ and can be simply found by applying (6) and (31) to (66) and using that in our risk neutral valuation context $\delta^* = -r$. Here, the probability level p is determined by solving $Q_p[S_T^c] = \frac{T \cdot K}{A(0)}$ which can be done invoking equation (29). Unfortunately, it holds that $AC^c(K, T)$ only exists when $r < \frac{\sigma^2}{2}$ and this makes this upper bound of relatively little use.

We can also find a lower bound for $AC(K, T)$, obtained by conditioning on $\Lambda = \sum_{i=1}^n \gamma(t_i)Z(t_i)$ as defined in Section 3.3.3. This lower bound price will be denoted by $AC^{l,s}(K, T)$, and a closed-form expression for it can be readily found by applying expression (6) and (48) to (66) where $\delta^* = -r$. In this case p will follow from solving $Q_p[S_T^{l,s}] = \frac{T \cdot K}{A(0)}$ which can be done using (44). Note that $AC^{l,s}(K, T)$ always exists and unlike the existing lower bound approaches that are based on conditioning does not require the evaluation of an integral.

In Table 3 we compare for an arithmetic Asian option our lower bound $AC^{l,s}(K, T)$ with the lower bound price from Rogers and Shi (1995) using Monte Carlo simulations as the benchmark. While Rogers & Shi's analysis was valid for any choice for the conditioning rv Λ , in their numerical investigations they proposed $\Lambda = \int_0^T Z(\tau)d\tau$ as suitable choice to derive the lower bound prices. In contrast, we start by taking $\Lambda_1 = \sum_{i=1}^n e^{rt_i} Z(t_i)$ with $t_i = \frac{t}{n}$ ($i = 1, 2, \dots, n$) as choice for our conditioning rv. This choice is strongly inspired by the maximal variance approximation that we discussed in Section 2.2. The results for $AC^{l,s}(K, T)$ that are obtained using this choice for Λ will for convenience be labelled as 'Maximal variance'. We also rely on the discussion in Section 2.2 to test the locally optimal $\Lambda_2 = \sum_{i=1}^n e^{rt_i} \cdot \Phi'(r_i^1 \sigma \sqrt{t_i} - \Phi^{-1}(p))Z_{t_i}$ as conditioning random variable where r_i^1 is the correlation between Λ_1 and Z_{t_i} . In this case we label the corresponding results for $AC^{l,s}(K, T)$ as 'Max CTE'.

Furthermore, we assume $\sigma = 0.1$, an initial stock price of $A(0) = 100$ and as expiry time we take $T = 1$. We will also take $n = 36$ which essentially means that we use the information of the underlying stock on a 10-daily basis when deriving $AC^{l,s}(K, T)$. The results show that the 'Maximal variance' and 'Maximal CTE' analytic bounds provide equally accurate results as in Rogers & Shi (1995) and also that these are very close to the results obtained by Monte Carlo simulations. However the tied bound of Rogers & Shiu (1995) or, equivalently, the one that was proposed in Thompson (1999) require the evaluation of a double or single integral, respectively. In contrast, our lower bound formula involves a sum with a fairly limited amount of terms only.

In Table 4 we repeat the exercise but now assuming $\sigma = 0.3$. Also in this case we find a close correspondence between the results of Rogers & Shi (1995) and our results. Our numerical experiments revealed that increasing n or making other choices for the evaluation times t_i did not improve the obtained bounds further.

In contrast, Table 5 where we take $\sigma = 0.5$ provides some support for the observations made in Vanduffel *et al.* (2007) that for out-of-the money options and in case the volatility is large, the 'Maximal CTE' lower bound approximations outperform the other approximations.

6 Concluding remarks

In Dhaene *et al.* (2002) it is shown that using comonotonic approximations for risk measures for constant continuous annuities in a Black&Scholes market gives rise

$\sigma = 0.1$					
r	Strike	Maximal variance	Maximal CTE	Rogers and Shi	Monte Carlo
0.05	90	11.951	11.951	11.951	11.953 [0.0018]
	100	3.641	3.641	3.641	3.639 [0.0013]
	110	0.331	0.331	0.331	0.331 [0.0004]
0.09	90	13.385	13.385	13.385	13.388 [0.0018]
	100	4.915	4.915	4.915	4.916 [0.0015]
	110	0.630	0.630	0.630	0.629 [0.0006]
0.15	90	15.399	15.399	15.399	15.402 [0.0017]
	100	7.027	7.027	7.028	7.027 [0.0016]
	110	1.413	1.413	1.413	1.412 [0.0009]

Table 3: The table compares the prices of Continuous Asian Call Options using the ‘maximal variance’ and ‘maximal CTE’ lower bounds, results in Roger and Shi (1995) and the Monte Carlo simulations. The figures between brackets represent the standard error on the Monte Carlo result (T=1, averaging starts immediately).

$\sigma = 0.3$					
r	Strike	Maximal variance	Maximal CTE	Rogers and Shi	Monte Carlo
0.05	90	13.951	13.951	13.952	13.949 [0.0047]
	100	7.943	7.943	7.944	7.937 [0.0038]
	110	4.070	4.070	4.070	4.071 [0.0028]
0.09	90	14.981	14.981	14.983	14.984 [0.0047]
	100	8.826	8.826	8.827	8.823 [0.0039]
	110	4.695	4.695	4.695	4.690 [0.0030]
0.15	90	16.510	16.510	16.512	16.511 [0.0048]
	100	10.208	10.208	10.208	10.207 [0.0041]
	110	5.728	5.728	5.728	5.730 [0.0032]

Table 4: The table compares the prices of Continuous Asian Call Options using the ‘maximal variance’ and ‘maximal CTE’ lower bounds, results in Roger and Shi (1995) and the Monte Carlo simulations. The figures between brackets represent the standard error on the Monte Carlo result. (T=1, averaging starts immediately).

$\sigma = 0.5$					
r	Strike	Maximal variance	Maximal CTE	Rogers and Shi	Monte Carlo
0.09	90	18.178	18.181	18.183	18.187[0.0076]
	100	13.019	13.020	13.022	13.027[0.0068]
	110	9.117	9.117	9.118	9.115 [0.0059]
	150	1.927	1.930	1.924	1.931 [0.0029]
	200	0.2566	0.2596	0.2548	0.2589[0.0011]

Table 5: The table compares the prices of Continuous Asian Call Options using the ‘maximal variance’ and ‘maximal CTE’ lower bounds, results in Rogers & Shi (1995) and the Monte Carlo simulations. The figures between brackets represent the standard error on the Monte Carlo result (T=1, averaging starts immediately).

to expressions that only involve the evaluation of a single integral. In this paper we showed that these integrals can sometimes be evaluated explicitly, hence yielding analytic approximations and bounds in this case. We compared these approximations with available explicit results for perpetuities.

We use our results to derive analytical formulas for upper and lower bound prices for continuously sampled Asian Options. The lower bound prices are at least as sharp as those from Rogers & Shi (1995) but since they do not require any longer the evaluation of an integral they are extremely fast to obtain.

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Appendix A

In this appendix, we prove Equation (29). From equation (24) ,it follows that:

$$Q_p [S_t^c] = \int_0^t e^{-\delta\tau + \sigma\sqrt{\tau}\Phi^{-1}(p)} d\tau$$

By substituting $y = \sqrt{\tau}$ we find that:

$$\begin{aligned} Q_p [S_t^c] &= \int_0^{\sqrt{t}} 2ye^{-\delta y^2 + \sigma\Phi^{-1}(p)y} dy, \\ &= 2 \int_0^{\sqrt{t}} ye^{-\frac{\left(y - \frac{\sigma\Phi^{-1}(p)}{2\delta}\right)^2}{1/\delta} + \frac{\sigma^2(\Phi^{-1}(p))^2}{4\delta}} dy. \end{aligned}$$

Now let $z = \left(y - \frac{\sigma\Phi^{-1}(p)}{2\delta}\right) / \left(1/\sqrt{2\delta}\right)$ and $a = \frac{\sigma\Phi^{-1}(p)}{\sqrt{2\delta}}$. Then, we obtain that:

$$\begin{aligned}
Q_p[S_t^c] &= 2e^{a^2/2} \int_{-a}^{\sqrt{2\delta t}-a} \left(\frac{1}{\sqrt{2\delta}}z + \frac{\sigma\Phi^{-1}(p)}{2\delta}\right) e^{-z^2/2} \frac{1}{\sqrt{2\delta}} dz, \\
&= e^{a^2/2} \int_{-a}^{\sqrt{2\delta t}-a} \left(\frac{1}{\delta}z + \frac{1}{\delta}a\right) e^{-z^2/2} dz, \\
&= \frac{1}{\delta} e^{a^2/2} \left[\int_{-a}^{\sqrt{2\delta t}-a} -de^{-z^2/2} + \int_{-a}^{\sqrt{2\delta t}-a} ae^{-z^2/2} dz \right], \\
&= \frac{1}{\delta} e^{a^2/2} \left[\left(-e^{-z^2/2} \Big|_{-a}^{\sqrt{2\delta t}-a}\right) + a\sqrt{2\pi} \int_{-a}^{\sqrt{2\delta t}-a} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \right], \\
&= \frac{1}{\delta} - \frac{1}{\delta} e^{-\delta t + \sigma\sqrt{t}\Phi^{-1}(p)} + \frac{1}{\delta} \sqrt{2\pi} a e^{a^2/2} \left(\Phi \left[\sqrt{2\delta t} - a \right] - \Phi[-a] \right)
\end{aligned}$$

with Φ the cdf of a standard normal random variable and this completes the proof.