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COMONOTONICITY AND MAXIMAL STOP-LOSS PREMIUMS

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Abstract

In this paper, we investigate the relationship between comonotonicity and stop-loss order. We prove our main results by using a characterization of stop-loss order within the framework of Yaari's (1987) dual theory of choice under risk. Wang and Dhaene (1997) explore related problems in the case of bivariate random variables. We extend their work to an arbitrary sum of random variables and present several examples illustrating our results.

1 Introduction

The stop-loss transform is an important tool for studying the riskiness of an insurance portfolio. In this paper, we consider the individual risk theory model, where the aggregate claims of the portfolio are modeled as the sum of the claims of the individual risks. We investigate the aggregate stop-loss transform of such a portfolio without making the usual assumption of mutual independence of the individual risks. Wang and Dhaene (1997) explore related problems in the case of bivariate random variables. We extend their work to an arbitrary sum of random variables.

To prove results concerning ordering of risks, one often uses characterizations of these orderings within the framework of expected utility theory, see e.g. Kaas et al. (1994). We, however, rely on the framework of Yaari's (1987) dual theory of choice under risk. Our results are easier to obtain in this dual setting.

In Section 2, we provide notation and a brief introduction to Yaari's dual theory of risk. We introduce a special type of dependency between the individual risks, the notion

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of "comonotonicity". Loosely speaking, risks are comonotonic if they "move in the same direction". In Section 3, we consider stop-loss order. It is well-known that stop-loss order is the order induced by all risk-averse decision makers whose preferences among risks obey the axioms of utility theory. We show that the class of decision makers, whose preferences obey the axioms of Yaari's dual theory of risk and who have increasing concave distortion functions, also induces stop-loss order. From this characterization of stop-loss order, we find the following result: If risk X_i is smaller in stop-loss order than risk Y_i, for $i = 1, \ldots, n$, and if the risks Y_i are mutually comonotonic, then the respective sums of risks are also stop-loss ordered. In Section 4, we characterize the stochastic dominance order within Yaari's theory. In Section 5, we consider the case that the marginal distributions of the individual risks are given. We derive an expression for the maximal aggregate stop-loss premium in terms of the stop-loss premiums of the individual risks. Finally, in Section 6, we present several examples to illustrate our results.

We remark that Wang and Young (1997) further consider ordering of risks under Yaari's theory. They extend first and second stochastic dominance orderings to higher orderings in this dual theory of choice under risk.

2 Distortion Functions and Comonotonicity

For a risk X (i.e. a non-negative real valued random variable with a finite mean), we denote its cumulative distribution function (cdf) and its decumulative distribution function (ddf) by F_X and S_X respectively:

$$F_X(x) = \Pr\{X \leq x\}, \quad 0 \leq x < \infty,$$

$$S_X(x) = \Pr\{X > x\}, \quad 0 \leq x < \infty.$$

In general, both F_X and S_X are not one-to-one so that we have to be cautious in defining their inverses. We define F_X^{-1} and S_X^{-1} as follows:

$$\begin{aligned} F_X^{-1}(p) &= \inf\{x : F_X(x) \geq p\}, \quad 0 < p \leq 1, \quad F_X^{-1}(0) = 0, \\ S_X^{-1}(p) &= \inf\{x : S_X(x) \leq p\}, \quad 0 \leq p < 1, \quad S_X^{-1}(1) = 0. \end{aligned}$$

where we adopt the convention that $\inf \phi = \infty$. We remark that F_X^{-1} is non-decreasing, S_X^{-1} is non-increasing and $S_X^{-1}(p) = F_X^{-1}(1-p)$.

Starting from axioms for preferences between risks, Von Neumann and Morgenstern (1947) developed utility theory. They showed that, within this axiomatic framework, a decisionmaker has a utility function u such that he or she prefers risk X to risk Y (or is indifferent between them) if and only if $E\left(u(-X)\right) \geq E\left(u(-Y)\right)$.

Yaari (1987) presents a dual theory of choice under risk. In this dual theory, the concept of "distortion function" emerges. It can be considered as the parallel to the concept of "utility function" in utility theory.

Definition 1 *A distortion function g is a non-decreasing function $g : [0,1] \rightarrow [0,1]$ with $g(0) = 0$ and $g(1) = 1$.*

Starting from an axiomatic setting parallel to the one in utility theory, Yaari shows that there exists a distortion function g such that the decision maker prefers risk X to risk Y (or is indifferent between them) if and only if $H_g(X) \leq H_g(Y)$, where for any risk X, the "certainty equivalent" $H_g(X)$ is defined as

$$H_g(X) = \int_0^\infty g[S_X(x)]dx = \int_0^1 S_X^{-1}(q)dg(q).$$

We remark that $H_g(X) = E(X)$ if g is the identity. For a general distortion function g, the certainty equivalent $H_g(X)$ can be interpreted as a "distorted" expectation of X. See Wang and Young (1997) for a discussion of Yaari's axioms in an insurance context.

In the following sections, we use two special families of distortion functions for proving some of our results. In the following lemma, we derive expressions for the certainty equivalents $H_g(X)$ of these families of distortion functions. For a subset A of the real numbers, we use the notation I_A for the indicator function which equals 1 if $x \in A$ and 0 otherwise.

Lemma 1 *(a) Let the distortion function g be defined by $g(x) = I(x > p)$, $0 \leq x \leq 1$, for an arbitrary, but fixed, $p \in [0,1]$. Then for any risk X the certainty equivalent $H_g(X)$ is given by*

$$H_g(X) = S_X^{-1}(p).$$

(b) Let the distortion function g be defined by $g(x) = min(x/p, 1)$, $0 \leq x \leq 1$, for an arbitrary, but fixed, $p \in (0,1]$. Then for any risk X, the certainty equivalent $H_g(X)$ is given by

$$H_g(X) = S_X^{-1}(p) + \frac{1}{p}\int_{S_X^{-1}(p)}^\infty S_X(x)dx.$$

Proof. (a) First let g be defined by $g(x) = I(x > p)$. As we have for any $x \geq 0$ that $S_X(x) \leq p \Leftrightarrow S_X^{-1}(p) \leq x$, we find

$$g(S_X(x)) = \begin{cases} 1, & x < S_X^{-1}(p), \\ 0, & x \geq S_X^{-1}(p), \end{cases}$$

from which we immediately obtain the expression for the certainty equivalent.

(b) Now let g be defined by $g(x) = min\,(x/p, 1)$. In this case we find

$$g(S_X(x)) = \begin{cases} 1, & x < S_X^{-1}(p), \\ S_X(x)/p, & x \geq S_X^{-1}(p), \end{cases}$$

from which we immediately obtain the desired result. ■

Yaari's axiomatic setting only differs from the axiomatic setting of expected utility theory by modifying the independence axiom. This modified axiom can be expressed in terms of "comonotonic" risks.

Definition 2 *The risks $X_1, X_2, ..., X_n$ are said to be mutually comonotonic if any of the following equivalent conditions hold:*
(1) The cdf $F_{X_1,X_2,...,X_n}$ of $(X_1, X_2, ..., X_n)$ satisfies

$$F_{X_1,X_2,...,X_n}(x_1, ..., x_n) = \min\left[F_{X_1}(x_1), ..., F_{X_n}(x_n)\right] \qquad \textit{for all } x_1, ..., x_n \geq 0.$$

(2) There exists a random variable Z and non-decreasing functions $u_1, ..., u_n$ on R such that $(X_1, ..., X_n) \stackrel{d}{=} (u_1(Z), ..., u_n(Z))$.
(3) For any uniformly distributed random variable U on $[0, 1]$, we have that

$$(X_1, ..., X_n) \stackrel{d}{=} \left(F_{X_1}^{-1}(U), ..., F_{X_n}^{-1}(U)\right).$$

In the definition above, the notation "$\stackrel{d}{=}$" is used to indicate that the two multivariate random variables are equal in distribution. The proof for the equivalence of the three conditions is a straightforward generalization of the proof for the bivariate case considered in Wang and Dhaene (1997).

The following theorem states that the certainty equivalent of the sum of mutually comonotonic risks is equal to the sum of the certainty equivalents of the different risks.

Theorem 2 *If the risks $X_1, X_2, ..., X_n$ are mutually comonotonic, then*

$$H_g(X_1 + X_2 + ... + X_n) = \sum_{i=1}^{n} H_g(X_i).$$

Proof. A proof for the bivariate case can be found in Wang (1996). A generalisation to the multivariate case follows immediately by considering the fact if $X_1, X_2, ..., X_n$ are mutually comonotonic, then also $X_1 + X_2 + ... + X_{n-1}$ and X_n are mutually comonotonic. ■

If we restrict to the class of concave distortion functions, then the certainty equivalent is subadditive, which means that the certainty equivalent of a sum of risks is smaller than or equal to the sum of the certainty equivalents. This property is stated in the following theorem.

Theorem 3 *If the distortion function g is concave, then for any risks $X_1, X_2, ..., X_n$ we have that*

$$H_g(X_1 + X_2 + ... + X_n) \leq \sum_{i=1}^{n} H_g(X_i).$$

The theorem above is a straightforward generalization of the bivariate case considered in Wang and Dhaene (1997).

3 Stop-Loss Order and Comonotonicity

For any risk X and any $d \geq 0$, we define $(X-d)_+ = \max(0, X-d)$. The stop-loss premium with retention d is then given by $\mathrm{E}(X-d)_+$.

Definition 3 *A risk X is said to precede a risk Y in stop-loss order, written $X \leq_{sl} Y$, if for all retentions $d \geq 0$, the stop-loss premium for risk X is smaller than that for risk Y:*

$$\mathsf{E}(X-d)_+ \leq \mathsf{E}(Y-d)_+.$$

In the following theorem, we derive characterizations of stop-loss order, within the framework of Yaari's dual theory of choice under risk.

Theorem 4 *For any risks X and Y, the following conditions are equivalent:*
(1) $X \leq_{sl} Y$.
(2) For all concave distortions functions, we have that $H_g(X) \leq H_g(Y)$.
(3) For all distortion functions g defined by $g(x) = min\,(x/p, 1)$, $p \in (0,1]$, we have that $H_g(X) \leq H_g(Y)$.

Proof.

(1) $\Rightarrow$ (2) : Relying on the fact that stop-loss order is the transitive closure of the order in dangerousness (Müller, 1996), and on the dominated convergence theorem, we only have to prove that if X and Y are ordered in dangerousness, written $X \leq_D Y$, then $H_g(X) \leq H_g(Y)$ for all concave distortion functions. Hence, let us assume that $X \leq_D Y$; that is, $E(X) \leq E(Y)$ and there exists a real number $c \geq 0$ such that

$$\begin{aligned} S_X(x) &\geq S_Y(x) \text{ for all } x < c, \\ S_X(x) &\leq S_Y(x) \text{ for all } x \geq c. \end{aligned}$$

Now let g be a distortion function. As g is non-decreasing, we immediately find

$$\begin{aligned} g\left(S_X(x)\right) &\geq g\left(S_Y(x)\right) \text{ for all } x < c, \\ g\left(S_X(x)\right) &\leq g\left(S_Y(x)\right) \text{ for all } x \geq c. \end{aligned}$$

Also assume that g is concave in $[0,1]$. Thus, for each y in $[0,1]$, there exists a line $l(x) = a_y x + b$, with $l(y) = g(y)$ and $l(x) \geq g(x)$ for all x in $[0,1]$. As $l(y) = g(y)$, we find that $l(x) = a_y(x-y) + g(y)$. Hence, $l(x) \geq g(x)$ can be written as

$$g(x) - g(y) \leq a_y(x-y) \text{ for all } x \text{ in } [0,1].$$

As g is non-decreasing, we find that $a_y \geq 0$. Further, a_y is a non-increasing function of y.

By substituting $S_X(x)$ and $S_Y(x)$ for x and y in the above inequality, we obtain

$$g\left(S_X(x)\right) - g\left(S_Y(x)\right) \leq a_{S_Y(x)}\left(S_X(x) - S_Y(x)\right) \text{ for all } x \geq 0.$$

From the crossing condition for $g\left(S_X(x)\right) - g\left(S_Y(x)\right)$ and the fact that $a_{S_Y(x)}$ is a non-decreasing function of x, we find

$$g\left(S_X(x)\right) - g\left(S_Y(x)\right) \leq a_{S_Y(c)}\left(S_X(x) - S_Y(x)\right) \text{ for all } x \geq 0.$$

Taking the integral over both members of the inequality above leads to

$$\int_0^\infty \left[g\left(S_X(x)\right) - g\left(S_Y(x)\right)\right] dx \leq a_{S_Y(c)} \int_0^\infty \left(S_X(x) - S_Y(x)\right) dx \leq 0$$

where the last inequality holds because $E(X) \leq E(Y)$. Hence, we have proven that condition (1) implies condition (2).

(2) $\Rightarrow$ (3) : This follows immediately, because $g(x) = min\,(x/p, 1)$ defines a concave distortion function.

(3) $\Rightarrow$ (1) : For any distortion function g defined by $g(x) = min\,(x/p, 1)$, $p \in (0,1]$, we find from Lemma 1 that $H_g(X) \leq H_g(Y)$ is equivalent to

$$p\ S_X^{-1}(p) + \int_{S_X^{-1}(p)}^d S_X(x)dx + \mathsf{E}(X-d)_+ \leq p\ S_Y^{-1}(p) + \int_{S_Y^{-1}(p)}^d S_Y(x)dx + \mathsf{E}(Y-d)_+.$$

for all $d \geq 0$. We have to prove that $\mathsf{E}(X-d)_+ \leq \mathsf{E}(Y-d)_+$ for any $d \geq 0$.

If $S_X(d) = 0$, then $\mathsf{E}(X-d)_+ = 0$ so that $\mathsf{E}(X-d)_+ \leq \mathsf{E}(Y-d)_+$.

Now assume that $S_X(d) > 0$, and let $p = S_X(d)$. Note that in general $S_X^{-1}(p) \leq d$ and that for $S_X^{-1}(p) \leq x \leq d$ we have that $S_X(x) = p$. Hence, $H_g(X) \leq H_g(Y)$ can be rewritten as

$$\mathsf{E}(X-d)_+ \leq \int_{S_Y^{-1}(p)}^d \left(S_Y(x) - p\right) dx + \mathsf{E}(Y-d)_+.$$

As $S_Y^{-1}(p) \leq x \Leftrightarrow S_Y(x) \leq p$, we find that the integral in the inequality above is always negative, from which it follows that $\mathsf{E}(X-d)_+ \leq \mathsf{E}(Y-d)_+$. As the proof holds for any $d \geq 0$, we find that condition (1) follows from condition (3). ■

Within the framework of expected utility theory, stop-loss order of two risks is equivalent to saying that one risk is preferred over the other by all risk-averse decision makers. From the theorem above, we see that we have a similar interpretation for stop-loss order within the framework of Yaari's theory of choice under risk: Stop-loss order of two risks is equivalent to saying that one risk is preferred over the other by all decision makers who have non-decreasing concave distortion functions. See Wang and Young (1997) for related results. Note that our Theorem 4 is more general than the corresponding result of Wang and Young (1997) because we do not assume that the distortions are differentiable.

It is well-known that stop-loss order is preserved under convolution of mutually independent risks, see e.g. Goovaerts et al. (1990). In the following theorem we consider the case of mutually comonotonic risks.

Theorem 5 *If $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ are sequences of risks with $X_i \leq_{sl} Y_i$ ($i = 1, ..., n$) and with $Y_1, Y_2, ..., Y_n$ mutually comonotonic, then*

$$\sum_{i=1}^{n} X_i \leq_{sl} \sum_{i=1}^{n} Y_i.$$

Proof. Using Theorems 2, 3 and 4 we find that for any concave distortion function g,

$$H_g(X_1 + X_2 + ... + X_n) \leq \sum_{i=1}^{n} H_g(X_i) \leq \sum_{i=1}^{n} H_g(Y_i) = H_g(Y_1 + Y_2 + ... + Y_n).$$

which proves the theorem. ■

Note that in the theorem above, we make no assumption concerning the dependency among the risks X_i. This means that the theorem is valid for any dependency among these risks.

For any risk X and any uniformly distributed random variable U on $[0, 1]$, we have that $X \stackrel{d}{=} F_X^{-1}(U)$. From this fact, we obtain the following corollary to Theorem 5.

Corollary 6 *For any random variable U, uniformly distributed on $[0, 1]$, and any risks $X_1, X_2, ..., X_n$, we have*

$$\sum_{i=1}^{n} X_i \leq_{sl} \sum_{i=1}^{n} F_{X_i}^{-1}(U).$$

Another proof for this corollary, in terms of "supermodular order", can be found in Müller (1997).

Note that $(X_1, X_2, ..., X_n)$ and $\left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), ..., F_{X_n}^{-1}(U)\right)$ have the same marginal distributions, while the risks $F_{X_i}^{-1}(U)$, $i = 1, ..., n$, are mutually comonotonic. Hence, Corollary 1 states that, within the class of all multivariate risk with given marginals $X_1, X_2, ..., X_n$, the stop-loss premiums of $X_1 + X_2 + ... + X_n$ are maximal if the risks X_i are mutually comonotonic.

4 Stochastic Dominance and Comonotonicity

In this section, we first examine whether Theorem 5, which holds for stop-loss order, also holds in the case of stochastic dominance, i.e. if "$\leq_{sl}$"is replaced by "$\leq_{st}$".

Definition 4 *A risk Y is said to stochastically dominate a risk X, written $X \leq_{st} Y$, if the following condition holds:*

$$S_X(x) \leq S_Y(x) \text{ for all } x \geq 0.$$

Let X_1, X_2, Y_1 and Y_2 be uniformly distributed random variables defined on $[0,1]$, with $X_2 \equiv 1 - X_1$ and $Y_1 \equiv Y_2$. Then we have that Y_1 and Y_2 are comonotonic. Further, $X_i \leq_{st} Y_i$ $(i = 1, 2)$. After some straightforward calculations, we find that

$$\begin{array}{rcll} F_{X_1+X_2}(x) & \leq & F_{Y_1+Y_2}(x) & \text{if } 0 \leq x < 1, \\ F_{X_1+X_2}(x) & \geq & F_{Y_1+Y_2}(x) & \text{if } x \geq 1. \end{array}$$

Hence, $X_1 + X_2$ is not stochastically dominated by $Y_1 + Y_2$ so that Theorem 5 does not hold in the case of stochastic dominance. However, stochastic dominance implies stop-loss order, so we should have that $X_1 + X_2 \leq_{sl} Y_1 + Y_2$. This follows indeed from the crossing condition.

Theorem 7 *For any risks X and Y, the following conditions are equivalent:*
(1) $X \leq_{st} Y$.
(2) For all distortion functions g we have that $H_g(X) \leq H_g(Y)$.
(3) $S_X^{-1}(p) \leq S_Y^{-1}(p)$ for all $p \in [0,1]$.

Proof.

(1) $\Rightarrow$ (2) : Straightforward.

(2) $\Rightarrow$ (3) : Let $p \in [0,1]$ and consider the distortion function g defined by $g(x) = I(x > p)$, $0 \leq x \leq 1$. The proof then follows from Lemma 1.

(3) $\Rightarrow$ (1) : For a fixed $x \geq 0$, let $p = S_Y(x)$. From $S_X^{-1}(p) \leq S_Y^{-1}(p)$, we have that $S_X\left(S_Y^{-1}(p)\right) \leq p = S_Y(x)$. Note that in general, $S_Y^{-1}(p) \leq x$. As S_X is non-increasing, we find

$$S_X(x) \leq S_X\left(S_Y^{-1}(p)\right) \leq S_Y(x).$$

As the proof holds for any $x \geq 0$, we have proven that condition (3) implies condition (1). ■

Within the framework of utility theory, it is well-known that stochastic dominance of two risks is equivalent to saying that one risk is preferred over the the other by all decision makers who prefer more to less. From the theorem above, we see that, within the framework of Yaari's theory of choice under risk, stochastic dominance of risk Y over risk X holds if and only if all decision makers with (non-decreasing) distortion function prefer risk X.

Actually, preservation of stochastic dominance is an axiom in both utility theory and Yaari's dual theory. Hence, the fact that condition (1) implies condition (2) is a direct result of this axiom.

5 Maximal Stop-Loss Premiums in the Multivariate Case

From Corollary 6, we concluded that in the class of all multivariate risks with given marginals $(X_1, X_2, ..., X_n)$, the stop-loss premiums are maximal if the risks $X_i, i = 1, \dots, n$, are mutually comonotonic. For comonotonic risks X_i, the stop-loss premium with retention d is given by

$$\mathsf{E}(X_1 + \cdots X_n - d)_+ = \int_0^1 \left[F_{X_1}^{-1}(p) + \cdots + F_{X_n}^{-1}(p) - d\right]_+ dp$$

Now we will derive another expression for this upper bound.

Theorem 8 *Let $X_1, \cdots, X_n$ be mutually comonotonic risks. Then for any retention $d \geq 0$, we have*

$$E(X_1 + \cdots + X_n - d)_+ = \sum_{i=1}^{n} E(X_i - d_i)_+ - \left[d - S_X^{-1}\left(S_X(d)\right)\right] \; S_X(d)$$

where $X = X_1 + \cdots + X_n$ and the d_i are defined by $d_i = S_{X_i}^{-1}(S_X(d))$.

Proof. If $S_X(d) = 0$, then the inequality trivially holds.

Now assume that $S_X(d) > 0$. Let $p \equiv S_X(d)$ and define a distortion function g by $g(x) = min\,(x/p, 1)$ for $0 \leq x \leq 1$. As $X_1, \cdots, X_n$ are mutually comonotonic, we find from Theorem 2 that

$$H_g(X) \;= \textstyle\sum_{i=1}^n H_g(X_i).$$

Using Lemma 1 this equality can be written as

$$S_X^{-1}(p) + \frac{1}{p}E\left(X - S_X^{-1}(p)\right)_+ = \sum_{i=1}^{n} S_{X_i}^{-1}(p) + \frac{1}{p}\sum_{i=1}^{n} E\left(X_i - S_{X_i}^{-1}(p)\right),$$

from which we find

$$E\left(X - S_X^{-1}(p)\right)_+ = \sum_{i=1}^{n} E\left(X_i - d_i\right)_+,$$

because $S_X^{-1}(p) = \sum_{i=1}^n S_{X_i}^{-1}(p)$ for comonotonic risks, see Denneberg (1994) or Wang (1996).

On the other hand, we have that

$$E(X - d)_+ = E\left(X - S_X^{-1}(p)\right) - \left[d - S_X^{-1}\left(S_X(d)\right)\right] \; S_X(d).$$

Now combine these two equalities to obtain the desired result. ■

From Theorem 8 we see that, apart from a correction factor, any stop-loss premium for the sum of comonotonic risks can be written as a sum of stop-loss premiums for the individual risks involved.

Note that in general we have that $S_X^{-1}(S_X(d)) \leq d$. However, if $S_X(x) > S_X(d)$ for all $x < d$, then $S_X^{-1}(S_X(d)) = d$, so that in this case

$$E(X_1 + \cdots + X_n - d)_+ = \sum_{i=1}^{n} E(X_i - d_i)_+$$

with the d_i as defined in Theorem 8. In this case, we also have that $\sum_{i=1}^{n} d_i = d$.

6 Examples

In this final section, we show by example how to evaluate stop-loss premiums for the sum $X = X_1 + X_2 + ... + X_n$ of the mutually comonotonic risks $X_1, X_2, ..., X_n$. We first consider the case for which all risks have a two-point distribution and then three cases for which all risks have continuous distributions.

Example 1: The Individual Life Model

Assume that each risk $X_i, (i = 1, \cdots, n)$ has a two-point distribution in 0 and $a_i > 0$ with $\Pr(X_i = a_i) = q_i$. The ddf of X_i is then given by

$$S_{X_i}(x) = \begin{cases} q_i, & \text{if } 0 \leq x < a_i, \\ 0, & \text{if } x \geq a_i, \end{cases}$$

from which we find

$$S_{X_i}^{-1}(p) = \begin{cases} a_i, & \text{if } 0 \leq p < q_i \\ 0, & \text{if } q_i \leq p \leq 1. \end{cases}$$

Without loss of generality, we assume that the random variables X_i are ordered such that $q_1 \geq \cdots \geq q_n$. Now assume that the risks are comonotonic, then we have

$$S_X^{-1}(p) = \sum_{i=1}^{n} S_{X_i}^{-1}(p) = \begin{cases} a_1 + \cdots + a_n, & \text{if } 0 \leq p < q_n, \\ a_1 + \cdots + a_j, & \text{if } q_{j+1} \leq p < q_j, \\ 0. & \text{if } q_1 \leq p < 1. \end{cases}$$

Hence,

$$S_X(x) = \begin{cases} q_1, & \text{if } 0 \leq x < a_1, \\ q_{j+1}, & \text{if } a_1 + \cdots + a_j \leq x < a_1 + \cdots + a_{j+1}, \\ 0, & \text{if } x \geq a_1 + \cdots + a_n. \end{cases}$$

which means that X is a discrete random variable with point-masses in $0, a_1,\ a_1+a_2,\ a_1+a_2+a_3, \cdots,\ a_1+a_2+\cdots+a_n$.

For d such that $a_1+\cdots+a_j \leq d < a_1+\cdots+a_{j+1}$, we find

$$S_{X_i}^{-1}(S_X(d)) = S_{X_i}^{-1}(q_{j+1}) = \begin{cases} a_i, & \text{if } i < j+1, \\ 0, & \text{if } i \geq j+1, \end{cases}$$

so that

$$S_X^{-1}(S_X(d)) = S_X^{-1}(q_{j+1}) = \sum_{i=1}^{n} S_{X_i}^{-1}(q_{j+1}) = a_1+\cdots+a_j.$$

We finally find from Theorem 8 that

$$E(X-d)_+ = \begin{cases} \sum_{i=1}^n q_i\ a_i, & \text{if } 0 \leq d < a_1, \\ \sum_{i=j+1}^n q_i\ a_i - \left(d - \sum_{i=1}^j a_i\right) q_{j+1} & \text{if } \sum_{i=1}^j a_i \leq d < \sum_{i=1}^{j+1} a_i, \\ 0, & \text{if } d \geq \sum_{i=1}^n a_i. \end{cases}$$

This individual life model is more extensively considered in Dhaene and Goovaerts (1996).

Example 2: Exponential Marginals

Assume that each X_i, $(i = 1, \cdots, n)$ is distributed according to the Exponential (b_i) distribution $(b_i > 0)$ with ddf given by

$$S_{X_i}(x) = e^{-x/b_i},\ x > 0.$$

For comonotonic X_i, the inverse ddf of their sum X is

$$S_X^{-1}(p) = -b \ln p,$$

in which $b = \sum_{i=1}^n b_i$. Thus,

$$S_X(x) = e^{-x/b}, \qquad x > 0.$$

In other words, the comonotonic sum of exponential random variables is exponentially distributed. Heilmann (1986) considers the case of $n = 2$.

One can easily verify that the stop-loss premium with retention d is given by

$$E(X-d)_+ = be^{-d/b}.$$

Example 3: Pareto Marginals

Assume that each X_i $(i = 1, \cdots, n)$ is distributed according to the Pareto (a, b_i) distribution $(a, b_i > 0)$ with ddf given by

$$S_{X_i}(x) = \left(\frac{b_i}{b_i + x}\right)^a,\ x > 0.$$

For comonotonic X_i, the inverse ddf of their sum X is

$$S_X^{-1}(p) = b\left(p^{-1/a} - 1\right),$$

in which $b = \sum_{i=1}^{n} b_i$. Thus,

$$S_X(x) = \left(\frac{b}{b+x}\right)^a, \qquad x > 0.$$

In other words, the comonotonic sum of Pareto random variables (with identical first parameter) is a Pareto random variable.

One can easily verify that for any $d \geq 0$ we have that

$$E(X-d)_+ = \left(\frac{b}{b+d}\right)^{a-1} \frac{b}{a-1}, \qquad a > 1.$$

Example 4: Exponential-Inverse Gaussian Marginals

Assume that each X_i, $(i = 1, \cdots, n)$ is distributed according to the exponential-inverse Gaussian (b_i, c_i) distribution $(b_i, c_i > 0)$ with ddf given by

$$S_{X_i}(x) = exp\left[-2\sqrt{c_i}\left(\sqrt{x+b_i} - \sqrt{b_i}\right)\right] \qquad x > 0,$$

see Hesselager, Wang and Willmot (1997). In this case the inverse ddf of X_i is

$$S_{X_i}^{-1}(p) = \frac{1}{4c_i}(ln\ p)^2 - \sqrt{\frac{b_i}{c_i}} ln\ p.$$

Thus, for comonotonic X_i, the inverse ddf of their sum X is

$$S_X^{-1}(p) = \frac{1}{4c}(ln\ p)^2 - \sqrt{\frac{b}{c}} ln\ p.$$

in which $c = \left(\sum_{i=1}^{n} \frac{1}{c_i}\right)^{-1}$, and $b = c\left(\sum_{i=1}^{n} \sqrt{\frac{b_i}{c_i}}\right)^2$. Thus

$$S_X(x) = exp\left[-2\sqrt{c}\left(\sqrt{x+b} - \sqrt{b}\right)\right], \qquad x > 0.$$

In other words, the comonotonic sum of exponential-inverse Gaussian random variables is also an exponential-inverse Gaussian random variable.

One can easily verify that for any $d \geq 0$ we have that

$$E(X-d)_+ = exp\left[-2\sqrt{c}\left(\sqrt{d+b} - \sqrt{b}\right)\right]\left[\sqrt{\frac{d+b}{c}} + \frac{1}{2c}\right].$$

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