Available online at www.sciencedirect.com
ScienceDirect
Insurance: Mathematics and Economics 42 (2008) 540–547
INSURANCE MATHEMATICS & ECONOMICS
www.elsevier.com/locate/ime

Actuarial risk measures for financial derivative pricing☆

Marc J. Goovaerts[a,b], Roger J.A. Laeven[a,*]

[a] *University of Amsterdam, Department of Quantitative Economics, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands*
[b] *Catholic University of Leuven, Center for Risk and Insurance Studies, Naamsestraat 69, B-3000 Leuven, Belgium*

Received August 2006; received in revised form April 2007; accepted 22 April 2007

Abstract

We present an axiomatic characterization of price measures that are superadditive and comonotonic additive for normally distributed random variables. The price representation derived involves a probability measure transform that is closely related to the Esscher transform, and we call it the *Esscher–Girsanov transform*. In a financial market in which the primary asset price is represented by a stochastic differential equation with respect to Brownian motion, the price mechanism based on the Esscher–Girsanov transform can generate approximate-arbitrage-free financial derivative prices.

JEL classification: D81; G12; G13

MSC: 91B06; 91B28; 91B30

Keywords: Derivative pricing; Incomplete markets; Stochastic ordering; Esscher transform; Girsanov's theorem; Comonotonicity; Equivalent martingale measure; Feynman–Kac integration

1. Introduction

Risk measures for actuarial pricing are usually justified, either directly or indirectly, by means of an axiomatic characterization; see, e.g., Goovaerts et al. (1984) and, more recently, Denuit et al. (2006) and Laeven and Goovaerts (2007). Financial derivative pricing usually relies on principles of no arbitrage. Various attempts to connect the two approaches are available in the literature; the interested reader is referred to Embrechts (2000) for a review. This paper establishes a new connection.

The connection is based on the time-honored *Esscher transform*. The Esscher transform has proven to be a valuable tool for the pricing of insurance and financial products. In Bühlmann (1980), a premium principle based on the Esscher transform is derived within a general equilibrium model in which decision makers have negative exponential utility functions; see Iwaki et al. (2001) for an extension of that model to a multi-period setting. Gerber and Goovaerts (1981) established an axiomatic characterization of an additive premium principle that involves a mixture of Esscher transforms.

In a financial environment, Gerber and Shiu (1994, 1996) use the Esscher transform to construct equivalent martingale measures for Lévy processes (with independent and stationary increments). Inspired by this, Bühlmann et al. (1996) more generally use *conditional Esscher transforms* to construct equivalent martingale measures for classes of semi-martingales.

In this paper, the approach of establishing risk evaluation mechanisms by means of an axiomatic characterization is used to characterize a price mechanism that can generate approximate-arbitrage-free financial derivative prices. In particular, this paper presents a representation theorem for price measures that are superadditive and comonotonic additive for normally distributed random variables. The price representation derived involves a probability measure transform that is

☆ This paper is largely based on a chapter in the Ph.D. thesis of the second author [Laeven, Roger J.A., 2005. Essays on risk measures and stochastic dependence. Tinbergen Institute Research Series 360. Thela Thesis. Amsterdam] (Laeven, 2005). A first version of this paper was circulated in spring 2004.

* Corresponding author. Tel.: +31 20 525 7317; fax: +31 20 525 4349.
E-mail address: R.J.A.Laeven@uva.nl (R.J.A. Laeven).

0167-6687/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.insmatheco.2007.04.001

closely related to the Esscher transform, and which we call the *Esscher–Girsanov* transform. We demonstrate that in a financial market in which the primary asset price is represented by a stochastic differential equation with respect to Brownian motion, approximate-arbitrage-free financial derivative prices coincide with the price representation derived.

The axioms imposed to establish the representation theorem can be formulated as follows:

1. Ordered *Esscher–Girsanov transforms* implies ordered prices. If the price measure is applied to normally distributed random variables, this axiom is equivalent to “respect for second-order stochastic dominance”.
2. The price measure is appropriately normalized such that the price of c non-random units is equal to c non-random units.
3. Additivity for sums of *Esscher–Girsanov transforms*. If the price measure is applied to normally distributed random variables, this axiom is equivalent to “superadditivity and comonotonic additivity of the price measure”, thus capturing the benefits of diversification.
4. Continuity conditions, which are necessary for establishing the mathematical proofs.

The outline of this paper is as follows: in Section 2, we consider the Esscher transform, we study some stochastic order relations derived from it and we discuss the axiomatization of the mixed Esscher principle. In Section 3, we introduce the *Esscher–Girsanov transform* and axiomatize a price measure induced by it. Section 4 addresses the pricing of financial derivatives by means of Esscher–Girsanov transforms.

2. Stochastic ordering and the Esscher transform

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In this paper, unless otherwise stated, a random variable (r.v.) represents net income or profit at a future point in time. Throughout, we assume that for any r.v. defined on the probability space, its *moment generating function* exists, i.e., for any r.v. $X : \Omega \to \mathbb{R}$

$$\mathbb{E}[e^{hX}] < +\infty, \quad h \in \mathbb{R}. \tag{1}$$

For the cumulative distribution function (cdf) $F_X(\cdot)$ with differential $dF_X(\cdot)$, corresponding to a given r.v. X, we define by

$$dF_X^{(h)}(x) = \frac{e^{hx} dF_X(x)}{\mathbb{E}\left[e^{hX}\right]}, \quad h \in \mathbb{R}, \tag{2}$$

its *Esscher transform* with parameter h. Esscher (1932) suggested using the transform in (2) instead of the original cdf, to apply the well-known Edgeworth approximation to; see also Gerber (1979). The reason was that the Edgeworth approximation performs well in the vicinity of the expectation, but performs worse in the tails. Notice that for $h = 0$, the original differential appears, and that $F_X(\cdot)$ and its Esscher transform $F_X^{(h)}(\cdot)$ are *equivalent distributions* in the sense that they have the same null sets. It is not difficult to verify that for a normal cdf with expectation μ and variance σ^2, its Esscher transform is a normal cdf with expectation $\mu+h\sigma^2$ and variance σ^2.

Next, for a given r.v. X, we define the real-valued function $\psi_X(\cdot)$ as follows:

$$\psi_X(h) = \int_{-\infty}^{+\infty} x\, dF_X^{(h)}(x) = \frac{\mathbb{E}\left[X e^{hX}\right]}{\mathbb{E}\left[e^{hX}\right]}. \tag{3}$$

The number $\psi_X(h)$ is known as the *Esscher premium* with parameter h; see Bühlmann (1980) and Goovaerts et al. (1984). Notice that $\psi_X(h)$ is non-decreasing in h. This can be proved easily using the Hölder inequality and will be used later; also, observe that the derivative of the last expression in (3) can be interpreted as a variance.

In the following, we denote by the functional $\pi[\cdot]$ a *risk measure* or – since X is interpreted as net income or profit – rather a *price measure* that assigns a real number to any r.v. or its cdf. We introduce a set of axioms that $\pi[\cdot]$ must satisfy:

A1. If $\psi_X(h) \le \psi_Y(h)$ for all $h \le 0$, then $\pi[X] \le \pi[Y]$.
A2. $\pi[c] = c$, for all c.
A3. $\pi[X+Y] = \pi[X] + \pi[Y]$ when X and Y are independent.
A4. If X_n converges weakly to X, with $\min[X_n] \to \min[X]$, then $\lim_{n\to+\infty} \pi[X_n] = \pi[X]$.

In a general setting, axiom A1 can be criticized. Gerber (1981) already pointed out that the Esscher premium is not monotonic, i.e., it does not hold that if X is first-order stochastically dominated by Y, denoted by $X \le_{\mathrm{st}} Y$, then $\psi_X(h) \le \psi_Y(h)$ for all $h \in \mathbb{R}$ (or even all $h \le 0$). Hence, axiom A1 does not guarantee monotonicity of the functional $\pi[\cdot]$.

Goovaerts et al. (2004) replaced axiom A1 by the more restrictive axiom of respect for *Laplace transform* order, which does guarantee monotonicity of the functional $\pi[\cdot]$. We say that X is smaller than Y in Laplace transform order if $\mathbb{E}[e^{hX}] \ge \mathbb{E}[e^{hY}]$ for all $h \le 0$. We write $X \le_{\mathrm{Lt}} Y$. Indeed, $X \le_{\mathrm{st}} Y$ implies $X \le_{\mathrm{Lt}} Y$. In the expected utility model, the Laplace transform order represents preferences of decision makers with a *negative exponential* utility function given by

$$U(x) = -\frac{1}{h}\left(1 - e^{hx}\right), \quad h < 0. \tag{4}$$

Here, $-h$ is the Arrow–Pratt measure of absolute risk aversion. The interested reader is referred to Denuit (2001) for a comprehensive treatment of the Laplace transform order.

In the following sections, normally distributed r.v.’s are of particular interest. Suppose that X and Y are normally distributed. Then the condition that

$$\psi_X(h) \le \psi_Y(h), \quad h \le 0, \tag{5}$$

is equivalent to the condition that both $\mu_X \le \mu_Y$ and $\sigma_X \ge \sigma_Y$. To verify this statement, notice that for normally distributed r.v.’s

$$\psi_X(h) = \mu_X + h\sigma_X^2. \tag{6}$$

Furthermore, it is not difficult to verify that if X and Y are normally distributed, then $X \le_{\mathrm{Lt}} Y$ if and only if condition (5) is satisfied (or equivalently if and only if both $\mu_X \le \mu_Y$ and $\sigma_X \ge \sigma_Y$).

More generally, it is well known that if X and Y are normally distributed with $\mu_X \leq \mu_Y$ and $\sigma_X \geq \sigma_Y$, then X is second-order stochastically dominated by Y and hence Y is preferred to X by any risk averse expected utility decision maker (with concave utility function); see Hadar and Russell (1969) and Rothschild and Stiglitz (1970) for the original work on second-order stochastic dominance. In particular, notice that for normally distributed r.v.'s, $X \leq_{\rm st} Y$ if and only if $\mu_X \leq \mu_Y$ and $\sigma_X = \sigma_Y$. Hence, axiom A1 is appealing for the case of normally distributed r.v.'s.

In the economics literature, axiom A2 is sometimes referred to as the *certainty equivalent* condition. Notice that c plays two roles in axiom A2: a degenerate r.v. at c on the left-hand side and a real number on the right-hand side.

The desirability of price additivity for independent r.v.'s, as imposed by axiom A3, was already pointed out by Borch (1962), p. 429; see also Bühlmann (1985).

Axiom A4 is a continuity condition on the price measure $\pi[\cdot]$. We state the following lemma:

Lemma 2.1. *A price measure $\pi[\cdot]$ satisfies the set of axioms* A1–A4 *if and only if there exists some non-decreasing function $H : [-\infty, 0] \to [0, 1]$ such that*

$$\pi[X] = \int_{[-\infty,0]} \psi_X(h) \mathrm{d}H(h). \tag{7}$$

Proof. The proof of this lemma is similar to the proof of Theorem 2 of Gerber and Goovaerts (1981); see Goovaerts et al. (2004) for comments on that proof.[1] We therefore simply identify the notation used in Gerber and Goovaerts (1981) with our notation: The function $\phi_X(\cdot)$ in Gerber and Goovaerts (1981) is our function $\psi_X(\cdot)$; their principle $H[\cdot]$ is our price measure $\pi[\cdot]$; and their mixture function $F(\cdot)$ is our mixture function $H(\cdot)$.

Whereas we impose that $\pi[X] \leq \pi[Y]$ whenever $\psi_X(h) \leq \psi_Y(h)$ for all $h \leq 0$, Gerber and Goovaerts (1981) impose the (weaker) condition that $\pi[X] \leq \pi[Y]$ whenever $\psi_X(h) \leq \psi_Y(h)$ for all h. As a consequence, the domain of our mixture function $H(\cdot)$ is restricted to $[-\infty, 0]$ whereas the domain of the mixture function in Gerber and Goovaerts (1981) is $[-\infty, +\infty]$. □

Some remarks:

Remark 2.1. Gerber and Goovaerts (1981) established an axiomatic characterization of the mixed Esscher principle. Goovaerts et al. (2004) axiomatized the mixed exponential principle. It is straightforward to verify that for normally distributed r.v.'s any mixed Esscher premium is a mixed exponential premium, and vice versa. In general, it only holds that any mixed exponential premium is a mixed Esscher premium; see Goovaerts et al. (2004). □

[1] In Goovaerts et al. (2004) it is demonstrated that a true equivalence statement formally requires an extension of the class of functions for which axioms A1–A4 should hold; see Goovaerts et al. (2004) for details.

Remark 2.2. The mixture function $H(\cdot)$ can be regarded as a cdf, supported on $(-\infty, 0]$ and possibly defective with a jump at $-\infty$. It can serve as a *prior distribution* for the Arrow–Pratt measure of absolute risk aversion; see in this respect Savage (1954). To see why the parameter $-h$ involved in the Esscher transform can be interpreted as the Arrow–Pratt measure of absolute risk aversion corresponding to a decision maker with a negative exponential utility function, we refer the reader to Bühlmann (1980) and Goovaerts et al. (1984), pp. 84–86. □

Remark 2.3. The price measure $\pi[\cdot]$ characterized in Lemma 2.1 can be expressed as $\pi[X] = \mathbb{E}^*[X]$, where the expectation is calculated using the differential

$$\mathrm{d}F_X^{(H(\cdot))}(x) = \left(\int_{h\in[-\infty,0]} \frac{\mathrm{e}^{hx}\mathrm{d}H(h)}{\mathbb{E}\left[\mathrm{e}^{hX}\right]} \right) \mathrm{d}F_X(x). \quad \square$$

3. The Esscher–Girsanov transform

In the previous section, we presented a representation theorem for price measures that are additive for independent r.v.'s. The price representation derived can be regarded as an expectation under a (mixed) Esscher transformed probability measure. In this section, we introduce a closely related probability measure transform and axiomatize a price measure induced by it.

For a given r.v. X, we define the extended real-valued function $\phi_X(\cdot)$ as follows:

$$\phi_X(x) = \Phi^{-1}(F_X(x)), \tag{8}$$

in which $\Phi^{-1}(\cdot)$ denotes the inverse distribution function of the standard normal distribution. It is well known that if F_X is continuous, then the r.v. $\phi_X(X)$ is normally distributed with mean 0 and variance 1. In the remainder of this section, unless otherwise stated, we restrict ourselves to r.v.'s with a continuous cdf. We state the following definition:

Definition 3.1 (*Esscher–Girsanov Transform*). For the cdf $F_X(\cdot)$ with differential $\mathrm{d}F_X(\cdot)$ corresponding to a given r.v. X, and a given real number v, we define by

$$\mathrm{d}F_X^{(h,v)}(x) = \frac{\mathrm{e}^{hv\phi_X(x)}}{\mathbb{E}[\mathrm{e}^{hv\phi_X(X)}]}\mathrm{d}F_X(x) = \mathrm{e}^{hv\phi_X(x)-\frac{1}{2}h^2v^2}\mathrm{d}F_X(x), \quad h \in \mathbb{R}, \tag{9}$$

its Esscher–Girsanov transform with parameters h and v (absolute risk aversion and penalty parameter, respectively). □

The name of Igor V. Girsanov is attached to the probability measure transform defined above to emphasize the close resemblance between the Radon–Nikodym derivative used in (9) and the Radon–Nikodym derivative used in Girsanov's Theorem; see, e.g., Karatzas and Shreve (1988).[2]

[2] Independently, Wang (2003) studies a probability measure transform closely related to (9). While Wang (2003) focuses on its connection to distorted

At this stage, only the product of h and v seems relevant. However, below the two parameters will play two distinct roles. In accordance with Bühlmann (1980), h could be interpreted as the coefficient of absolute risk aversion while $v\phi_X(X)$ could capture *aggregate market risk*. By virtue of the CLT, in usual circumstances, aggregate market risk can be well approximated by a normal r.v. Moreover, when only normal individual risks are considered (as in Section 4, at least infinitesimally) aggregate market risk is exactly normal.

It is not difficult to verify that for a normal cdf with expectation μ and variance σ^2, its Esscher–Girsanov transform is a normal cdf with expectation $\mu + hv\sigma$ and variance σ^2. In particular, if $v = \sigma$, we trivially find that the Esscher–Girsanov transform is an ordinary Esscher transform. Hence, for a normal cdf, the Esscher–Girsanov transform, just like the ordinary Esscher transform, changes the mean while preserving the variance. Notice that for the change of the mean, the value of the mean is irrelevant.

In the following, we let v be strictly positive and temporarily fixed and restrict the domain of h to $h \leq 0$.

We introduce the real-valued function $\psi_X^v(\cdot)$ defined by

$$\psi_X^v(h) = \int_{-\infty}^{+\infty} x \mathrm{d}F_X^{(h,v)}(x) = \mathbb{E}\left[X\mathrm{e}^{hv\phi_X(X)-\frac{1}{2}h^2v^2}\right], \quad h \leq 0. \tag{10}$$

Henceforth, the number $\psi_X^v(h)$ is called the *Esscher–Girsanov price* of the r.v. X, with parameters $h \leq 0$ and $v > 0$. Notice that given v, there exists a unique correspondence between X and its Esscher–Girsanov price in the sense that $X = Y$ in distribution if and only if

$$\psi_X^v(h) = \psi_Y^v(h), \quad h \leq 0. \tag{11}$$

To verify this statement, notice that

$$\psi_X^v(h) = \int_{-\infty}^{+\infty} F_X^{-1}(\Phi(y))\mathrm{e}^{hvy-\frac{1}{2}h^2v^2}\mathrm{d}\Phi(y), \tag{12}$$

which can be regarded as a Laplace transform, so that the one-to-one correspondence between $\psi_X^v(\cdot)$ and $F_X^{-1}(\cdot)$ follows. The derivative of $\psi_X^v(h)$ with respect to h is given by

$$v\left(\frac{\mathbb{E}\left[X\phi_X(X)\mathrm{e}^{hv\phi_X(X)}\right]}{\mathbb{E}\left[\mathrm{e}^{hv\phi_X(X)}\right]} - \frac{\mathbb{E}\left[X\mathrm{e}^{hv\phi_X(X)}\right]}{\mathbb{E}\left[\mathrm{e}^{hv\phi_X(X)}\right]}\frac{\mathbb{E}\left[\phi_X(X)\mathrm{e}^{hv\phi_X(X)}\right]}{\mathbb{E}\left[\mathrm{e}^{hv\phi_X(X)}\right]}\right), \tag{13}$$

in which the expression between brackets can be regarded as the *Esscher–Girsanov covariance* of X and $\phi_X(X)$ and is non-negative.

As was pointed out in Goovaerts et al. (2004), the price measure characterized in Lemma 2.1 has a counterpart that assigns a real number to the function $\psi_X(\cdot)$. Similarly, we denote by $\rho^v[\cdot]$ a functional that assigns a real number to any function $\psi_X^v(\cdot)$, we let the price measure $\pi^v[\cdot]$ be defined by

$$\pi^v[X] = \rho^v\left[\psi_X^v\right], \tag{14}$$

and state the following set of axioms that $\rho^v[\cdot]$ should satisfy:

B1. If $\psi_X^v(h) \leq \psi_Y^v(h)$ for all $h \leq 0$, then $\rho^v\left[\psi_X^v\right] \leq \rho^v\left[\psi_Y^v\right]$.
B2. $\rho^v[c] = c$, for all c.
B3. $\rho^v\left[\psi_X^v + \psi_Y^v\right] = \rho^v\left[\psi_X^v\right] + \rho^v\left[\psi_Y^v\right]$.
B4. If $\psi_{X_n}^v(h)$ converges to $\psi_X^v(h)$ for all $h \in [-\infty, 0]$, then $\lim_{n\to+\infty} \rho^v\left[\psi_{X_n}^v\right] = \rho^v\left[\psi_X^v\right]$.

Notice that axioms B2 and B4 are similar to axiom A2 and A4. Notice furthermore that $\psi_{cX}^v(h) = c\psi_X^v(h)$ for all $c \geq 0$. Hence, axioms B3 and B4 imply that the price of a portfolio cX equals c times the price of X. This is an intuitive condition whenever financial markets are sufficiently liquid.

We note that for normally distributed r.v.'s, axiom B1 is similar to axiom A1 and gives rise to the appealing second-order stochastic dominance preserving property for $\pi^v[\cdot]$. One easily verifies that if X and Y are two normally distributed r.v.'s with linear correlation coefficient ρ_{XY}, then

$$\psi_{X+Y}^v(h) = \mu_X + \mu_Y + hv\sqrt{\sigma_X^2 + 2\rho_{XY}\sigma_X\sigma_Y + \sigma_Y^2}. \tag{15}$$

Hence, for normally distributed r.v.'s, axiom B3 is equivalent to the condition that the price of the portfolio $X+Y$ is equal to the price of a normally distributed r.v. $\tilde{X}$ with mean $\alpha(\mu_X+\mu_Y)$ and standard deviation $\beta\sqrt{\sigma_X^2 + 2\rho_{XY}\sigma_X\sigma_Y + \sigma_Y^2}$ plus the price of a normally distributed r.v. $\tilde{Y}$ with mean $(1-\alpha)(\mu_X+\mu_Y)$ and standard deviation $(1-\beta)\sqrt{\sigma_X^2 + 2\rho_{XY}\sigma_X\sigma_Y + \sigma_Y^2}$, for any pair (α, β) with $0 \leq \alpha, \beta \leq 1$.

For later reference, we state the following equivalent definitions for a pair of r.v.'s to be *comonotonic*; we follow Denneberg (1994), Proposition 4.5:

Definition 3.2 (*Comonotonicity*). We say that a pair of r.v.'s $X, Y : \Omega \to \mathbb{R}$ is comonotonic, denoted by (X^c, Y^c), if

1. there exists no pair ω_1, ω_2 such that $X(\omega_1) < X(\omega_2)$ while $Y(\omega_1) > Y(\omega_2)$;
2. there exists a r.v. $Z : \Omega \to \mathbb{R}$ and non-decreasing functions $a(\cdot)$ and $b(\cdot)$ on $\mathbb{R}$ such that

$$X(\omega) = a(Z(\omega)), \qquad Y(\omega) = b(Z(\omega)), \quad \text{for all } \omega \in \Omega.$$

□

It is well known that for a bivariate normal comonotonic couple (X^c, Y^c) it holds that $\rho_{X^cY^c} = 1$. Hence, using (15) and axioms B1 and B3, respectively, one easily verifies that for a bivariate normal couple (X, Y)

$$\rho^v[\psi_{X+Y}^v] \geq \rho^v[\psi_{X^c+Y^c}^v] = \rho^v[\psi_X^v] + \rho^v[\psi_Y^v], \tag{16}$$

recalling that $h \leq 0$ and that $v > 0$. This means that for normally distributed r.v.'s, axiom B3 is equivalent to superadditivity and comonotonic additivity of the price measure $\pi^v[\cdot]$, which captures the diversification benefit of pooling.

Then we state the following theorem:

probability measures, our main focus is on the representation theorem involving the Esscher–Girsanov transform (Theorem 3.1) and the approximate-arbitrage-free financial derivative prices generated by it (Section 4 below). We thank Shaun Wang for pointing this out to us.

Theorem 3.1. *A functional* $\rho^v[\cdot]$ *satisfies the set of axioms* B1–B4 *if and only if there exists some non-decreasing function* $H:[-\infty,0]\to[0,1]$ *such that*

$$\rho^v\left[\psi_X^v\right]=\int_{[-\infty,0]}\psi_X^v(h)\mathrm{d}H(h). \tag{17}$$

Proof. Just as the proof of Lemma 2.1, the proof of this theorem is similar to the proof of Theorem 2 of Gerber and Goovaerts (1981); see again Goovaerts et al. (2004) for comments on that proof as well as footnote 1 of this paper. We therefore simply (re)identify the notation used in Gerber and Goovaerts (1981) with our notation: The function $\phi_X(\cdot)$ in Gerber and Goovaerts (1981) is our function $\psi_X^v(\cdot)$; their principle $H[\cdot]$ is our functional $\rho^v[\cdot]$; and their mixture function $F(\cdot)$ is our mixture function $H(\cdot)$. □

Since the next section will consider stochastic processes instead of r.v.'s, the definition of the Esscher–Girsanov transform has to be generalized. In the remainder of this section, we consider a discrete-time stochastic process $X=(X_i:i=1,2,\ldots)$, $X_0=x_0$, with independent increments. Clearly, it holds that

$$\mathrm{d}F_{X_n|X_0}(x_n|x_0)=\int_{x_{n-1}}\cdots\int_{x_1}\mathrm{d}F_{X_n|X_{n-1}}(x_n|x_{n-1})\times\mathrm{d}F_{X_{n-1}|X_{n-2}}(x_{n-1}|x_{n-2})\cdots\mathrm{d}F_{X_1|X_0}(x_1|x_0).$$

We state the following definition:

Definition 3.3 (*Discrete-Time Esscher–Girsanov Transform*). For the cdf $F_{X_n}(\cdot)$ with differential $\mathrm{d}F_{X_n}(\cdot)$ corresponding to a given continuous r.v. X_n, and a given strictly positive function $v(\cdot)$, we define by

$$\mathrm{d}F_{X_n|X_0}^{(h,v(\cdot))}(x_n|x_0)=\int_{x_{n-1}}\cdots\int_{x_1}\mathrm{e}^{h\sum_{j=0}^{n-1}v(x_j)\phi_{X_{j+1}|X_j}(x_{j+1}|x_j)-\frac{1}{2}h^2v(x_j)^2}\times\mathrm{d}F_{X_n|X_{n-1}}(x_n|x_{n-1})\mathrm{d}F_{X_{n-1}|X_{n-2}}(x_{n-1}|x_{n-2})\cdots\mathrm{d}F_{X_1|X_0}(x_1|x_0),\quad h\le 0, \tag{18}$$

its discrete-time Esscher–Girsanov transform with parameter h and penalty function $v(\cdot)$. □

The discrete-time Esscher–Girsanov transform can be regarded as a particular example of a *conditional Esscher transform* (see Bühlmann et al. (1996)), though there is a subtle difference being that, in accordance with the economic interpretation and axiomatization, we use a constant Arrow–Pratt measure of absolute risk aversion.

4. Financial derivative pricing by means of Esscher–Girsanov transforms

In this section, we will show that in a financial market in which the primary asset is represented by a stochastic differential equation (SDE) with respect to Brownian motion, the price mechanism based on the Esscher–Girsanov transform can generate approximate-arbitrage-free financial derivative prices.

We consider a finite time horizon $T<+\infty$. The flow of information is represented by the completed and right continuous filtration $\mathbb{F}=(\mathcal{F}_t:0\le t\le T)$, with for all $s\le t\le T$, $\mathcal{F}_s\subset\mathcal{F}_t\subset\mathcal{F}_T=\mathcal{F}$. Henceforth, for a given r.v. X, we denote by $\mathbb{E}_t[X]=\mathbb{E}[X|\mathcal{F}_t]$ the conditional expectation of X given $\mathcal{F}_t$.

We consider a time-homogeneous primary asset process $S=(S_t:0\le t\le T)$, defined by a stochastic differential equation of the form

$$S_0=s_0,\qquad \mathrm{d}S_t=\mu(S_t)\mathrm{d}t+\sigma(S_t)\mathrm{d}B_t, \tag{19}$$

in which $\mu:\mathbb{R}\to\mathbb{R}$, $\sigma:\mathbb{R}\to\mathbb{R}$ and $B=(B_t:0\le t\le T)$ denotes a standard Brownian motion. Henceforth, we understand $\sigma(S_t)\mathrm{d}B_t$ in the usual "Itô sense" (i.e., left point discretization). Under well-known regularity conditions on $\mu(\cdot)$ and $\sigma(\cdot)$ (see, e.g., Duffie (1996) or Karatzas and Shreve (1988)) there exists a unique Itô process S that solves (19) for each starting point s_0. We note that in general S need not be positive as it represents an arbitrary primary asset. If however the application that one has in mind requires positive primary asset processes, additional conditions on $\mu(\cdot)$ and $\sigma(\cdot)$ can be imposed.

Next, we consider a bond price process $\beta=(\beta_t:0\le t\le T)$, defined by the SDE

$$\beta_0>0,\qquad \mathrm{d}\beta_t=\beta_t r(S_t)\mathrm{d}t, \tag{20}$$

in which $r:\mathbb{R}\to\mathbb{R}$ is sufficiently smooth for the existence of the integral

$$\exp\left[\int_0^t r(S_\tau)\mathrm{d}\tau\right],\qquad t\in(0,T]. \tag{21}$$

Although we restrict ourselves to time-homogeneous primary asset processes, a generalization to general diffusion processes is feasible. Notice, however, that most of the well-known diffusion processes (e.g., the (geometric) Wiener process, the Ornstein–Uhlenbeck process, the Cox–Ingersoll–Ross model or the Bessel process) are already contained in (19).

We introduce a function $v:\mathbb{R}\to\mathbb{R}_+$. We assume henceforth that $v(\cdot)$ satisfies *Novikov's condition*:

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T v(S_\tau)^2\mathrm{d}\tau\right)\right]<+\infty.$$

Furthermore, given S and $v(\cdot)$, we introduce the r.v. $Z_t^T(S,h,v(\cdot))$ defined by

$$Z_t^T(S,h,v(\cdot))=h\int_t^T v(S_\tau)\mathrm{d}B_\tau-\frac{1}{2}h^2\int_t^T v(S_\tau)^2\mathrm{d}\tau,\qquad h\le 0. \tag{22}$$

It is well known that

$$\mathbb{E}_t\left[\mathrm{e}^{Z_t^T(S,h,v(\cdot))}\right]=1. \tag{23}$$

Recall Definition 3.3. Notice that the r.v. $\mathrm{e}^{Z_t^T(S,h,v(\cdot))}$ can be regarded as the continuous-time analog of the Radon–Nikodym derivative used on the right-hand side of (18). Hence, the r.v. $\mathrm{e}^{Z_t^T(S,h,v(\cdot))}$ can be used to establish the continuous-time analog of the discrete-time Esscher–Girsanov transform.

Consider a financial derivative security defined by the payoff $g(S_T)$ at time T, for some continuous function $g : \mathbb{R} \to \mathbb{R}$. Notice that in the case where S is a traded asset, we remain in a *complete* financial market setting, whereas if S is a non-traded asset the financial market is *incomplete*; see, e.g., Duffie (1996), p. 113, for a definition of a *complete* financial market. We introduce a function $\varphi^{(h)} : \mathbb{R} \times [0, T) \to \mathbb{R}$, with $\varphi^{(h)} \in C^{2,1}(\mathbb{R} \times [0, T))$. Let $\varphi^{(h)}(\cdot, \cdot)$ satisfy the boundary condition $\varphi^{(h)}(S_T, T) = g(S_T)$.

At time $t \in [0, T]$, the price $\pi_t^{v(\cdot)}[g(S_T)]$ of the derivative security $g(S_T)$ based on the Esscher–Girsanov transform, including the time-discount factor, is then given by

$$\pi_t^{v(\cdot)}[g(S_T)] = \int_{[-\infty,0]} \mathbb{E}_t\left[\mathrm{e}^{-\int_t^T r(S_\tau)\mathrm{d}\tau}\varphi^{(h)}(S_T, T)\mathrm{e}^{Z_t^T(S,h,v(\cdot))}\right]\mathrm{d}H(h), \tag{24}$$

for some non-decreasing function $H : [-\infty, 0] \to [0, 1]$.

Remark 4.1. The right-hand side of expression (24) can be regarded as a Feynman–Kac (path) integral (i.e., a probabilistic integral representation); see Feynman and Hibbs (1965) and Karatzas and Shreve (1988). □

Remark 4.2. The function $\varphi^{(h)}(\cdot, \cdot)$ will be chosen such that the calculation of the Feynman–Kac integral on the right-hand side of (24) becomes feasible. Whatever function $\varphi^{(h)}(\cdot, \cdot)$ is introduced, the right-hand side of (24) only depends on the terminal value $\varphi^{(h)}(S_T, T) = g(S_T)$. □

Then we state the following theorem:

Theorem 4.1. *The Esscher–Girsanov price of the derivative security $g(S_T)$, given in (24), satisfies the martingale property*

$$\pi_t^{v(\cdot)}[g(S_T)] = \int_{[-\infty,0]} \varphi^{(h)}(S_t, t)\mathrm{d}H(h), \tag{25}$$

whenever $\varphi^{(h)}(x, \tau)$ is the solution to the partial differential equation (PDE)

$$\frac{\partial\varphi^{(h)}(x,\tau)}{\partial\tau} + (\mu(x) + hv(x)\sigma(x))\frac{\partial\varphi^{(h)}(x,\tau)}{\partial x} + \frac{1}{2}\sigma(x)^2\frac{\partial^2\varphi^{(h)}(x,\tau)}{\partial x^2} = r(x)\varphi^{(h)}(x,\tau), \quad \tau \in (t, T]. \tag{26}$$

Proof. A proof of this theorem, based on Feynman–Kac integration, is provided in the Appendix. □

Remark 4.3. Notice that depending on the mixture function $H(\cdot)$ and the function $v(\cdot)$, the price mechanism in (24) can generate an infinite number of prices. □

In *approximate-arbitrage-free* financial markets (see Duffie (1996), p. 121, for a definition) there exists a probability measure that is equivalent to the "real" probability measure and under which all discounted price processes are martingales. For the original work on the relation between the condition of no arbitrage and the existence of an equivalent martingale measure, we refer the reader to Harrison and Kreps (1979) and Harrison and Pliska (1981). The basic idea of valuation by adjusting the primary asset process is from Cox and Ross (1976). If the financial market is complete, in addition to being approximate-arbitrage-free, the equivalent martingale measure under which all discounted price processes are martingales is unique, and is found in Theorem 4.2 stated below. If the financial market is incomplete, which is the usual case for (securitized) insurance risks, the derivative price processes cannot be hedged and no arbitrage arguments do not in general supply a unique equivalent martingale measure. In that case, as can be seen from Theorem 4.1, the Esscher–Girsanov transform is a tool for establishing a particular (axiomatically justified) equivalent martingale measure.

Theorem 4.2. *Suppose that S is a traded asset. If $v(x) = \frac{\mu(x)-r(x)x}{\sigma(x)}$ and*

$$H(h) = \begin{cases} 1, & h \geq -1 \\ 0, & \text{otherwise}, \end{cases}$$

then $\pi_t^{v(\cdot)}[g(S_T)]$ coincides with the approximate-arbitrage-free price of the financial derivative $g(S_T)$ at time $t \in [0, T]$.

Proof. The well-known PDE characterization of approximate-arbitrage-free financial derivative prices in a complete financial market (see, e.g., Duffie (1996), p. 90) coincides with the PDE in (26) in the case where $v(x) = \frac{\mu(x)-r(x)x}{\sigma(x)}$ and $h = -1$. This proves the stated result. □

Remark 4.4. The mixture function $H(\cdot)$ derived in Theorem 4.2 can be regarded as a cdf corresponding to a r.v. degenerate at -1. From an economic point of view, this mixture function corresponds to a representative agent with Arrow–Pratt measure of absolute risk aversion equal to 1. Notice however that if in the economy considered there exists a representative agent with a negative exponential utility function and Arrow–Pratt measure of absolute risk aversion equal to $-h$, the function $v(\cdot)$ derived in Theorem 4.2 can be scaled accordingly. □

Acknowledgements

We thank Marc Decamps, Hans Gerber, Elias Shiu, Shaun Wang and an anonymous referee for valuable comments. Marc Goovaerts acknowledges the financial support of the Onderzoeksfonds K.U. Leuven (GOA/02: Actuariële, financiële en statistische aspecten van afhankelijkheden in verzekerings- en financiële portefeuilles). Roger Laeven acknowledges the support of the Netherlands Organization for Scientific Research (No. NWO 42511013 and VENI Grant 2006).

Appendix

Proof of Theorem 4.1. We introduce the notation

$$\mathbb{E}_t\left[\mathrm{e}^{-\int_t^T r(S_\tau)\mathrm{d}\tau}\varphi^{(h)}(S_T, T)\mathrm{e}^{Z_t^T(S,h,v(\cdot))}\right] = \mathbb{I}_t\left[\mathrm{e}^{-\int_t^T r(s_\tau)\mathrm{d}\tau}\varphi^{(h)}(s_T, T)\mathrm{e}^{z_t^T(s,h,v(\cdot))}\right],$$

where the operator $\mathbb{I}_t[\cdot]$ denotes the distribution weighted integral.

To prove the theorem, we perform a substitution for S_τ in the Feynman–Kac integral on the right-hand side of (24). We introduce a continuous and twice differentiable function $u : \mathbb{R} \to \mathbb{R}$. Using Itô's formula,

$$\mathrm{d}u(S_t) = \left(\mu(S_t)u'(S_t) + \frac{1}{2}\sigma(S_t)^2 u''(S_t)\right)\mathrm{d}t + \sigma(S_t)u'(S_t)\mathrm{d}B_t,$$

where $u'(\cdot)$ and $u''(\cdot)$ denote the first and second derivatives of $u(\cdot)$ in the usual way. We denote by $u_0(\cdot,\cdot)$ a solution to

$$\left(\mu(S_\tau)u'(S_\tau) + \frac{1}{2}\sigma(S_\tau)^2 u''(S_\tau)\right)\mathrm{d}\tau = 0, \quad \tau \in [t, T].$$

We define the functions $\tilde{\varphi}^{(h)} : \mathbb{R} \times [0, T] \to \mathbb{R}$, $\tilde{v} : \mathbb{R} \to \mathbb{R}_+$ and $\tilde{r} : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{\varphi}^{(h)}(u_0(x), \tau) = \varphi^{(h)}(x, \tau),$$
$$\tilde{v}(u_0(x)) = v(x),$$
$$\tilde{r}(u_0(x)) = r(x),$$

and let $\tilde{Z}_t^T(u_0(S), h, \tilde{v}(\cdot)) = Z_t^T(S, h, v(\cdot))$. In the following, we write $u_0(s_t) = u_t$ and $s_t = w(u_t)$. Furthermore, we write

$$f(u_t) = \sigma(w(u_t))u_0'(w(u_t)).$$

We introduce a sequence of partitions P_n given by

$$P_n = \{t_{0,n}, t_{1,n}, \ldots, t_{n-1,n}, t_{n,n}\}, \quad n = 1, 2, \ldots,$$

in which $t_{m,n}, m = 0, 1, \ldots, n$ are real numbers satisfying $t = t_{0,n} < t_{1,n} < \cdots < t_{n-1,n} < t_{n,n} = T$, with $(t_{n,n} - t_{0,n})/n = \varepsilon_n$ and $t_{m,n} = t_{m-1,n} + \varepsilon_n$, and hence $\lim_{n\to+\infty} \max_{1\le m\le n} |t_{m,n} - t_{m-1,n}| = 0$. Below we use the expansion

$$\begin{aligned}\tilde{\varphi}^{(h)}(u_T, T) = {}& \tilde{\varphi}^{(h)}(u_{T-\varepsilon_n}, T-\varepsilon_n) + (u_T - u_{T-\varepsilon_n}) \times \frac{\partial \tilde{\varphi}^{(h)}(u_{T-\varepsilon_n}, T-\varepsilon_n)}{\partial u_{T-\varepsilon_n}} \\ &+ \frac{1}{2} f(u_{T-\varepsilon_n})^2 \varepsilon_n \frac{\partial^2 \tilde{\varphi}^{(h)}(u_{T-\varepsilon_n}, T-\varepsilon_n)}{\partial u_{T-\varepsilon_n}^2} \\ &+ \varepsilon_n \frac{\partial \tilde{\varphi}^{(h)}(u_{T-\varepsilon_n}, T-\varepsilon_n)}{\partial (T-\varepsilon_n)} + o(\varepsilon_n),\end{aligned}$$

where as usual $\limsup_{n\to+\infty} \frac{o(\varepsilon_n)}{\varepsilon_n} = 0$.

Then the right-hand side of (24) can be written as follows:

$$\begin{aligned}\pi_t^{v(\cdot)}[g(S_T)] = {}& \int_{[-\infty,0]} \Bigg(\lim_{n\to+\infty} \int_{-\infty}^{+\infty} \Bigg((u_T - u_{T-\varepsilon_n}) \\ &\times \frac{\partial \tilde{\varphi}^{(h)}(u_{T-\varepsilon_n}, T-\varepsilon_n)}{\partial u_{T-\varepsilon_n}} \\ &+ \frac{1}{2} f(u_{T-\varepsilon_n})^2 \varepsilon_n \frac{\partial^2 \tilde{\varphi}^{(h)}(u_{T-\varepsilon_n}, T-\varepsilon_n)}{\partial u_{T-\varepsilon_n}^2} \\ &+ \varepsilon_n \frac{\partial \tilde{\varphi}^{(h)}(u_{T-\varepsilon_n}, T-\varepsilon_n)}{\partial (T-\varepsilon_n)} \\ &- \tilde{r}(u_{T-\varepsilon_n}) \varepsilon_n \tilde{\varphi}^{(h)}(u_{T-\varepsilon_n}, T-\varepsilon_n) \Bigg) \\ &\times \frac{1}{\sqrt{2\pi\varepsilon_n} f(u_{T-\varepsilon_n})} \mathrm{e}^{h\tilde{v}(u_{T-\varepsilon_n}) \frac{(u_T - u_{T-\varepsilon_n})}{f(u_{T-\varepsilon_n})} - \frac{1}{2} h^2 \tilde{v}(u_{T-\varepsilon_n})^2 \varepsilon_n} \\ &\times \mathrm{e}^{-\frac{1}{2} \frac{(u_T - u_{T-\varepsilon_n})^2}{f(u_{T-\varepsilon_n})^2 \varepsilon_n}} \mathrm{d}u_T \times \mathbb{I}_t \left[\mathrm{e}^{-\int_t^{T-\varepsilon_n} \tilde{r}(u_\tau)\mathrm{d}\tau} \mathrm{e}^{\tilde{z}_t^{T-\varepsilon_n}(u,h,\tilde{v}(\cdot))} \right] \\ &+ \mathbb{I}_t \left[\mathrm{e}^{-\int_t^{T-\varepsilon_n} \tilde{r}(u_\tau)\mathrm{d}\tau} \tilde{\varphi}^{(h)}(u_{T-\varepsilon_n}, T-\varepsilon_n) \times \mathrm{e}^{\tilde{z}_t^{T-\varepsilon_n}(u,h,\tilde{v}(\cdot))} \right] \Bigg) \mathrm{d}H(h).\end{aligned}$$

One easily verifies that

$$\begin{aligned}&\int_{-\infty}^{+\infty} \frac{(u_T - u_{T-\varepsilon_n})}{\sqrt{2\pi\varepsilon_n} f(u_{T-\varepsilon_n})} \mathrm{e}^{h\tilde{v}(u_{T-\varepsilon_n}) \frac{(u_T - u_{T-\varepsilon_n})}{f(u_{T-\varepsilon_n})} - \frac{1}{2} h^2 \tilde{v}(u_{T-\varepsilon_n})^2 \varepsilon_n} \times \mathrm{e}^{-\frac{1}{2} \frac{(u_T - u_{T-\varepsilon_n})^2}{f(u_{T-\varepsilon_n})^2 \varepsilon_n}} \mathrm{d}u_T \\ &= \int_{-\infty}^{+\infty} \frac{(u_T - u_{T-\varepsilon_n})}{\sqrt{2\pi\varepsilon_n} f(u_{T-\varepsilon_n})} \left(1 + h\tilde{v}(u_{T-\varepsilon_n}) \frac{(u_T - u_{T-\varepsilon_n})}{f(u_{T-\varepsilon_n})} \right) \\ &\quad \times \mathrm{e}^{-\frac{1}{2} h^2 \tilde{v}(u_{T-\varepsilon_n})^2 \varepsilon_n - \frac{1}{2} \frac{(u_T - u_{T-\varepsilon_n})^2}{f(u_{T-\varepsilon_n})^2 \varepsilon_n}} \mathrm{d}u_T + o(\varepsilon_n) \\ &= h\tilde{v}(u_{T-\varepsilon_n}) f(u_{T-\varepsilon_n}) \varepsilon_n \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\varepsilon_n} f(u_{T-\varepsilon_n})} \\ &\quad \times \mathrm{e}^{-\frac{1}{2} h^2 \tilde{v}(u_{T-\varepsilon_n})^2 \varepsilon_n - \frac{1}{2} \frac{(u_T - u_{T-\varepsilon_n})^2}{f(u_{T-\varepsilon_n})^2 \varepsilon_n}} \mathrm{d}u_T + o(\varepsilon_n) \\ &= h\tilde{v}(u_{T-\varepsilon_n}) f(u_{T-\varepsilon_n}) \varepsilon_n + o(\varepsilon_n).\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\pi_t^{v(\cdot)}[g(S_T)] = {}& \int_{[-\infty,0]} \Bigg(\lim_{n\to+\infty} \varepsilon_n \Bigg(h\tilde{v}(u_{T-\varepsilon_n}) f(u_{T-\varepsilon_n}) \\ &\times \frac{\partial \tilde{\varphi}^{(h)}(u_{T-\varepsilon_n}, T-\varepsilon_n)}{\partial u_{T-\varepsilon_n}} + \frac{1}{2} f(u_{T-\varepsilon_n})^2 \\ &\times \frac{\partial^2 \tilde{\varphi}^{(h)}(u_{T-\varepsilon_n}, T-\varepsilon_n)}{\partial u_{T-\varepsilon_n}^2} + \frac{\partial \tilde{\varphi}^{(h)}(u_{T-\varepsilon_n}, T-\varepsilon_n)}{\partial (T-\varepsilon_n)} \\ &- \tilde{r}(u_{T-\varepsilon_n}) \tilde{\varphi}^{(h)}(u_{T-\varepsilon_n}, T-\varepsilon_n) \Bigg) \\ &\times \mathbb{I}_t \left[\mathrm{e}^{-\int_t^{T-\varepsilon_n} \tilde{r}(u_\tau)\mathrm{d}\tau} \mathrm{e}^{\tilde{z}_t^{T-\varepsilon_n}(u,h,\tilde{v}(\cdot))} \right] \\ &+ \mathbb{I}_t \left[\mathrm{e}^{-\int_t^{T-\varepsilon_n} \tilde{r}(u_\tau)\mathrm{d}\tau} \tilde{\varphi}^{(h)}(u_{T-\varepsilon_n}, T-\varepsilon_n) \times \mathrm{e}^{\tilde{z}_t^{T-\varepsilon_n}(u,h,\tilde{v}(\cdot))} \right] \Bigg) \mathrm{d}H(h).\end{aligned}$$

Now, we let $\tilde{\varphi}^{(h)}(\cdot,\cdot)$ be the solution of the PDE

$$\frac{\partial \tilde{\varphi}^{(h)}(u,\tau)}{\partial \tau} + h\tilde{v}(u) f(u) \frac{\partial \tilde{\varphi}^{(h)}(u,\tau)}{\partial u} + \frac{1}{2} f(u)^2 \frac{\partial^2 \tilde{\varphi}^{(h)}(u,\tau)}{\partial u^2} = \tilde{r}(u) \tilde{\varphi}^{(h)}(u,\tau), \quad \tau \in [t, T]. \tag{27}$$

We finally note that if $\tilde{\varphi}^{(h)}(\cdot,\cdot)$ satisfies the PDE (27), then $\varphi^{(h)}(\cdot,\cdot)$ satisfies the PDE (26). Then iterative application of the above procedure yields the stated result.

The PDE in (26) coincides with the well-known Kolmogorov Backward Equation (see, e.g., Duffie (1996), p. 294) of $\varphi^{(h)}(x,\tau)$, with the drift function adjusted for the change of probability measure as established by the Esscher–Girsanov transform. □

References

Borch, Karl, 1962. Equilibrium in a reinsurance market. Econometrica 3, 424–444.

Bühlmann, Hans, 1980. An economic premium principle. Astin Bulletin 11, 52–60.

Bühlmann, Hans, 1985. Premium calculation from top down. Astin Bulletin 15, 89–101.

Bühlmann, Hans, Delbaen, Freddy, Embrechts, Paul, Shiryaev, Albert N., 1996. No-arbitrage, change of measure and conditional Esscher transforms. CWI Quarterly 9, 291–317.

Cox, John, Ross, Stephen, 1976. The valuation of options for alternative stochastic processes. Journal of Financial Economics 3, 145–166.

Denneberg, Dieter, 1994. Non-Additive Measure and Integral. Kluwer Academic Publishers, Boston.

Denuit, Michel, 2001. Laplace transform ordering of actuarial quantities. Insurance: Mathematics and Economics 29, 83–102.

Denuit, Michel, Dhaene, Jan, Goovaerts, Marc J., Kaas, Rob, Laeven, Roger J.A., 2006. Risk measurement with equivalent utility principles, in: Rüschendorf, Ludger (Ed.), Risk Measures: General Aspects and Applications. Statistics and Decisions 24, 1–26 (special issue).

Duffie, Darrell, 1996. Dynamic Asset Pricing Theory, 2nd edn. Princeton University Press, Princeton.

Embrechts, Paul, 2000. Actuarial versus financial pricing of insurance. Risk Finance 1, 17–26.

Esscher, Fredrik, 1932. On the probability function in the collective theory of risk. Scandinavian Actuarial Journal 15, 175–195.

Feynman, Richard P., Hibbs, Albert R., 1965. Quantum Mechanics and Path Integrals. McGraw-Hill, New York.

Gerber, Hans U., 1979. An Introduction to Mathematical Risk Theory. In: Huebner Foundation Monograph, vol. 8. Homewood, Illinois.

Gerber, Hans U., Goovaerts, Marc J., 1981. On the representation of additive principles of premium calculation. Scandinavian Actuarial Journal 4, 221–227.

Gerber, Hans U., 1981. The Esscher premium principle: A criticism, comment. Astin Bulletin 12, 139–140.

Gerber, Hans U., Shiu, Elias S.W., 1994. Option pricing by Esscher transforms. Transactions of the Society of Actuaries 46, 99–191.

Gerber, Hans U., Shiu, Elias S.W., 1996. Actuarial bridges to dynamic hedging and option pricing. Insurance: Mathematics and Economics 18, 183–218.

Goovaerts, Marc J., De Vylder, F. Etienne C., Haezendonck, Jean, 1984. Insurance Premiums. North Holland Publishing, Amsterdam.

Goovaerts, Marc J., Kaas, Rob, Laeven, Roger J.A., Tang, Qihe, 2004. A comonotonic image of independence for additive risk measures. Insurance: Mathematics and Economics 35, 581–594.

Hadar, Joseph, Russell, William R., 1969. Rules for ordering uncertain prospects. American Economic Review 59, 25–34.

Harrison, J. Michael, Kreps, David M., 1979. Martingales and arbitrage in multiperiod securities markets. Journal of Economic Theory 20, 381–408.

Harrison, J. Michael, Pliska, Stanley R., 1981. Martingales and stochastic integrals in the theory of continuous trading. Stochastic Processes and their Applications 11, 215–260.

Iwaki, Hideki, Kijima, Masaaki, Morimoto, Yuji, 2001. An economic premium principle in a multiperiod economy. Insurance: Mathematics and Economics 28, 325–339.

Karatzas, Ioannis, Shreve, Steven E., 1988. Brownian Motion and Stochastic Calculus. Springer-Verlag, New York.

Laeven, Roger J.A., 2005. Essays on risk measures and stochastic dependence. Tinbergen Institute Research Series 360. Thela Thesis. Amsterdam.

Laeven, Roger J.A., Goovaerts, Marc J., 2007. Premium calculation and insurance pricing. In: Encyclopedia of Quantitative Risk Assessment. Wiley (in press).

Rothschild, Michael, Stiglitz, Joseph E., 1970. Increasing risk I: A definition. Journal of Economic Theory 2, 225–243.

Savage, Leonard J., 1954. The Foundations of Statistics. Wiley, New York.

Wang, Shaun S., 2003. Equilibrium pricing transforms: New results using Bühlmann's 1980 economic model. Astin Bulletin 33, 57–73.