



KATHOLIEKE  
UNIVERSITEIT  
LEUVEN

# DEPARTEMENT TOEGEPASTE ECONOMISCHE WETENSCHAPPEN

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THE HURDLE-RACE PROBLEM

by

S. VANDUFFEL

J. DHAENE

M. GOOVAERTS

R. KAAS

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# The hurdle-race problem

S. Vanduffel<sup>1</sup>      J. Dhaene<sup>2</sup>      M. Goovaerts<sup>2</sup>

R. Kaas<sup>2</sup>

<sup>1</sup>K.U. Leuven

<sup>2</sup>K.U.Leuven and University of Amsterdam

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## Abstract

We consider the problem of how to determine the required level of the current provision in order to be able to meet a series of future deterministic payment obligations, in case the provision is invested according to a given random return process. Approximate solutions are derived, taking into account imposed minimum levels of the future random values of the reserve. The paper ends with numerical examples illustrating the presented approximations.

## 1 Introduction

Consider the problem of how to determine at current time 0 the amount  $R_0$  that will enable us to pay the amounts  $\alpha_i$  at times  $i$  ( $i = 1, 2, \dots, n$ ). We will call  $R_0$  the provision or the reserve at time 0.

A first way of determining the provision follows from noting that these future liabilities are equivalent to the liabilities associated with selling  $n$  zero coupon bonds with face values  $\alpha_1, \alpha_2, \dots, \alpha_n$  respectively. The provision is then determined as the price of this zero-coupon bond portfolio. Investing the provision in this sequence of zero-coupon bonds will exactly generate the amounts  $\alpha_i$  at times  $i$  ( $i = 1, 2, \dots, n$ ), so that we will be able to meet our future obligations with certainty under this investment strategy.

In this paper however, we will determine the level of the provision under a given investment strategy. We will assume that the provision will be invested

such that it generates stochastic yearly returns  $Y_1, Y_2, \dots, Y_n$  in the coming years. The provision will be determined such that the probability that we will be able to meet our future obligations will be sufficiently large. Conversely, if the level of the provision is given, our methodology will enable us to compute the probability that we will be able to meet our future obligations under the given investment strategy.

Note that the latter approach also allows to determine the optimal investment strategy by comparing the required provision (for a given level of certainty) under different investment vectors  $(Y_1, Y_2, \dots, Y_n)$ . The optimal investment strategy is the one for which the provision is minimal. On the other hand, if the level of the provision is given, the optimal investment strategy could be determined as the one leading to the maximal probability that we will be able to meet our future obligations.

An example of a situation where this methodology is appropriate is an insurer who wants to determine the current reserve or capital required to meet his future obligations. An example in the framework of personal finance is an individual who wants to invest an amount today that will provide him with a predetermined fixed income during the coming 20 years. In Section 4 we illustrate our results by several numerical examples.

## 2 Determining the provision for future payment obligations

Let  $R_0$  be the provision at time 0, and consider the stochastic return process  $(Y_1, \dots, Y_n)$ , i.e. 1 unit invested at time 0 is assumed to grow to  $e^{Y_1 + \dots + Y_j}$  at time  $j$  ( $j = 1, 2, \dots, n$ ). Let  $R_j$  be defined recursively by

$$R_j = R_{j-1} e^{Y_j} - \alpha_j, \quad j = 1, \dots, n. \quad (1)$$

Hence,  $R_j$  is the (stochastic) provision that will be available at time  $j$ , after the payment of  $\alpha_j$ , given that  $R_0$  is the provision at time 0. The realization of  $R_j$  will be known at time  $j$ , and depends on the investment returns (stochastic part) and on the payments (deterministic part) in the past years.

One could determine the initial provision as the minimal amount such that  $R_n$  will be non-negative with a probability of at least  $1 - \varepsilon_n$ , with  $\varepsilon_n$  sufficiently small. This means that we determine the initial provision in such a way that we will be able to "reach the finish" with a predefined (high)

probability:

$$\bar{R}_0 = \inf \{R_0 \mid \Pr [R_n \geq 0 \mid R_0] \geq 1 - \varepsilon_n\}. \quad (2)$$

From the recursion (1) for the provisions, we find the following explicit expression for  $R_j$ :

$$R_j = R_0 e^{Y_1 + \dots + Y_j} - \sum_{i=1}^j \alpha_i e^{Y_{i+1} + \dots + Y_j}, \quad j = 1, \dots, n. \quad (3)$$

We always will conventionally take  $\sum_{i=m}^n b_i = 0$  if  $m > n$ . We find

$$\Pr [R_n \geq 0 \mid R_0] = \Pr [S \leq R_0] \quad (4)$$

with  $S$  defined by

$$S = \sum_{i=1}^n \alpha_i e^{-(Y_1 + \dots + Y_i)}. \quad (5)$$

The random variable  $S$  is the stochastically discounted value of all future liabilities.

Hence,  $\bar{R}_0$  is given by

$$\bar{R}_0 = \inf \{R_0 \mid \Pr [S \leq R_0] \geq 1 - \varepsilon_n\}, \quad (6)$$

which means that the initial reserve is determined as the  $(1 - \varepsilon_n)$ -quantile of  $S$ :

$$\bar{R}_0 = F_S^{-1}(1 - \varepsilon_n). \quad (7)$$

In general, it is impossible to determine the distribution function and the quantiles of  $S$  analytically, because in any realistic model for the return process  $(Y_1, \dots, Y_n)$  the random variable  $S$  will be a sum of strongly dependent random variables. Approximations for the distribution function of sums of dependent random variables have been considered extensively in the actuarial literature. These approximations are based on the concept of "comonotonicity" which describes a strong positive dependency between random variables, see e.g. Heilmann (1986), Dhaene & Goovaerts (1996), Dhaene & Goovaerts (1997), Müller (1997), Bäurle & Müller (1998), Wang & Dhaene (1998), Wang & Young (1998), Goovaerts & Dhaene (1999), Goovaerts & Redant

(1999), Dhaene, Wang, Young & Goovaerts (2000), Embrechts, McNeil & Straumann (2000), Goovaerts, Dhaene & De Schepper (2000), Kaas, Dhaene & Goovaerts (2000), Vyncke, Goovaerts & Dhaene (2000), Goovaerts & Kaas (2001), Kaas, Dhaene, Vyncke, Goovaerts & Denuit (2001).

A drawback of the determination of the provision  $R_0$  as a quantile of  $S$  is that the only goal that has to be met is "reaching the finish". In a situation where first the  $\alpha_j$  are positive (hence payments) and moderate in absolute value, but the last ones are negative (hence incomes) and large in absolute value, this may lead to a situation where the  $R_j$  in the first years become negative (or below a predefined level), which may be an undesirable situation.

In this paper, we will present an (approximate) way of determining the initial provision, which does not only take into account the goal of "reaching the finish", but also the conditions that at each time  $j$  the provision  $R_j$  is larger than a given deterministic value  $V_j$  with a sufficiently large probability. These additional requirements are the "hurdles" that have to be taken. In the case of an insurer establishing his reserve, these "hurdles" might be imposed by a supervisory authority or by internal policy. Hence, in the sequel of this paper the provision at time 0 is determined by

$$\bar{R}_0 = \inf \{ R_0 \mid R_0 \geq V_0; \Pr[R_j \geq V_j \mid R_0] \geq 1 - \varepsilon_j; j = 1, \dots, n \}, \quad (8)$$

for given hurdles  $V_0, V_1, \dots, V_n$  and given (small) probabilities  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ . Determining the initial provision as in (8) allows one to make the probability of taking the hurdles time-dependent. In situations where year-to-year adjustments of the level of the reserve are possible, the probabilities of taking the hurdles in the first years could be chosen larger than these probabilities in the later years.

Let us define the random variables  $S_{[0,j]}$ , ( $j = 1, \dots, n$ ) by

$$\begin{aligned} S_{[0,j]} &= \sum_{i=1}^{j-1} \alpha_i e^{-(Y_1 + \dots + Y_i)} + (V_j + \alpha_j) e^{-(Y_1 + \dots + Y_j)} \\ &= \sum_{i=1}^j \alpha_{i(j)} e^{-(Y_1 + \dots + Y_i)}, \quad j = 1, \dots, n \end{aligned} \quad (9)$$

with the  $\alpha_{i(j)}$  given by

$$\alpha_{i(j)} = \begin{cases} \alpha_i, & i \neq j \\ V_j + \alpha_j, & i = j \end{cases}. \quad (10)$$

The random variable  $S_{[0,j]}$  can be interpreted as the stochastically discounted value of the future obligations in the restricted time period  $[0, j]$ .

**Theorem 1** *The optimal initial provision  $\bar{W}_0$  defined in (8) is given by*

$$\bar{R}_0 = \max \left\{ V_0, F_{S_{[0,1]}}^{-1}(1 - \varepsilon_1), F_{S_{[0,2]}}^{-1}(1 - \varepsilon_2), \dots, F_{S_{[0,n]}}^{-1}(1 - \varepsilon_n) \right\} \quad (11)$$

**Proof.** From (3), it is straightforward to verify that

$$\Pr [R_j \geq V_j \mid R_0] = \Pr [S_{[0,j]} \leq R_0], \quad j = 1, \dots, n. \quad (12)$$

Hence,  $\bar{R}_0$  follows from

$$\begin{aligned} \bar{R}_0 &= \inf \left\{ R_0 \mid R_0 \geq V_0; \Pr [S_{[0,j]} \leq R_0] \geq 1 - \varepsilon_j, \quad j = 1, \dots, n \right\} \\ &= \inf \left\{ R_0 \mid R_0 \geq V_0; R_0 \geq F_{S_{[0,j]}}^{-1}(1 - \varepsilon_j), \quad j = 1, \dots, n \right\} \end{aligned}$$

This proves the stated result. ■

Under certain constraints, the initial provision defined in (8) coincides with the initial reserve defined in (4). This follows from the following corollary.

**Corollary 2** *If  $\alpha_j \geq 0$ ,  $V_j = 0$  and  $\varepsilon_j \geq \varepsilon_n$ , ( $j = 1, \dots, n$ ), then  $\bar{R}_0$  is given by*

$$\bar{R}_0 = \max \left\{ V_0, F_{S_{[0,n]}}^{-1}(1 - \varepsilon_n) \right\}. \quad (13)$$

**Proof.** Under the constraints of the corollary, we have that

$$F_{S_{[0,j]}}^{-1}(1 - \varepsilon_j) \leq F_{S_{[0,j]}}^{-1}(1 - \varepsilon_n) \leq F_{S_{[0,n]}}^{-1}(1 - \varepsilon_n)$$

where the last inequality follows from the fact that  $F_{S_{[0,j]}}(x) \geq F_{S_{[0,n]}}(x)$  holds for all values of  $x$ . This proves the stated result. ■

If in addition to the conditions in the Corollary, we also have that  $V_0 = 0$ , then the initial provision defined in (4) and (8) are identical. In practice, one will often choose the  $\varepsilon_i$  in the first years lower than the later ones because the conditions in the immediate future have to be met with the highest probability. In this case, the conditions of the corollary will not be fulfilled, and  $\bar{R}_0$  may be different from  $F_{S_{[0,n]}}^{-1}(1 - \varepsilon_n)$ .

### 3 Approximations for $\bar{R}_0$

In order to find an accurate approximation for  $\bar{R}_0$ , we will use the approximations proposed in Kaas, Dhaene & Goovaerts (2000), see also Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2002 a, b). We will illustrate our methodology for lognormal returns. Hence, in the remainder of this paper we will always assume that  $(Y_1, Y_2, \dots, Y_n)$  has a multivariate normal distribution. The random variables  $Y(i)$  are defined by

$$Y(i) = Y_1 + Y_2 + \dots + Y_i, \quad i = 1, \dots, n. \quad (14)$$

Also consider the random variable  $\Lambda_{(j)}$  defined by

$$\Lambda_{(j)} = \sum_{i=1}^j \beta_{i(j)} Y_i, \quad j = 1, \dots, n, \quad (15)$$

for some choice of the parameters  $\beta_{i(j)}$ . Finally, let  $r_{i(j)}$  be Pearson's correlation coefficient between  $Y(i)$  and  $\Lambda_j$ :

$$r_{i(j)} = \text{corr} [Y(i), \Lambda_{(j)}], \quad i = 1, \dots, j; j = 1, \dots, n. \quad (16)$$

In order to determine  $\bar{R}_0$ , one needs to compute quantiles of  $S_{[0,j]}$ , ( $j = 1, \dots, n$ ), which in general can not be determined exactly. Therefore, we will approximate the distribution functions of the random variables  $S_{[0,j]}$ ,  $j = 1, \dots, n$  by the respective distribution functions of the random variables  $S_{[0,j]}^l$  and  $S_{[0,j]}^c$  as suggested in Kaas, Dhaene & Goovaerts (2000):

$$\begin{aligned} S_{[0,j]}^l &= \sum_{i=1}^j E [\alpha_{i(j)} e^{-Y(i)} | \Lambda_{(j)}] \\ &= \sum_{i=1}^j \alpha_{i(j)} e^{-E[Y(i)] - r_{i(j)} \sigma_{Y(i)} \Phi^{-1}(U) + \frac{1}{2}(1 - r_{i(j)}^2) \sigma_{Y(i)}^2} \end{aligned} \quad (17)$$

$$\begin{aligned} S_{[0,j]}^c &= \sum_{i=1}^j F_{\alpha_{i(j)} e^{-Y(i)}}^{-1}(V) \\ &= \sum_{i=1}^j \alpha_{i(j)} e^{-E[Y(i)] + \text{sign}(\alpha_{i(j)}) \sigma_{Y(i)} \Phi^{-1}(V)} \end{aligned} \quad (18)$$

where  $U$  and  $V$  are uniformly distributed on the interval  $(0, 1)$  and  $\Phi$  is the cumulative distribution function of the  $N(0, 1)$  distribution.

One can verify that

$$E[S_{[0,j]}] = E[S_{[0,j]}^l] = E[S_{[0,j]}^c] \quad (19)$$

holds for any  $j$ . Furthermore, in Kaas, Dhaene & Goovaerts (2000) it is proven that

$$E[(S_{[0,j]}^l - d)_+] \leq E[(S_{[0,j]} - d)_+] \leq E[(S_{[0,j]}^c - d)_+] \quad (20)$$

holds for any  $j$  and any retention  $d$ . Hence,  $S_{[0,j]}^c$  can be considered as a stochastic upper bound (in the sense of convex order) for  $S_{[0,j]}$ , while  $S_{[0,j]}^l$  is a stochastic lower bound (in the sense of convex order) for  $S_{[0,j]}$ . This implies that  $E[u(-S_{[0,j]})] \geq E[u(-S_{[0,j]}^c)]$  holds for any concave utility function  $u$ . Hence, replacing (the distribution function of)  $S_{[0,j]}$  by (the distribution function of)  $S_{[0,j]}^c$  is a "safe" strategy in the sense that all risk averse decision makers will prefer liabilities  $S_{[0,j]}$  to  $S_{[0,j]}^c$ . Likewise they will prefer  $S_{[0,j]}^l$  to  $S_{[0,j]}$ .

Following Theorem 1 and the ideas in Kaas, Dhaene and Goovaerts (2000), we propose the following approximations for  $\bar{R}_0$ :

$$\bar{R}_0^l = \max \left\{ V_0; F_{S_{[0,j]}^l}^{-1}(1 - \varepsilon_j), j = 1, \dots, n \right\}, \quad (21)$$

$$\bar{R}_0^c = \max \left\{ V_0; F_{S_{[0,j]}^c}^{-1}(1 - \varepsilon_j), j = 1, \dots, n \right\}. \quad (22)$$

In Section 4 we will numerically illustrate that  $\bar{R}_0^l$ , and  $\bar{R}_0^c$  will often be good approximations for  $\bar{R}_0$ . Especially, the approximation  $\bar{R}_0^l$  for  $\bar{R}_0$  performs very good. In general, the quantiles  $F_{S_{[0,j]}^l}^{-1}(1 - \varepsilon_j)$  and  $F_{S_{[0,j]}^c}^{-1}(1 - \varepsilon_j)$  can easily be computed as is explained in Kaas, Dhaene & Goovaerts (2000) or Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2001, b), see also the numerical illustration section.

Note that (20) does not necessary imply  $\bar{R}_0^l \leq \bar{R}_0 \leq \bar{R}_0^c$ . Let us now assume in the remainder of this section that all  $\alpha_{i(j)}$  and all  $r_{i(j)}$  are non-negative. Then one can prove that the inequality  $\bar{R}_0^l \leq \bar{R}_0^c$  holds provided the



$\epsilon_j$  are sufficiently small. Indeed, in this case it is straightforward to verify that for all  $p \in (0, 1)$ , the condition

$$p \geq \Phi \left[ \frac{1}{2} \max \{ (1 + r_{i(j)}) \sigma_{Y(i)}; i = 1, \dots, j \} \right] \equiv p_{(j)}^+ \quad (23)$$

implies that  $F_{S_{[0,j]}^l}^{-1}(p) \leq F_{S_{[0,j]}^c}^{-1}(p)$ . Hence, if all  $\epsilon_j$  are chosen such that

$$\epsilon_j \leq 1 - p_{(j)}^+, \quad j = 1, \dots, n, \quad (24)$$

the inequality  $\bar{R}_0^l \leq \bar{R}_0^c$  will hold.

If the  $Y_i$  are i.i.d. with variance  $\sigma^2$ , then the conditions

$$\epsilon_j \leq 1 - \Phi \left[ \sigma \sqrt{j} \right], \quad j = 1, \dots, n, \quad (25)$$

ensure that  $\bar{R}_0^l \leq \bar{R}_0^c$  holds. As we will see in the next section, these conditions are often fulfilled in practical applications.

Similarly, one can prove that the condition

$$p \leq \Phi \left[ \frac{1}{2} \min \{ (1 + r_{i(j)}) \sigma_{Y(i)}; i = 1, \dots, j \} \right] \equiv p_{(j)}^- \quad (26)$$

implies that  $F_{S_{[0,j]}^c}^{-1}(p) \leq F_{S_{[0,j]}^l}^{-1}(p)$ . Note that  $p_{(j)}^- \geq 0.5$ .

We can also conclude that the distribution functions of  $S_{[0,j]}^l$  and  $S_{[0,j]}^c$  can only cross in the region where their distribution functions take a value that is contained in the interval  $[p_{(j)}^-, p_{(j)}^+]$ . In the region  $p \leq p_{(j)}^-$ , the distribution function of  $S_{[0,j]}^c$  will lay above the distribution function of  $S_{[0,j]}^l$ , while in the region  $p \geq p_{(j)}^+$ , the distribution function of  $S_{[0,j]}^c$  will lay under the distribution function of  $S_{[0,j]}^l$ .

## 4 Numerical examples

### 4.1 The return process

In this section, we will numerically evaluate the approximations  $\bar{R}_0^l$  and  $\bar{R}_0^c$  for the provision  $\bar{R}_0$  in various applications. In order to judge the accurateness

of these approximations, we have to compare them with the exact initial provision. As  $\bar{R}_0$  cannot be determined analytically, we will determine it by simulation.

We will assume that the returns  $Y_i$  are i.i.d. and  $N(\mu - \frac{\sigma^2}{2}, \sigma^2)$ . This implies that  $E[Y(i)] = i(\mu - \frac{\sigma^2}{2})$  and  $Var[Y(i)] = i\sigma^2$ . In order to determine  $\bar{R}_0^l$ , we will use conditioning random variables  $\Lambda_{(j)}$  as defined in (15). In this case we find

$$Var[\Lambda_{(j)}] = \sigma^2 \sum_{k=1}^j \beta_{k(j)}^2, \quad j = 1, \dots, n, \quad (27)$$

$$r_{i(j)} = \frac{\sum_{k=1}^i \beta_{k(j)}}{\sqrt{i \sum_{k=1}^j \beta_{k(j)}^2}}, \quad i = 1, \dots, j; j = 1, \dots, n. \quad (28)$$

In particular, we will choose the coefficients  $\beta_{i(j)}$ ,  $j = 1, \dots, n$  as follows:

$$\beta_{i(j)} = \sum_{k=i}^j \alpha_{k(j)} e^{-k(\mu - \frac{\sigma^2}{2})}, \quad i = 1, \dots, j; j = 1, \dots, n. \quad (29)$$

This choice makes  $\Lambda_j$  a linear transformation of a first order approximation to  $S_{[0,j]}$ , ( $j = 1, \dots, n$ ). Indeed,

$$\begin{aligned} S_{[0,j]} &= \sum_{k=1}^j \alpha_{k(j)} e^{-k(\mu - \frac{\sigma^2}{2}) - \sum_{i=1}^k (Y_i - (\mu - \frac{\sigma^2}{2}))} \approx \sum_{k=1}^j \alpha_{k(j)} e^{-k(\mu - \frac{\sigma^2}{2})} [1 - \sum_{i=1}^k (Y_i - (\mu - \frac{\sigma^2}{2}))] \\ &= C - \sum_{k=1}^j \alpha_{k(j)} e^{-k(\mu - \frac{\sigma^2}{2})} \sum_{i=1}^k Y_i = C - \sum_{i=1}^j Y_i \sum_{k=i}^j \alpha_{k(j)} e^{-k(\mu - \frac{\sigma^2}{2})}, \end{aligned}$$

where  $C$  is the appropriate constant. By this choice of the coefficients  $\beta_{i(j)}$ , the distribution function of  $S_{[0,j]}^l$  will be “close” to the distribution function of  $S_{[0,j]}$ , provided  $(Y_i - (\mu - \frac{\sigma^2}{2}))$  is sufficiently small, or equivalently,  $\sigma$  is sufficiently small, see Section 4.1 in Kaas, Dhaene & Goovaerts (2000). Note that if all  $\alpha_{i(j)}$  are non-negative, then this particular choice of the  $\beta_{i(j)}$  implies that all correlation coefficients  $r_{i(j)}$  are non-negative.

## 4.2 Example 1

Consider a person who invests at time 0 an amount of 10 in a fund with a yearly expected return  $\mu = \ln 1.10$  and a yearly volatility  $\sigma = 0.10$ . Each of

the coming 10 years, he wants to withdraw 0.8 units of this initial amount. In addition, at the end of the 10 year period he wants to recover his initially invested capital of 10. The investor believes that it is very likely that his investment will meet his requirements because the investment period is long enough to eliminate the downside risk.

In terms of the notations introduced in Section 2, the investor has obligations  $\alpha_i = 0.8$ ,  $i = 1, \dots, 10$ , with a final hurdle  $V_{10} = 10$ , while his initial provision equals  $R_0 = 10$ . Let us assume that his belief can be expressed as  $\Pr [R_n \geq 10 \mid R_0 = 10] \geq 0.995$ .

In order to judge the investor's belief, we determine

$$\Pr [R_n \geq 10 \mid R_0 = 10] = \Pr [S_{[0,10]} \leq 10].$$

From the expressions (17) and (18) one finds that the (approximate) probability that the investor will take the last hurdle successfully if his initial investment is 10 equals 65.28% if we perform the calculations with  $S_{[0,10]}^l$  and 64.53% if we use  $S_{[0,10]}^c$ . In order to judge the quality of these approximations, we also determine  $\Pr [S_{[0,10]} \leq 10]$  by simulating 100,000 values for the return vector  $(Y_1, Y_2, \dots, Y_n)$ . This simulation leads to a value of 65.35% for this probability, with a standard error equal to 0.077%. This indicates that especially the approximation  $S_{[0,10]}^l$  will perform very well. Moreover, the investors belief of reaching the finish with a probability of at least 99.5% is certainly not true.

Let us now determine  $\bar{R}_0$  which is the smallest  $R_0$  such that the investor will take the final hurdle  $V_{10} = 10$  with a probability of at least 99.5% :

$$\begin{aligned} \bar{R}_0 &= \inf \{R_0 \mid \Pr [R_{10} \geq 10 \mid R_0] \geq 0.995\} \\ &= \inf \{R_0 \mid \Pr [S_{[0,10]} \leq 10] \geq 0.995\} \\ &= F_{S_{[0,10]}^l}^{-1} (0.995). \end{aligned}$$

Using  $S_{[0,10]}^l$ , we find  $\bar{R}_0 = F_{S_{[0,10]}^l}^{-1} (0.995) = 16.98$ , while for  $S_{[0,10]}^c$ , we find  $\bar{R}_0^c = F_{S_{[0,10]}^c}^{-1} (0.995) = 17.872$ . The value for  $\bar{R}_0$  obtained by the simulation of 100,000 paths for the returns equals 16.985, which is again very close to the approximation  $S_{[0,10]}^l$ .

Finally, we determine the maximal amount  $\bar{V}_{10}$  that will be guaranteed at time 10 with a probability of at least 99.5% if the initial invested amount

equals 10 :

$$\begin{aligned}\bar{V}_{10} &= \sup \{V_{10} \mid \Pr [R_{10} \geq V_{10} \mid R_0 = 10] \geq 0.995\} \\ &= \sup \{V_{10} \mid \Pr [S_{[0,10]} \leq 10] \geq 0.995\} \\ &= \sup \left\{ V_{10} \mid F_{S_{[0,10]}^l}^{-1} (0.995) \geq 10 \right\}.\end{aligned}$$

This means that we want to determine the hurdle  $\bar{V}_{10}$  as the maximal amount such that the 99.5% percentile of  $S_{[0,10]}$  is at least equal to 10. Using the approximation  $S_{[0,10]}^l$ , we obtain  $\bar{V}_{10}^l = 2.184$ , while for  $S_{[0,10]}^c$  we find  $\bar{V}_{10}^c = 1.399$ . The simulation of 100,000 paths leads to a value of 2.148 for the hurdle at time 10.

### 4.3 Example 2

Consider a person who invests an amount of 10 in the fund as described in Example 1, with a yearly expected return  $\mu = \ln 1.10$  and a yearly volatility  $\sigma = 0.10$ . At the end of each of the coming 40 years he wants to withdraw 0.8 units of the fund. Moreover he would like to be sure that the value of the invested amount will never become lower than 10. He wonders if his requirements can be met by the given investment strategy.

In terms of the notations introduced in Section 2, the investor has obligations  $\alpha_i = 0.8$ ,  $i = 1, \dots, 40$  with an initial provision  $R_0 = 10$ . All hurdles  $V_i$  are equal to the initial invested amount of 10. We assume that the investor wants to take each of these hurdles with a probability of at least 99,5%.

As before, we can use the approximations  $F_{S_{[0,j]}^l}^{-1} (0.995)$  and  $F_{S_{[0,j]}^c}^{-1} (0.995)$  for the percentiles  $F_{S_{[0,j]}^l}^{-1} (0.995)$ ,  $j = 1, \dots, 40$ . Note that each  $F_{S_{[0,j]}^l}^{-1} (0.995)$  represents the initial required amount needed to take the hurdle at time  $j$ . The quantiles  $F_{S_{[0,j]}^l}^{-1} (0.995)$ ,  $F_{S_{[0,j]}^c}^{-1} (0.995)$  and the corresponding quantiles ' $F_{S_{[0,j]}^l}^{-1} (0.995)$ ' obtained by simulating 50,000 paths are presented in Tables 1 and 2.

If the investor wants to pass all hurdles with the predetermined degree of certainty, the approximations (21) and (22) for the initial required amount  $\bar{R}_0$  as defined in (8) are given by  $\bar{R}_0^l = 17.55$  and  $\bar{R}_0^c = 20.10$  respectively. The value for  $\bar{R}_0$  obtained by simulating 50,000 paths equals 17.57.

$j$	$F_{S_{[0,j]}^l}^{-1} (0.995)$	$F_{S_{[0,j]}^{-1}} (0.995)$	$F_{S_{[0,j]}^e}^{-1} (0.995)$
1	12.77	12.78	12.77
2	13.86	13.93	13.92
3	14.63	14.58	14.78
4	15.22	15.16	15.46
5	15.68	15.63	16.02
6	16.06	16.07	16.51
7	16.36	16.38	16.92
8	16.61	16.66	17.28
9	16.81	16.81	17.60
10	16.98	16.91	17.87
11	17.11	16.96	18.12
12	17.22	17.13	18.33
13	17.31	17.11	18.53
14	17.38	17.23	18.70
15	17.43	17.25	18.85
16	17.47	17.16	18.99
17	17.50	17.20	19.11
18	17.53	17.25	19.22
19	17.54	17.44	19.32
20	17.55	17.31	19.41

Table 1: Approximate and simulated values for the quantiles in Example 2

$j$	$F_{S_{[0,j]}'}^{-1}(0.995)$	$'F_{S_{[0,j]}'}^{-1}(0.995)'$	$F_{S_{[0,j]}^e}^{-1}(0.995)$
21	17.55	17.33	19.49
22	17.55	17.45	19.56
23	17.55	17.45	19.62
24	<b>17.55</b>	17.49	19.68
25	17.54	17.50	19.73
26	17.53	<b>17.57</b>	19.78
27	17.52	17.54	19.82
28	17.51	17.54	19.86
29	17.50	17.51	19.89
30	17.49	17.56	19.92
31	17.48	17.52	19.95
32	17.47	17.45	19.97
33	17.46	17.54	19.99
34	17.45	17.47	20.01
35	17.44	17.42	20.03
36	17.43	17.49	20.05
37	17.42	17.44	20.06
38	17.41	17.39	20.08
39	17.40	17.39	20.09
40	17.39	17.41	<b>20.10</b>

Table 2: Approximate and simulated values for the quantiles in Example 2 (cont'd)

In case of  $\bar{R}_0^l$  the binding hurdles are the ones corresponding to years 20 to 24. For  $\bar{R}_0^c$  however the binding hurdle is the one at time 40. The binding hurdle for the simulation is the one after 26 years.

A probability of 90% for taking the hurdles would have resulted in an initial reserve equal to  $\bar{R}_0^l = 12.36$  corresponding to the binding hurdle at time 12. For  $\bar{R}_0^c$ , we find a value of 12.84 for the initial reserve and a binding hurdle at time 23, while by simulation we obtain an initial reserve of 12.39 and a binding hurdle at time 12.

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