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Optimal Portfolio Selection for Cash-Flows with Bounded Capital at Risk

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ABSTRACT

We consider a continuous-time Markowitz type portfolio problem that consists of minimizing the discounted cost of a given cash-flow under the constraint of a restricted Capital at Risk. In a Black-Scholes setting, upper and lower bounds are obtained by means of simple analytical expressions that avoid the classical simulation approach for this type of problems. The problem is easily extended to cope with more general discount processes.

Keywords: Black-Scholes model, Capital at Risk, portfolio optimization, Value at Risk.

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I. INTRODUCTION

The portfolio selection algorithm as introduced in Markowitz (1959) uses a mean-variance analysis to find *optimal* portfolios. In this method, a portfolio is called optimal if it yields the largest return among all portfolios with the same variance or, vice versa, if it has the smallest variance among all portfolios with the same return. For long term investments, however, the use of the variance as a risk measure leads to a smaller proportion of risky assets in a portfolio than one would expect. Based on the empirical observation that stock indices are growing faster than riskless rates in the long run, the proportion of risky assets should increase with the duration of the investment period. But since the variance increases with time, the proportion of risky assets will decrease. Therefore, Emmer *et al* (2001) propose to use the *Capital at Risk* (CaR) as an alternative risk measure and derive a closed-form formula to calculate the optimal CaR-constrained portfolio.

The Capital at Risk of a portfolio is commonly defined as the difference between the mean of the profit-loss distribution and a small quantile of this distribution (the so-called Value at Risk), but Emmer *et al* (2001) use a different definition which limits the possibility of excess losses over the riskless investment. We will also follow this approach and show how optimal CaR-constrained portfolios can be obtained for cash-flows in a Black-Scholes setting.

In a Black-Scholes market the stock prices $\{S_i(t)\}_{t \geq 0}$ for $i = 1, \ldots, d$, evolve from the following equations:

$$dS_i(t) = S_i(t)\left(\mu_i dt + \sum_{j=1}^{d} \sigma_{ij} dW_j(t)\right), \quad S_\mathrm{i}(0) = s_i, \quad i = 1, \ldots, d$$

where $\boldsymbol{W}(t)$ is a standard d-dimensional Brownian motion, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d)'$ is the vector of stock-appreciation rates, and $\boldsymbol{\sigma} = (\sigma_{ij})_{1 \leq i, j \leq d}$ is the matrix of stock-volatilities.

Denoting the fraction of the wealth $X(\boldsymbol{\pi}, t)$ that is invested in asset i by $\pi_i(t)$, the wealth process of an initial unit amount follows the dynamic

$$dX(\boldsymbol{\pi}, t) = X(\boldsymbol{\pi}, t)\,(\boldsymbol{\pi}'\boldsymbol{\mu} dt + \boldsymbol{\pi}'\boldsymbol{\sigma} d\boldsymbol{W}(t)), \quad X(\boldsymbol{\pi}, 0) = 1. \tag{1}$$

As in Emmer *et al* (2001), we assume that the fractions in the different stocks remain constant on $[0, T]$, i.e. $\boldsymbol{\pi}(t) = \boldsymbol{\pi} = (\pi_1, \ldots, \pi_d)'$. So, since the stock prices evolve randomly, one has to follow a dynamic trading strategy to keep the fractions of wealth invested in the different stocks constant. Solving the SDE (1) yields

$$X(\boldsymbol{\pi}, t) = \exp\{Y(\boldsymbol{\pi}, t)\} \tag{2}$$

where $Y(\boldsymbol{\pi}, t)$ equals

$$Y(\boldsymbol{\pi}, t) = \boldsymbol{\pi}'\boldsymbol{\mu}t - \boldsymbol{\pi}'\boldsymbol{\sigma}^2\boldsymbol{\pi}\frac{t}{2} + \boldsymbol{\pi}'\boldsymbol{\sigma}\boldsymbol{W}(t). \tag{3}$$

In the following section we extend some results obtained by Emmer *et al* (2001) to an actuarial context where consecutive payments have to be made. But implementing the mean-CaR criterion in a multi-period model inevitably turns the portfolio selection problem into a very complex problem for which no analytical solutions exist. Consequently, this type of portfolio selection problem is generally tackled by means of simulation of the corresponding stochastic processes. In section III we construct close approximations to the model in order to avoid this excessive time-consuming approach. Section IV concludes with a numerical illustration of the approximations.

II. EXTENSION TO CASH-FLOWS

In an insurance setting, at certain points in time an amount is withdrawn from or added to the money invested. We consider a cash-flow c_t denoting the total payments for each year t (e.g. pensions in a pension fund). Throughout the paper we will assume that $c_t \geq 0$ ($t = 1, \ldots, T$).

The present value of the cash-flow equals

$$V = \sum_{t=1}^{T} c_t e^{-Y(\boldsymbol{\pi}, t)}. \tag{4}$$

with expected value given by

$$V_0 = \mathrm{E}[V] = \sum_{t=1}^{T} c_t e^{-m(\boldsymbol{\pi})t + s^2(\boldsymbol{\pi})t} \tag{5}$$

where $m(\boldsymbol{\pi}) = \boldsymbol{\pi}'\boldsymbol{\mu}$ and $s^2(\boldsymbol{\pi}) = \boldsymbol{\pi}'\boldsymbol{\sigma}^2\boldsymbol{\pi}$. Obviously, we want to select a portfolio that minimizes V_0. Note however that there are restrictions on the amounts of risky stocks that can be bought, due to the requirements of control authorities. In addition, also due to regulators, the probability of "ruin" has to be restricted. Let ε denote the maximum probability of "ruin allowed". Denoting the $1-\varepsilon$ quantile of the discounted cash-flow by $Q_\varepsilon(\boldsymbol{\pi})$, i.e.

$$\Pr\left[\sum_{t=1}^{T} c_t e^{-Y(\boldsymbol{\pi},\, t)} \leq Q_\varepsilon(\boldsymbol{\pi})\right] = 1-\varepsilon \tag{6}$$

we define the Capital at Risk as

$$\mathrm{CaR} = Q_\varepsilon(\boldsymbol{\pi}) - \sum_{t=1}^{T} c_t e^{-rt} \tag{7}$$

where r is a constant reference interest rate, e.g. the riskless interest rate.

So, if we assume that the provision for the future payment obligations is the $1-\varepsilon$ quantile, then the CAR is the required provision in excess of the required provision in case of riskless investments.
By taking into account the extra cost of the CaR, we come to the following optimization problem:

$$\boxed{\min_{\boldsymbol{\pi}} V_0 + u(\mathrm{CaR}), \quad \text{subject to } \pi_j \geq 0, \sum_{j=1}^{d} \pi_j = 1, \quad \mathrm{CaR} \leq C}$$

where the increasing function $u(\cdot)$ denotes the supplementary cost of the Capital at Risk and where C denotes the maximum CaR allowed.

Since the quantile $Q_\varepsilon(\boldsymbol{\pi})$ is very hard (or even impossible) to obtain due to the dependency structure between the random variables $Y(\boldsymbol{\pi}, t)$, $t = 1, \ldots, T$, in (4), it seems impossible to solve this optimization problem without using Monte Carlo simulation. In the next section, we will show how this excessive time-consuming approach can be avoided by using easily computable approximations to $Q_\varepsilon(\boldsymbol{\pi})$.

III. AVOIDING SIMULATION

Instead of calculating the exact quantile of the distribution, we will look for bounds, in the sense of "more favourable/less dangerous" and "less favourable/more dangerous", with a simpler structure. This technique is common practice in the actuarial literature. When lower and upper bounds are close to each other, together they can provide reliable information about the original and more complex variable. The notion "less favourable" or "more dangerous" will be defined by means of the convex order.

Definition 1 *A random variable V is smaller than a random variable W in convex order if*

$$E\,[u(V)] \leq E\,[u(W)], \tag{8}$$

for all convex functions $u: \mathbb{R} \to \mathbb{R} : x \mapsto u(x)$, provided the expectations exist. This is denoted as

$$V \leq_{cx} W. \tag{9}$$

Since convex functions are functions that take on their largest values in the tails, the variable W is more likely to take on extreme values than the variable V, and thus W is more dangerous.

In Vyncke *et al* (2001) and Kaas *et al* (2001) upper and lower bounds for present value functions are constructed. These bounds in convex order turn out to be rather close to the exact present value distribution.

Proposition 1 *Consider a sum of random variables*

$$V = X_1 + X_2 + \ldots + X_n,$$

and define the related stochastic quantities

$$V_u = F^{-1}_{X_1}(U) + F^{-1}_{X_2}(U) + \ldots + F^{-1}_{X_n}(U) \tag{10}$$

$$V_l = E[X_1|Z] + E[X_2|Z] + \ldots + E[X_n|Z], \tag{11}$$

where U is a random variable, uniformly distributed on [0,1], *and where Z can be any random variable for which the expectations exist. The following relations then hold*:

$$V_l \leq_{cx} V \leq_{cx} V_u.$$

Proof: see Vyncke *et al* (2001) and Dhaene *et al* (2002).

It is clear that the lower bound V_l will perform at best if Z and V are very similar, so we choose

$$Z = \sum_{t=1}^{T} c_t e^{-\boldsymbol{\pi}'\boldsymbol{\mu} t}\, Y(\boldsymbol{\pi}, t) \tag{12}$$

which can be seen to be a first order approximation of V.

For the $(1 - \varepsilon)$-quantiles of V_u and V_l we find

$$Q_\varepsilon^*(\boldsymbol{\pi}) = \sum_{t=1}^{T} c_t \exp\left\{-m(\boldsymbol{\pi})t + \rho(\boldsymbol{\pi}, t)\, s(\boldsymbol{\pi})\, \sqrt{t}\Phi^{-1}(1 - \varepsilon) + \left(1 - \frac{\rho^2(\boldsymbol{\pi}, t)}{2}\right) s^2(\boldsymbol{\pi})t\right\} \tag{13}$$

with $m(\boldsymbol{\pi}) = \boldsymbol{\pi}'\boldsymbol{\mu}$, $s^2(\boldsymbol{\pi}) = \boldsymbol{\pi}'\boldsymbol{\sigma}^2\boldsymbol{\pi}$ and where the parameters $\rho(\boldsymbol{\pi}, t)$ are given by

$$\rho(\boldsymbol{\pi}, t) = \mathrm{Corr}(Y(\boldsymbol{\pi}, t), Z) = \frac{\sum_{i=1}^{t}\beta_i}{\sqrt{t \sum_{i=1}^{T}\beta_i^2}}, \quad \text{with} \quad \beta_i = \sum_{t=i}^{T} c_t e^{-m(\boldsymbol{\pi})t}$$

in case of the lower bound V_l, and $\rho(\boldsymbol{\pi}, t) \equiv 1$ in case of the upper bound V_u. Note that $\rho(\boldsymbol{\pi}, t)$ depends on $\boldsymbol{\pi}$ only through $m(\boldsymbol{\pi})$. Since $0 \leq \rho(\boldsymbol{\pi}, t) \leq 1$ and $s(\boldsymbol{\pi}) \geq 0$, the quantile $Q_\varepsilon^*(\boldsymbol{\pi})$ is an increasing function of $s(\boldsymbol{\pi})$. This implies that the adjusted Capital at Risk

$$\mathrm{CaR}^* = Q_\varepsilon^*(\boldsymbol{\pi}) - \sum_{t=1}^{T} c_t e^{-rt} \tag{14}$$

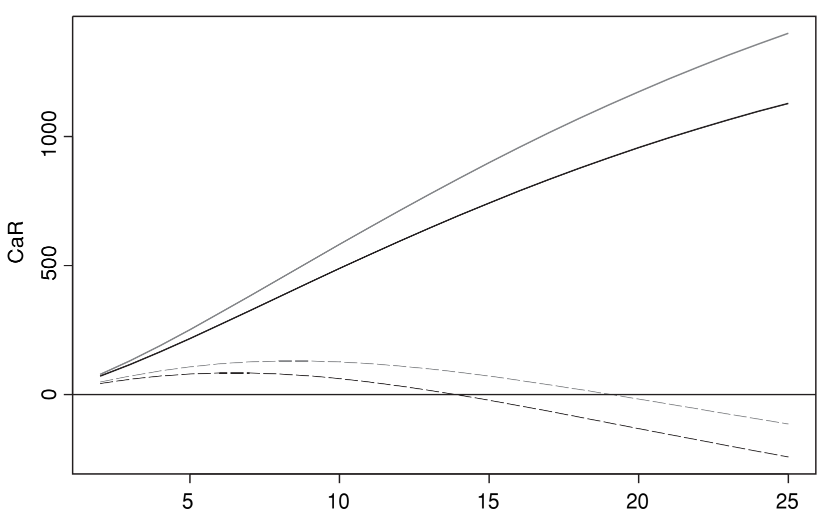
is also increasing in $s(\boldsymbol{\pi})$ for both approximations. Note, however, that the adjusted CaR isn't necessarily increasing with the planning horizon T. Figure 1 shows the adjusted CaR ($\varepsilon = 0.05$) for a pure stock portfolio, i.e. a portfolio consisting of one asset with a strictly positive volatility, for a cash-flow c_t=100 (t=1, …, T) for different values of T. In case the return μ equals 0.1, the adjusted CaR increases with T (for both upper and lower bound), but if $\mu = 0.18$ then the CaR* first increases and then decreases with time. As in Emmer *et al* (2001), at some point in time the CaR* even becomes negative which means that the pure stock strategy should be preferred over the risk-free strategy if the planning horizon is beyond that point in time.

From (5) it can be seen that also V_0 is increasing in $s(\boldsymbol{\pi})$. So logically assuming that the cost function $u(\cdot)$ is an increasing function, we see that the adjusted objective function $V_0 + u(\text{CaR}^*)$ is increasing in $s(\boldsymbol{\pi})$. This implies that the adjusted optimization problem

$$\boxed{\min_{\boldsymbol{\pi}} V_0 + u(\text{CaR}^*), \quad \text{subject to } \pi_j \geq 0, \sum_{j=1}^{d} \pi_j = 1, \text{Car}^* \leq C}$$

FIGURE 1

Capital at Risk of the pure stock portfolio as a funstion of the planning horizon. Both upper (grey) and lower bound (black) are depicted for μ = 0.10 (solid line) and μ = 0.18 (dashed line). The volatility equals 0.20 and the risk free interest rate equals 0.05.



can be solved by minimizing $s^2(\boldsymbol{\pi})$ for each $m(\boldsymbol{\pi}) = m$, and choosing the solution which minimizes the adjusted objective function. So, solving this optimization problem boils down to successively solving a quadratic program. Because of the specific nature of the optimization problem, the solution will also be part of the mean-variance efficient frontier.

IV. NUMERICAL ILLUSTRATION

In this section we illustrate the method by considering a portfolio consisting of 5 risky stocks and 1 riskless bond. The stock-appreciation rates μ and stock-volatilities σ are given by

stock	1	2	3	4	5
μ	0.1346	0.1659	0.1895	0.2014	0.095
σ	0.1585	0.2293	0.3368	0.4299	0.0686

and their correlation matrix equals

$$\begin{pmatrix} 1 & 0.7217 & 0.2571 & -0.0719 & 0.408 \\ 0.7217 & 1 & 0.1436 & -0.083 & 0.1419 \\ 0.2571 & 0.1436 & 1 & 0.0255 & -0.0829 \\ -0.0719 & -0.083 & 0.0255 & 1 & -0.1154 \\ 0.408 & 0.1419 & -0.0829 & -0.1154 & 1 \end{pmatrix}$$

The riskless bond yields a 0.05 return and we assume that the cost function is given by

$$u(x) = \begin{cases} 0.2x & x \geq 0 \\ 0.05x & x \leq 0 \end{cases}$$

This can be interpreted as follows: the insurance company has to pay a dividend of 20% to its shareholders if the CaR is positive and earns 5% (the risk-free interest rate) if the CaR is negative.

First, we consider a cash-flow $c_t = 100$ (t=1, …, 20). For $\varepsilon = 0.05$, the proportions (in %) based on the upper bound V_u are very close to those based on V_l, as can be seen from the following table. Note that π_0 indicates the proportion that is invested in the riskless bond.

appr. ε	V_u 0.05	V_l 0.05	V_u 0.01	V_l 0.01
π_0	0.00	0.00	0.00	0.00
π_1	0.00	0.00	0.00	0.00
π_2	48.28	48.01	37.28	45.59
π_3	27.16	27.30	20.44	24.35
π_4	24.57	24.70	18.39	21.97
π_5	0.00	0.00	23.90	8.10
m	18.10	18.11	16.03	17.37
s	18.23	18.25	13.90	16.67
V_0	595.13	595.10	621.03	602.11
CaR^*	-139.10	-235.56	52.46	-0.12
cost	588.17	583.33	631.52	602.10

In Figure 2 the optimal portfolios based on V_u and V_l are indicated by a circle and a rectangle respectively. For $\varepsilon = 0.01$, the proportions appear to be less efficient for this kind of cash-flow (see also Figure 3).

Next, we consider an increasing cash-flow $c_t = 5t$ (t=1, …, 20). Apart from rounding errors, we see that the proportions for the V_u approximation equal those of the V_l approximation in case of $\varepsilon = 0.05$. For $\varepsilon = 0.01$, the largest difference in proportion is approximately 8% (see also Figures 4 and 5).

appr. ε	V_u 0.05	V_l 0.05	V_u 0.01	V_l 0.01
π_0	0.00	0.00	0.00	0.00
π_1	0.00	0.00	0.00	0.00
π_2	48.01	48.01	45.73	48.28
π_3	27.30	27.30	24.42	27.16
π_4	24.70	24.70	22.04	24.57
π_5	0.00	0.00	7.82	0.00
m	18.11	18.11	17.39	18.10
s	18.25	18.25	16.72	18.23
V_0	183.89	183.89	187.21	183.91
CaR^*	-155.02	-184.89	-0.19	-32.75
cost	176.14	174.67	187.20	182.27

Finally, we consider a decreasing cash-flow $c_t = 105\text{-}5t$ (t=1, …, 20). For $\varepsilon = 0.05$ (see Figure 6) as well as for $\varepsilon = 0.01$ (see Figure 7), the method appears to perform quite well.

appr. ε	V_u 0.05	V_l 0.05	V_u 0.01	V_l 0.01
π_0	0.00	0.00	0.00	0.00
π_1	0.00	0.00	0.00	0.00
π_2	48.52	48.28	34.96	40.04
π_3	25.73	27.16	19.35	21.74
π_4	23.24	24.57	17.40	19.59
π_5	2.52	0.00	28.30	18.64
m	17.84	18.10	15.66	16.48
s	17.67	18.23	13.15	14.81
V_0	442.18	440.98	459.03	451.36
CaR^*	0.61	-51.86	91.55	43.59
cost	442.30	438.38	477.34	460.08

REFERENCES

Dhaene J., Denuit M., Goovaerts M.J., Kaas R. & Vyncke D., 2002, The Concept of Comonotonicity in Actuarial Science and Finance: Theory, *Insurance: Mathematics and Economics*, to appear.

Emmer S., Klüppelberg C., Korn R., 2001. Optimal Portfolios with Bounded Capital at Risk, *Mathematical Finance* 11, 4, 365-384.

Kaas R., Dhaene J., Goovaerts M.J., 2000. Upper and Lower Bounds for Sums of Random Variables, *Insurance: Mathematics and Economics* 27, 151-168.

Markowitz H., 1959. Portfolio Selection: Efficient Diversification of Investments (John Wiley & Sons, New York).

Vyncke D., Goovaerts M. & Dhaene J., 2001. Convex Upper and Lower Bounds for Present Value Functions, *Applied Stochastic Models for Business and Industry*, 17, 149-164.

FIGURE 2

Optimal portfolios for c_t =100 (t=1, ..., 20) with ε = 0.05.

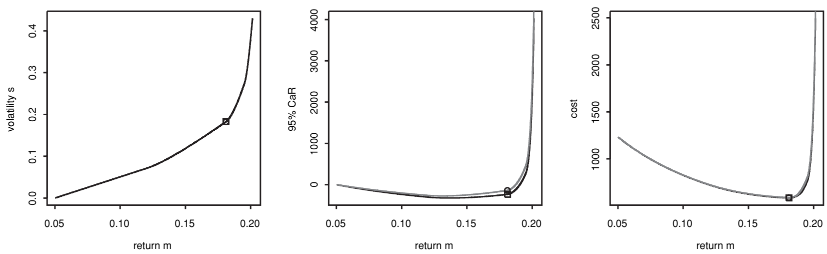


FIGURE 3

Optimal portfolios for c_t =100 (t=1, ..., 20) with ε = 0.01.

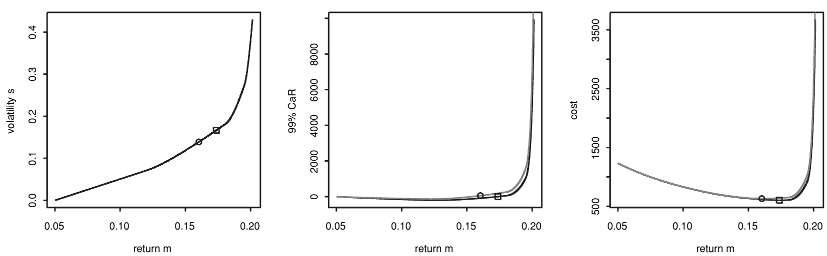


FIGURE 4

Optimal portfolios for c_t = 5t (t=1, ..., 20) with ε = 0.05.

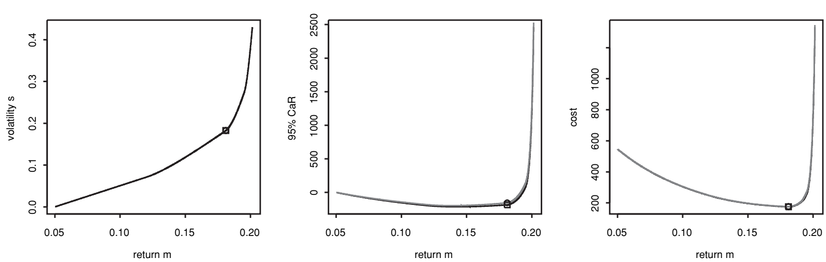


FIGURE 5

Optimal portfolios for $c_t = 5t$ (t=1, …, 20) with $\varepsilon = 0.01$.

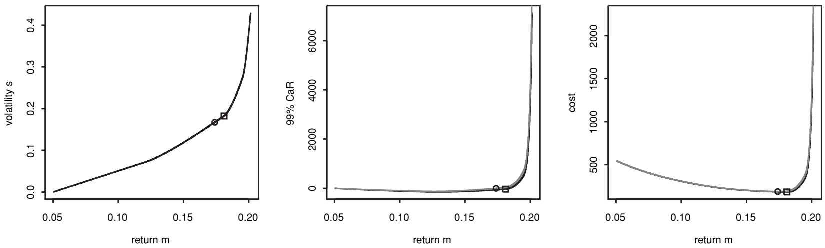


FIGURE 6

Optimal portfolios for $c_t = 105 - 5t$ (t=1, …, 20) with $\varepsilon = 0.05$.

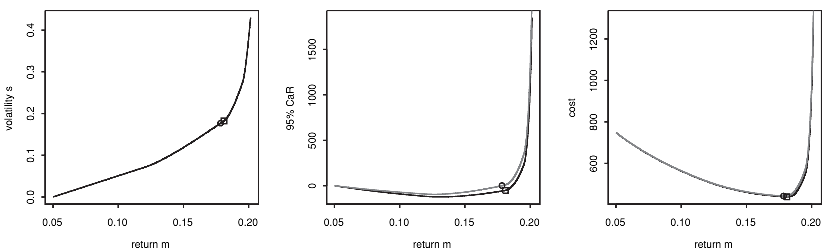


FIGURE 7

Optimal portfolios for $c_t = 105 - 5t$ (t=1, …, 20) with $\varepsilon = 0.01$.